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Research Article

Optimal Backward Perturbation Analysis for the Block Skew Circulant Linear Systems with Skew Circulant Blocks

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We first give the block style spectral decomposition of arbitrary block skew circulant matrix with skew circulant blocks. Secondly, we obtain the singular value of block skew circulant matrix with skew circulant blocks as well. Finally, based on the block style spectral decomposition, we deal with the optimal backward perturbation analysis for the block skew circulant linear system with skew circulant blocks.

1. Introduction

A block skew circulant matrix with skew circulant blocks with the first row $(a_{11}, \dots, a_{1m}, a_{21}, \dots, a_{2m}, \dots, a_{n1}, \dots, a_{nm})$ is meant a square matrix of the following form:

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_{n-1} & A_n \\ -A_n & A_1 & A_2 & \cdots & A_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -A_3 & \cdots & -A_n & A_1 & A_2 \\ -A_2 & -A_3 & \cdots & -A_n & A_1 \end{pmatrix}, \tag{1}$$

and for any $k = 1, 2, \dots, n$,

$$A_k = \begin{pmatrix} a_{k1} & a_{k2} & \cdots & a_{k(m-1)} & a_{km} \\ -a_{km} & a_{k1} & a_{k2} & \cdots & a_{k(m-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -a_{k3} & \cdots & -a_{km} & a_{k1} & a_{k2} \\ -a_{k2} & -a_{k3} & \cdots & -a_{km} & a_{k1} \end{pmatrix}, \tag{2}$$

denoted by $\mathrm{BC}_{-1,-1}^{n,m}(a_{11}, \dots, a_{1m}, \dots, a_{n1}, \dots, a_{nm})$. Note that in this paper all facts are based on real field.

Skew circulant matrices have important applications in various disciplines including image processing, signal processing, solving Toeplitz matrix problems, preconditioner, and solving least squares problems in [1–10].

Liu and Guo [11] gave the optimal backward perturbation analysis for a block circulant linear system. Li et al. [12] gave the style spectral decomposition of skew circulant matrix firstly and then dealt with the optimal backward perturbation analysis for the skew circulant linear system. The optimal backward perturbation bounds for underdetermined systems are studied by J.G. Sun and Z. Sun in [13]. Some new theorems generalizing a result of Oettli and Prager are applied to a posteriori analysis of the compatibility of a computed solution to the uncertain data of a linear system by Rigal and Gaches in [14]. Systems with BC structure appear in the context of multichannel signal estimation [15, 16], image restoration [17], cyclic convolution filter banks [18], texture synthesis and recognition [19], and so on.

The block skew circulant matrix with skew circulant blocks is an extension of skew circulant matrix and we believe the block skew circulant linear system with skew circulant blocks can be used in those fields as well. In this paper, firstly, by using the style spectral decomposition of a special skew circulant matrix C in [12], we get the block style spectral decomposition of arbitrary block skew circulant matrix with skew circulant blocks. Secondly, we obtain the singular value of block skew circulant matrix with skew circulant blocks as well. Finally, we deal with the optimal backward perturbation analysis for the block skew circulant linear system with skew circulant blocks on the base of its block style spectral decomposition.

2. The Block Style Spectral Decomposition of Block Skew Circulant Matrix with Skew Circulant Blocks

Let matrix A be a block skew circulant matrix with skew circulant blocks as in the form of (1); then by using the properties of Kronecker products in [20], the matrix A can be decomposed as

$$A = \sum_{k=1}^{n} \left(C_{n\times n}^{k-1} \otimes A_k \right), \tag{3}$$

where

$$C_{n\times n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}. \tag{4}$$

According to the style spectral decomposition of basic skew circulant (please refer to equations (10) and (11) in [12]), the style spectral decomposition of the matrix $C_{n\times n}^{k}$ is

$$C_{n\times n}^{k} = QC_0^kQ^T, \tag{5}$$

where Q is an orthogonal matrix, $\theta_j = ((2j-1)/n)\pi$,

$$C_0^k = \begin{pmatrix} C_1^k & & & \\ & C_2^k & & \\ & & \ddots & \\ & & & C_{n/2}^k \end{pmatrix}, \quad (n \text{ is even}),$$

$$C_0^k = \begin{pmatrix} C_1^k & & & \\ & \ddots & & \\ & & C_{(n-1)/2}^k & \\ & & & (-1)^k \end{pmatrix}, \quad (n \text{ is odd}), \tag{6}$$

$$C_j^k = \begin{pmatrix} \cos k\theta_j & \sin k\theta_j \\ -\sin k\theta_j & \cos k\theta_j \end{pmatrix},$$

$$j = \begin{cases} 1, 2, \ldots, \dfrac{n}{2}, & \text{when } n \text{ is even}, \\ 1, 2, \ldots, \dfrac{n-1}{2}, & \text{when } n \text{ is odd}. \end{cases} \tag{7}$$

Consider (3) and (5); the matrix A can be decomposed as

$$\begin{aligned} A &= \sum_{k=1}^{n} \left(C_{n\times n}^{k-1} \otimes A_k \right) = \sum_{k=1}^{n} \left(QC_0^{k-1}Q^T \right) \otimes A_k \\ &= \sum_{k=1}^{n} (Q \otimes I_m) \left(C_0^{k-1} \otimes A_k \right) \left(Q^T \otimes I_m \right) \\ &= (Q \otimes I_m) \left(\sum_{k=1}^{n} C_0^{k-1} \otimes A_k \right) \left(Q^T \otimes I_m \right). \end{aligned} \tag{8}$$

Noticing that $Q \otimes I_m$ is an orthogonal matrix, so (8) is the block style spectral decomposition of the matrix A.

3. The Structured Perturbation Analysis

In this section, we give the structured perturbation analysis for the block skew circulant linear systems with skew circulant blocks.

3.1. Condition Number and Relative Error of Block Skew Circulant Linear System with Skew Circulant Blocks.

Consider

$$Ax = b, \tag{9}$$

where A is defined in (1).

From (8) and the property of Kronecker products in [20], we express the matrix A by using the elements in its first row as

$$\begin{aligned} A &= \sum_{k=1}^{n} \left(C_{n\times n}^{k-1} \otimes A_k \right) \\ &= \sum_{k=1}^{n} \left[C_{n\times n}^{k-1} \otimes \left(\sum_{l=1}^{m} a_{kl} C_{m\times m}^{l-1} \right) \right] \\ &= \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl} \left(C_{n\times n}^{k-1} \otimes C_{m\times m}^{l-1} \right), \end{aligned} \tag{10}$$

where

$$C_{m\times m} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m}. \tag{11}$$

We denote the eigenvalues of matrix $C_{n\times n}$ as ε_i ($i = 1, 2, \ldots, n$), and denote the eigenvalues of matrix $C_{m\times m}$ as δ_j ($j = 1, 2, \ldots, m$); then the eigenvalues of A are (regarding more properties, please refer to [20, 21])

$$\lambda_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl} \varepsilon_i^{k-1} \delta_j^{l-1}. \tag{12}$$

Lemma 1. *A is an invertible matrix if and only if $f(\varepsilon_i, \delta_j) \neq 0$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$), where*

$$f\left(\varepsilon_i, \delta_j\right) = \lambda_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl} \varepsilon_i^{k-1} \delta_j^{l-1}. \tag{13}$$

Let

$$\sigma_{ij} = \left| f\left(\varepsilon_i, \delta_j\right) \right|, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m,$$
$$\mathscr{K} = \frac{\max\left\{\sigma_{ij}\right\}}{\min\left\{\sigma_{ij}\right\}}. \tag{14}$$

Theorem 2. *If $A = BC_{-1,-1}^{n,m}(a_{11}, \ldots, a_{1m}, \ldots, a_{n1}, \ldots, a_{nm})$ is a block skew circulant matrix with skew circulant blocks, then the singular values of matrix A are $\sigma_{11}, \ldots, \sigma_{1m}, \sigma_{21}, \ldots, \sigma_{2m}, \ldots, \sigma_{n1}, \ldots, \sigma_{nm}$.*

Proof. Obviously, the matrix A has a form as (1) and the conjugate transpose of A is

$$A^* = \begin{pmatrix} A_1^* & -A_n^* & \cdots & -A_3^* & -A_2^* \\ A_2^* & A_1^* & \ddots & \vdots & -A_3^* \\ \vdots & A_2^* & \ddots & -A_n^* & \vdots \\ A_{n-1}^* & \vdots & \ddots & A_1^* & -A_n^* \\ A_n^* & A_{n-1}^* & \cdots & A_2^* & A_1^* \end{pmatrix}. \quad (15)$$

Through a direct calculation, we can get $AA^* = A^*A$, and that means that A is a normal matrix. By using Theorem 2.5.4 in [22], we know that A is unitarily diagonalizable. That is, there is a unitary matrix $\mathbb{U} \in M_{mn}$ such that

$$\mathbb{U}^* A \mathbb{U} = \Lambda = \operatorname{diag}(\lambda_{11}, \ldots, \lambda_{1m}, \ldots, \lambda_{n1}, \ldots, \lambda_{nm}). \quad (16)$$

where λ_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ are the eigenvalues of A. Taking a conjugate transpose at both sides of (16),

$$\mathbb{U}^* A^* \mathbb{U} = \Lambda^* = \operatorname{diag}(\overline{\lambda}_{11}, \ldots, \overline{\lambda}_{1m}, \ldots, \overline{\lambda}_{n1}, \ldots, \overline{\lambda}_{nm}), \quad (17)$$

and so, we have

$$\begin{aligned} &\mathbb{U}^* (A^* A) \mathbb{U} \\ &\quad = (\mathbb{U}^* A^* \mathbb{U})(\mathbb{U}^* A \mathbb{U}) \\ &\quad = \operatorname{diag}(|\lambda_{11}|^2, \ldots, |\lambda_{1m}|^2, \ldots, |\lambda_{n1}|^2, \ldots, |\lambda_{nm}|^2). \end{aligned} \quad (18)$$

Hence, for any $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, the eigenvalues of matrix A^*A are

$$\lambda_{ij}(A^*A) = |\lambda_{ij}|^2. \quad (19)$$

Therefore, we can get the singular value of A as

$$\sigma_{ij}(A) = [\lambda_{ij}(A^*A)]^{1/2} = |\lambda_{ij}|. \quad (20)$$

Recalling (13) and (14), the proof is completed. □

Since the spectral norm of matrix A is defined as

$$\|A\|_2 = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} [\lambda_{ij}(A^*A)]^{1/2}, \quad (21)$$

by using Theorem 2, we have the following corollary.

Corollary 3. *Let $A = BC_{-1,-1}^{n,m}(a_{11}, \ldots, a_{1m}, \ldots, a_{n1}, \ldots, a_{nm})$ be a block skew circulant matrix with skew circulant blocks; then the spectral norm of matrix A is*

$$\|A\|_2 = \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} \{\sigma_{ij}\}. \quad (22)$$

By using equations (10) and (11) in [12], we can express the matrix $C_{n\times n}$ and $C_{m\times m}$ as

$$C_{n\times n} = Q_n C_{n0} Q_n^T, \qquad C_{m\times m} = Q_m C_{m0} Q_m^T, \quad (23)$$

where

$$\begin{gathered} C_{n0} = \begin{pmatrix} C_{11} & & & \\ & C_{22} & & \\ & & \ddots & \\ & & & C_{tt} \end{pmatrix}, \\ C_{m0} = \begin{pmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{ss} \end{pmatrix}, \\ C_{hh} = \begin{pmatrix} \cos\theta_h & \sin\theta_h \\ -\sin\theta_h & \cos\theta_h \end{pmatrix}, \quad h = 1, 2, \ldots, t. \\ D_{rr} = \begin{pmatrix} \cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r \end{pmatrix}, \quad r = 1, 2, \ldots, s. \\ t = \begin{cases} \dfrac{n}{2}, & \text{when } n \text{ is even}, \\ \dfrac{n-1}{2} + 1, & \text{when } n \text{ is odd}, \end{cases} \\ s = \begin{cases} \dfrac{m}{2}, & \text{when } m \text{ is even}, \\ \dfrac{m-1}{2} + 1, & \text{when } m \text{ is odd}. \end{cases} \end{gathered} \quad (24)$$

So, we can get

$$A = \mathbb{Q}\left(\sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl} C_{n0}^{k-1} \otimes C_{m0}^{l-1}\right)\mathbb{Q}^T, \quad (25)$$

where $\mathbb{Q} = (Q_n \otimes I_m)(I_n \otimes Q_m)$,

$$\sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl} C_{n0}^{k-1} \otimes C_{m0}^{l-1} = \begin{pmatrix} B_{11} & & \\ & \ddots & \\ & & B_{tt} \end{pmatrix}, \quad (26)$$

and $B_{pp} = \sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl} C_{pp} \otimes C_{m0}^{l-1}$, $p = 1, 2, \ldots, t$.

Let ΔA, Δb be the perturbation of the coefficient matrix A and vector b, respectively, where $\Delta A = BC_{-1,-1}^{n,m}(\delta a_{11}, \ldots, \delta a_{1m}, \ldots, \delta a_{n1}, \ldots, \delta a_{nm})$ is a block skew circulant matrix with skew circulant blocks, has the form as follows:

$$\begin{gathered} \Delta A = \begin{pmatrix} \Delta A_1 & \cdots & \Delta A_{n-1} & \Delta A_n \\ -\Delta A_n & \Delta A_1 & \cdots & \Delta A_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ -\Delta A_2 & \cdots & -\Delta A_n & \Delta A_1 \end{pmatrix}, \\ \Delta A_k = \begin{pmatrix} \delta a_{k1} & \cdots & \delta a_{k(m-1)} & \delta a_{km} \\ -\delta a_{kn} & \delta a_{k1} & \cdots & \delta a_{k(m-1)} \\ \vdots & \ddots & \ddots & \vdots \\ -\delta a_{k2} & \cdots & -\delta a_{km} & \delta a_{k1} \end{pmatrix}, \\ k = 1, 2, \ldots, n. \end{gathered} \quad (27)$$

Let

$$\widehat{A} = A + \Delta A, \qquad \widehat{b} = b + \Delta b,$$
$$\widehat{f}\left(\varepsilon_i, \delta_j\right) = \sum_{k=1}^{n}\sum_{l=1}^{m}\left(a_{kl} + \delta a_{kl}\right)\varepsilon_i^{k-1}\delta_j^{l-1}. \tag{28}$$

If

$$\sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right| < \min_{\substack{1\le i\le n\\1\le j\le m}}\left\{\sigma_{ij}\right\}, \tag{29}$$

then

$$\begin{aligned}\left|\widehat{f}\left(\varepsilon_i, \delta_j\right)\right| &= \left|\sum_{k=1}^{n}\sum_{l=1}^{m}\left(a_{kl} + \delta a_{kl}\right)\varepsilon_i^{k-1}\delta_j^{k-1}\right| \\ &\ge \left|\sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl}\varepsilon_i^{k-1}\delta_j^{l-1}\right| \\ &\quad - \sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right|\left|\varepsilon_i\right|^{k-1}\left|\delta_j\right|^{l-1} \\ &\ge \min_{\substack{1\le i\le n\\1\le j\le m}}\left\{\sigma_{ij}\right\} - \sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right| > 0.\end{aligned} \tag{30}$$

By using Lemma 1, we know that $\widehat{A}$ is an invertible matrix. Let

$$\sigma_{\min} = \min_{\substack{1\le i\le n\\1\le j\le m}}\left\{\sigma_{ij}\right\}, \qquad \Delta = \sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right|. \tag{31}$$

By $Ax = b$, $\widehat{A}\widehat{x} = \widehat{b}$, we get

$$\begin{aligned}\widehat{x} - x &= \widehat{A}^{-1}\widehat{b} - A^{-1}b \\ &= \widehat{A}^{-1}\left(b + \delta b\right) - A^{-1}b \\ &= \widehat{A}^{-1}\delta b + \left(\widehat{A}^{-1} - A^{-1}\right)b \\ &= \widehat{A}^{-1}\delta b + \left(\widehat{A}^{-1} - A^{-1}\right)Ax \\ &= \widehat{A}^{-1}\delta b + \widehat{A}^{-1}\left(A - \widehat{A}\right)x\end{aligned} \tag{32}$$
$$\|\widehat{x} - x\|_2 \le \left\|\widehat{A}^{-1}\right\|_2\|\delta b\|_2 + \left\|\widehat{A}^{-1}\right\|_2\left\|\widehat{A} - A\right\|_2\|x\|_2.$$

Since $\widehat{A}^{-1}\cdot\widehat{A} = I_{mn}$ and $\|I_{mn}\|_2 = 1$, so we have $\|\widehat{A}^{-1}\|_2\|\widehat{A}\|_2 \le 1$. Besides, we know that

$$\left\|\widehat{A}\right\|_2 = \max_{\substack{1\le i\le n\\1\le j\le m}}\left|\widehat{f}\left(\varepsilon_i, \delta_j\right)\right| \ge \sigma_{\min} - \Delta > 0. \tag{33}$$

So, we obtain

$$\left\|\widehat{A}^{-1}\right\|_2 \le \frac{1}{\left\|\widehat{A}\right\|_2} \le \frac{1}{\sigma_{\min} - \Delta}. \tag{34}$$

Hence,

$$\begin{aligned}\|\widehat{x} - x\|_2 &\le \frac{\|\delta b\|_2}{\sigma_{\min} - \Delta} + \frac{\left\|\widehat{A} - A\right\|_2\|x\|_2}{\sigma_{\min} - \Delta}, \\ \frac{\|\widehat{x} - x\|_2}{\|x\|_2} &\le \frac{\|\delta b\|_2}{\left(\sigma_{\min} - \Delta\right)\|x\|_2} + \frac{\left\|\widehat{A} - A\right\|_2}{\sigma_{\min} - \Delta} \\ &= \frac{\|A\|_2}{\sigma_{\min} - \Delta}\left[\frac{\|\delta b\|_2}{\|A\|_2\|x\|_2} + \frac{\left\|\widehat{A} - A\right\|_2}{\|A\|_2}\right] \\ &\le \frac{\|A\|_2}{\sigma_{\min} - \Delta}\left[\frac{\|\delta b\|_2}{\|b\|_2} + \frac{\left\|\widehat{A} - A\right\|_2}{\|A\|_2}\right],\end{aligned} \tag{35}$$

where

$$\|A\|_2 = \max_{\substack{1\le i\le n\\1\le j\le m}}\left\{\sigma_{ij}\right\}. \tag{36}$$

Notice that $\widehat{A} - A = \Delta A$ is a block skew circulant matrix with skew circulant blocks, and $\|\widehat{A} - A\|_2 = |-1|\|\widehat{A} - A\|_2 = \|\widehat{A} - A\|_2$. So, we get

$$\begin{aligned}\left\|\widehat{A} - A\right\|_2 &= \max_{\substack{1\le j\le n\\1\le j\le m}}\left|\sum_{k=1}^{n}\sum_{l=1}^{m}\delta a_{kl}\varepsilon_i^{k-1}\delta_j^{l-1}\right| \\ &\le \max_{\substack{1\le j\le n\\1\le j\le m}}\sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right|\left|\varepsilon_i\right|^{k-1}\left|\delta_j\right|^{l-1} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{m}\left|\delta a_{kl}\right| = \Delta.\end{aligned} \tag{37}$$

Hence, we have the following theorem.

Theorem 4. *Let $A, \widehat{A}, \delta b, \Delta, \sigma_{\min}$ be defined as above. If $\Delta < \sigma_{\min}$, then*

$$\frac{\|\widehat{x} - x\|_2}{\|x\|_2} \le \frac{\sigma_{\max}}{\sigma_{\min} - \Delta}\left(\frac{\|\delta b\|_2}{\|b\|_2} + \frac{\Delta}{\sigma_{\max}}\right), \tag{38}$$

where

$$\sigma_{\max} = \|A\|_2. \tag{39}$$

Remark 5. From (38) and (39), the condition number $\mathscr{K}$ of the block skew circulant linear system with skew circulant blocks can be easily computed, as well as the bound of perturbation (38).

3.2. Optimal Backward Perturbation Bound of the Block Skew Circulant Linear System with Skew Circulant Blocks. Let $\widehat{x}$ be an approximate solution to $Ax = b$ and let

$$\begin{aligned}\Omega &\equiv \left\{\left(\Delta A, \Delta b\right) \mid \left(A + \Delta A\right)\widehat{x} = b + \Delta b\right\}, \\ \eta\left(\widehat{x}\right) &\equiv \inf_{\left(\Delta A, \Delta b\right)\in\Omega}\|\Delta A, \Delta b\|,\end{aligned} \tag{40}$$
$$\left(A + \Delta A\right)\widehat{x} = b + \Delta b,$$

which is equivalent to

$$(\Delta A, \Delta b)\begin{pmatrix}\widehat{x}\\-1\end{pmatrix} = b - A\widehat{x}. \tag{41}$$

Due to [14], we have

$$\eta(\widehat{x}) = \frac{\|b - A\widehat{x}\|_2}{\sqrt{1 + \|\widehat{x}\|_2^2}}, \quad (\|\ \| \text{ being any unitary invariant norm}). \tag{42}$$

Let $\widehat{x}$ be an approximate solution to $Ax = b$, where A is defined in (1), as follows:

$$\begin{aligned}\Omega \equiv\ & \{(\Delta A, \Delta b) \mid (A + \Delta A)\widehat{x} = b + \Delta b,\\ & \Delta A \text{ is a block skew circulant matrix}\\ & \text{with skew circulant blocks}\}\\ \eta(\widehat{x}) \equiv\ & \inf_{(\Delta A, \Delta b)\in\Omega}\{\|\Delta A, \Delta b\|_F\}.\end{aligned} \tag{43}$$

Then $\Omega \neq \phi$ (such as $\Delta A = 0$ is a block skew circulant matrix with skew circulant blocks, $\Delta b = A\widehat{x} - b$)

$$\eta^2(\widehat{x}) = \inf_{(\Delta A, \Delta b)\in\Omega}\left\{\|\Delta A\|_F^2 + \|\Delta A\widehat{x} + A\widehat{x} - b\|_F^2\right\}. \tag{44}$$

Since

$$\begin{aligned}\|\Delta A\|_F^2 &= mn\sum_{k=1}^{n}\sum_{l=1}^{m}(\delta a_{kl})^2,\\ \Delta A &= \mathbb{Q}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\delta a_{kl}C_{n0}^{k-1}\otimes C_{m0}^{l-1}\right)\mathbb{Q}^T,\end{aligned} \tag{45}$$

so

$$\begin{aligned}&\|\Delta A\widehat{x} + A\widehat{x} - b\|_F^2\\ &= \left\|\mathbb{Q}\begin{pmatrix}\delta B_{11} & & \\ & \ddots & \\ & & \delta B_{tt}\end{pmatrix}\mathbb{Q}^T\widehat{x} + A\widehat{x} - b\right\|_F^2\\ &= \left\|\begin{pmatrix}\delta B_{11} & & \\ & \ddots & \\ & & \delta B_{tt}\end{pmatrix}\begin{pmatrix}x_1^{(0)}\\ \vdots\\ x_t^{(0)}\end{pmatrix} - r_0\right\|_F^2\\ &= \left\|\begin{pmatrix}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\delta a_{kl}C_{11}^{k-1}\otimes C_{m0}^{l-1}\right)x_1^{(0)}\\ \vdots\\ \left(\sum_{k=1}^{n}\sum_{l=1}^{m}\delta a_{kl}C_{tt}^{k-1}\otimes C_{m0}^{l-1}\right)x_t^{(0)}\end{pmatrix} - r_0\right\|_F^2\\ &= \left\|G_0(\delta a_{11}, \dots, \delta a_{1m}, \dots, \delta a_{n1}, \dots, \delta a_{nm})^T - r_0\right\|_F^2,\end{aligned} \tag{46}$$

where $r_0 = \mathbb{Q}^T(b - A\widehat{x})$, $\mathbb{Q}^T\widehat{x} = \begin{pmatrix}x_1^{(0)}\\ \vdots\\ x_t^{(0)}\end{pmatrix}$, $G_0 = (R_1, R_2, \dots, R_n) \in \mathbf{R}_{mn\times mn}$, and

$$\begin{aligned}R_k &= \begin{pmatrix}P_{1,k,1} & \cdots & P_{1,k,m}\\ \vdots & \ddots & \vdots\\ P_{t,k,1} & \cdots & P_{t,k,m}\end{pmatrix},\\ P_{p,k,l} &= C_{pp}^{k-1}\otimes C_{m0}^{l-1}x_p^{(0)},\end{aligned} \tag{47}$$

$p = 1, 2, \dots, t$, $k = 1, 2, \dots, n$, and $l = 1, 2, \dots, m$.

Let

$$\begin{aligned}&f(\delta a_{11}, \dots, \delta a_{nm})\\ &\quad = mn\sum_{k=1}^{n}\sum_{l=1}^{m}(\delta a_{kl})^2 + \left\|G_0\begin{pmatrix}\delta a_{11}\\ \vdots\\ \delta a_{nm}\end{pmatrix} - r_0\right\|_F^2,\end{aligned} \tag{48}$$

then

$$\frac{\partial f}{\partial \delta a_{kl}} = 0, \tag{49}$$

which is equivalent to

$$\begin{aligned}&\left(2mnI_{mn} + 2G_0^TG_0\right)\begin{pmatrix}\delta a_{11}\\ \vdots\\ \delta a_{nm}\end{pmatrix} - 2G_0^Tr_0 = 0,\\ &\frac{\partial^2 f}{\partial(\delta a_{kl})^2} = 2mnI_{mn} + 2G_0^TG_0 > 0.\end{aligned} \tag{50}$$

Hence f is a convex function about $(\delta a_{11}, \dots, \delta a_{nm})$, and the point of minimal value is

$$\begin{pmatrix}\delta a_{11}\\ \vdots\\ \delta a_{nm}\end{pmatrix} = \left(mnI_{mn} + G_0^TG_0\right)^{-1}G_0^Tr_0. \tag{51}$$

Substituting it into (48), we can get the following.

Theorem 6. *Let r_0 and G_0 be defined as above; then we have*

$$\begin{aligned}\eta(\widehat{x})^2 &= mnr_0^TG_0\left(mnI_{mn} + G_0^TG_0\right)^{-2}G_0^Tr_0\\ &\quad + \left\|\left[G_0\left(mnI_{mn} + G_0^TG_0\right)^{-1}G_0^T - I_{mn}\right]r_0\right\|_F^2.\end{aligned} \tag{52}$$

Let $G_0 = U\Sigma V^T$ be the singular value decomposition of G_0, where U and V are real orthogonal matrices, $\Sigma = \operatorname{diag}(\sigma'_1, \ldots, \sigma'_{nm})$, $\sigma'_j \geq 0$ $(j = 1, 2, \ldots, nm)$, so

$$\begin{aligned}
\eta(\hat{x})^2 &= mnr_0^T U\Sigma V^T \left(mnI_{mn} + \Sigma^2\right)^{-2} V\Sigma U^T r_0 \\
&\quad + \left\| \left[U\Sigma V^T \left(mnI_{mn} + \Sigma^2\right)^{-1} V\Sigma U^T - I_{mn} \right] r_0 \right\|_F^2 \\
&= mnr_1^T \Sigma \left(mnI_{mn} + \Sigma^2\right)^{-2} \Sigma r_1 \\
&\quad + \left\| \left[\Sigma \left(mnI_{mn} + \Sigma^2\right)^{-1} \Sigma - I_{mn} \right] r_0 \right\|_F^2 \\
&= mnr_1^T \Sigma \left(mnI_{mn} + \Sigma^2\right)^{-2} \Sigma r_1 \\
&\quad + \left\| \left[\Sigma \left(mnI_{mn} + \Sigma^2\right)^{-1} \Sigma - I_{mn} \right] r_1 \right\|_F^2 \\
&= mnr_1^T \Sigma \left(mnI_{mn} + \Sigma^2\right)^{-2} \Sigma r_1 \\
&\quad + m^2n^2 r_1^T \left(mnI_{mn} + \Sigma^2\right)^{-2} r_1 \\
&= r_1^T \operatorname{diag}\left(d_1, d_2, \ldots, d_{mn}\right) r_1,
\end{aligned} \tag{53}$$

where $r_1 = U^T r_0$, $d_j = (mn\sigma_j'^2 + m^2n^2)/(mn + \sigma_j'^2)^2 = mn/(mn + \sigma_j'^2)$, $j = 1, 2, \ldots, mn$.

Remark 7. By $\sigma_j^2 \leq \|G_0\|_F^2 = mn\|\hat{x}\|_2^2$, we get $1 + \sigma_j'^2/mn \leq 1 + \|\hat{x}\|_2^2$, and hence $mn/(mn + \sigma_j'^2) \geq 1/(1 + \|\hat{x}\|_2^2)$.

Algorithm 8. Consider the following.

Step 1. Form the block style spectral decomposition of the matrix $C_{n\times n}$ and $C_{m\times m}$,

$$C_{n\times n} = Q_n \begin{pmatrix} C_{11} & & & \\ & C_{22} & & \\ & & \ddots & \\ & & & C_{tt} \end{pmatrix} Q_n^T,$$
$$C_{m\times m} = Q_m \begin{pmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{ss} \end{pmatrix} Q_m^T. \tag{54}$$

Step 2. Compute $\mathbb{Q} = (Q_n \otimes I_m)(I_n \otimes Q_m)$.

Step 3. Compute $r = b - A\hat{x}$.

Step 4. Compute $r_0 = \mathbb{Q}^T r$.

Step 5. Compute $\mathbb{Q}^T \hat{x} = (x_1^{(0)}, \ldots, x_t^{(0)})^T$.

Step 6. Form G_0.

Step 7. Compute $\eta^2(\hat{x})$.

Table 1

	ϵ	$\mathscr{K}$	$\eta_1(\hat{x})$	$\eta_2(\hat{x})$
Case 0	0	6.472	0	0
Case 1	0.0694	6.193	0.1252	0.2487
Case 2	0.0075	6.438	0.0127	0.0223
Case 3	7.5122×10^{-4}	6.468	0.0013	0.0022

3.3. Numerical Example. In this section, we give a simple numerical example to verify the conclusions above. Suppose that $n = 3$, $m = 2$ in the following example.

If the coefficient matrix of block skew circulant linear system with skew circulant blocks is $A = \mathrm{BC}_{-1,-1}^{3,2}(1, 2, 4.5, 3, 4, 2.5)$, and the constant vector $b = (1, 2, 1, 0, 3, 4)^T$. Now, for comparative analysis, we give three perturbations in the following:

$$\begin{aligned}
\Delta A_1 &= \mathrm{BC}_{-1,-1}^{3,2}\left(0.01, 0.03, 0.02, -0.05, -0.03, 0.01\right), \\
\Delta b_1 &= \left(0.1, 0.3, 0, -0.2, 0.01, 0.04\right)^T, \\
\Delta A_2 &= \mathrm{BC}_{-1,-1}^{3,2}\left(0.01, 0.01, 0.01, 0.01, 0.01, 0.01\right), \\
\Delta b_2 &= \left(0.01, 0.01, 0.01, 0.01, 0.01, 0.01\right)^T, \\
\Delta A_3 &= \mathrm{BC}_{-1,-1}^{3,2}\left(0.001, 0.001, 0.001, 0.001, 0.001, 0.001\right), \\
\Delta b_3 &= \left(0.001, 0.001, 0.001, 0.001, 0.001, 0.001\right)^T,
\end{aligned} \tag{55}$$

then by equation $\widehat{A}\hat{x} = \hat{b}$, we get the approximate solution of $Ax = b$ correspondingly as

$$\begin{gathered}
x = \begin{pmatrix} -1.0505, \\ 0.6490, \\ 1.0280, \\ -1.1028, \\ -0.9215, \\ 1.2482 \end{pmatrix}, \qquad \hat{x}_1 = \begin{pmatrix} -1.1526, \\ 0.6358, \\ 1.1192, \\ -1.0611, \\ -1.0158, \\ 1.2286 \end{pmatrix}, \\
\hat{x}_2 = \begin{pmatrix} -1.0396, \\ 0.6473, \\ 1.0178, \\ -1.1003, \\ -0.9109, \\ 1.2484 \end{pmatrix}, \qquad \hat{x}_3 = \begin{pmatrix} -1.0494, \\ 0.6489, \\ 1.0270, \\ -1.1025, \\ -0.9204, \\ 1.2482 \end{pmatrix},
\end{gathered} \tag{56}$$

where x is the solution of $Ax = b$ and x_i $(i = 1, 2, 3)$ is the solution of $(A + \Delta A_i)x = b + \Delta b$.

According to the Algorithm, we obtain Table 1, where ϵ means relative error of block skew circulant linear system with skew circulant blocks, $\mathscr{K} = \max\{\sigma_{ij}\}/\min\{\sigma_{ij}\}$ is the condition number, $\eta_1(\hat{x}) = \|b - A\hat{x}\|_2/\sqrt{1 + \|\hat{x}\|_2^2}$ and η_2 are obtained from the Algorithm.

From the tabular, we know that the conclusions above are right and the Algorithm is efficient.

4. Conclusion

The related problems of block skew circulant matrix with skew circulant blocks are considered in this paper. We not only present block style spectral decomposition and singular value, but also study backward perturbation analysis for the block skew circulant linear system with skew circulant blocks. The reason why we focus our attentions on block skew circulant matrix with skew circulant blocks is to explore the application of block skew circulant matrix with skew circulant blocks in the related field of medicine and real-time tracking. On the basis of existing application situation [23], we conjecture that SVD decomposition of block skew circulant matrix with skew circulant blocks will play an important role in CT-perfusion imaging of human brain. On the basis of method [7] and ideas of [24], we will exploit real-time tracking with kernel matrix of block skew circulant matrix with skew circulant blocks structure. The circulant singular value decomposition (cSVD) techniques with a block-circulant deconvolution matrix [25–29] were used to perform the deconvolution calculation to obtain the [rCBF.$R(t)$] curve [25, 26]. The maximum value of the [rCBF.$R(t)$] curve was used as the rCBF. We will exploit the skew circulant singular value decomposition (scSVD) techniques of a block skew circulant matrix with skew circulant blocks deconvolution matrix to obtain the [rCBF.$R(t)$] curve.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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