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Research Article

On the Minimal Polynomials and the Inverses of Multilevel Scaled Factor Circulant Matrices

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Circulant matrices have important applications in solving various differential equations. The level-k scaled factor circulant matrix over any field is introduced. Algorithms for finding the minimal polynomial of this kind of matrices over any field are presented by means of the algorithm for the Gröbner basis of the ideal in the polynomial ring. And two algorithms for finding the inverses of such matrices are also presented. Finally, an algorithm for computing the inverse of partitioned matrix with level-k scaled factor circulant matrix blocks over any field is given by using the Schur complement, which can be realized by CoCoA 4.0, an algebraic system, over the field of rational numbers or the field of residue classes of modulo prime number.

1. Introduction

Circulant matrices play an important role in solving many different differential equations, such as ordinary, partial, matrix, linear second-order partial, bi-Hamiltonian partial, parameterized delay, fractional order, and singular perturbation delay. Lee et al. investigated a high-order compact (HOC) scheme for the general two-dimensional (2D) linear partial differential equation in [1] with a mixed derivative. Meanwhile, in order to establish the CCD2 scheme, they rewrote equation (1.1) into (2.1) in [1]. To write the CCD2 system in a concise style, they employed circulant matrix to obtain the corresponding whole CCD2 linear system (2.10), whose entries are circulant block. Using circulant matrix, Karasözen and Şimşek [2] considered periodic boundary conditions such that no additional boundary terms will appear after semidiscretization. Guo et al. concerned generic Dn-Hopf bifurcation to a delayed Hopfield-Cohen-Grossberg model of neural networks (5.17) in [3], where T denoted an interconnection matrix. In particular, they assumed that T is a symmetric circulant matrix. Trench considered nonautonomous systems of linear differential equations (1) in [4] with some constraints on the coefficient matrix $A(t)$. One case is that the $A(t)$ is a variable block circulant matrix. In [5], some Routh-Hurwitz stability conditions are generalized to the fractional order case. Ahmed et al. considered the 1-system CML (10) in [5]. They selected a circulant matrix, which reads a tridiagonal matrix. In [6], Jin et al. proposed the GMRES method with the Strang-type block-circulant preconditioner for solving singular perturbation delay differential equations. In [7], Claeyssen and Leal introduce factor circulant matrices: matrices with the structure of circulants, but with the entries below the diagonal multiplied by the same factor. The diagonalization of a circulant matrix and spectral decomposition are conveniently generalized to block matrices with the structure of factor circulants. Matrix and partial differential equations involving factor circulants are considered. Wilde [8] developed a theory for the solution of ordinary and partial differential equations whose structure involves the algebra of circulants. He showed how the algebra of 2×2 circulants is related to the study of the harmonic oscillator, Cauchy-Riemann equations, Laplace's equation, the Lorentz transformation, and the wave equation. And he used $n\times n$ circulants to suggest natural generalizations of these equations to higher dimensions.

With the development of the mathematical research, multilevel circulant matrix had been defined. And it has been used on network engineering, approximate calculation, and Image processing [9–12]. Jiang and Liu [13] introduced the level-m scaled circulant factor matrix over the complex number field and discussed its diagonalization and spectral decomposition and representation. Zhang et al. [14] gave algorithms for the minimal polynomial and the inverse of a level-$n(r_1, r_2, \ldots, r_n)$-block circulant matrix over any field by means of the algorithm for the Gröbner basis for the ideal of the polynomial ring over the field. Morhac and Matousek [15] present an efficient algorithm to solve a one-dimensional as well as n-dimensional circulant convolution system. Rezghi and Elden [16] defined tensors with diagonal and circulant structure and developed a framework for the analysis of such tensors. Georgiou and Koukouvinos [17] presented a new method for constructing multilevel supersaturated designs. Trench [18, 19] considered properties of unilevel block circulants and multilevel block α-circulants. Block [20] considered the property of circulants of level-k. Baker et al. discussed the structure of multiblock circulants in [21]. More details on multilevel circulant matrix can be found in [22–24].

This paper is devoted to study the level-k scaled factor circulant matrix, and it is organized as follows.

In Section 2, a level-k scaled factor circulant matrix over any field is introduced and its algebraic properties are given.

In Section 3, we first show that the ring of all level-k scaled factor circulant matrices over a field is isomorphic to a factor ring of a polynomial ring in k variables over the same field, and then we present an algorithm for finding the minimal polynomial of a level-k scaled factor circulant matrix by mean of the algorithm for the Gröbner basis for a kernel of a ring homomorphism.

In Section 4, we give a sufficient and necessary condition to determine whether a level-k scaled factor circulant matrix over a field is singular or not and then present an algorithm for finding the inverse of such a matrix over a field.

In Section 5, an algorithm for finding the inverse of partitioned matrix with level-k scaled factor circulant matrix blocks over a field is presented by using the Schur complement and Buchberger's algorithm.

We first introduce some terminologies and notations used in the equations. Let $\mathbb{F}$ be a field and $\mathbb{F}[x_1, \ldots, x_k]$ the polynomial ring of k variables over field $\mathbb{F}$. By Hilbert basis Theorem, we know that every ideal $\mathbf{I}$ in $\mathbb{F}[x_1, \ldots, x_k]$ is finitely generated. Fixing a term order in $\mathbb{F}[x_1, \ldots, x_k]$, a set of nonzero polynomials $\mathbf{G} = \{g_1, \ldots, g_t\}$ in an ideal $\mathbf{I}$ is called a Gröbner basis for $\mathbf{I}$ if and only if, for all nonzero $f \in \mathbf{I}$, there exists $i \in \{1, \ldots, t\}$ such that $lp(g_i)$ divides $lp(f)$, where $lp(g_i)$ and $lp(f)$ are the leading power products of g_i and f, respectively. A Gröbner basis $\mathbf{G} = \{g_1, \ldots, g_t\}$ is called a reduced Gröbner basis if and only if, for all i, $lc(g_i) = 1$ and g_i is reduced with respect to $\mathbf{G} - g_i$; that is, for all i, no nonzero term in g_i is divisible by any $lp\,(g_j)$ for any $j \neq i$, where $lc(g_i)$ is the leading coefficient of g_i.

In this paper, we set $A^0 = I$ for a square matrix A, and $\langle f_1, \ldots, f_m \rangle$ denotes an ideal of $\mathbb{F}[x_1, \ldots, x_k]$ generated by polynomials $f_1, \ldots, f_m$.

2. Level-k Scaled Factor Circulant Matrices

If R is an $n \times n$ matrix over field $\mathbb{F}$ which is the product of a diagonal matrix D and a circulant permutation matrix C, this is

$$D = \operatorname{diag}\left(d_1, d_2, \ldots, d_n\right),$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}. \tag{1}$$

Then, the matrix $R = DC$ is called a scaled circulant permutation matrix over field $\mathbb{F}$.

When field $\mathbb{F}$ is the complex field, this kind of matrix is the same as in [25].

For the remainder of the paper, the indices $1, 2, \ldots, n$ are congruence classes modulo n. We will use 0 instead of n. For convenience, we will refer to such a matrix as an SCPMF.

As $R = DC$ is a scaled circulant permutation matrix over field $\mathbb{F}$, then

$$\begin{gathered} \det R = (-1)^{n-1} \prod_{j=1}^{n} d_j, \\ R^n = \left(\prod_{j=1}^{n} d_j\right) I_n. \end{gathered} \tag{2}$$

In this paper, focus on the case where $R_i = D_i C_i$ is nonsingular SCPMF, where

$$\begin{gathered} D_i = \operatorname{diag}\left(d_{i1}, d_{i2}, \ldots, d_{in_i}\right), \\ C_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n_i\times n_i}, \quad i = 1, 2, \ldots, k. \end{gathered} \tag{3}$$

It is easy to show that the polynomial $x_i^{n_i} - \prod_{j_i=1}^{n_i} d_{ij_i}$ is both the minimal polynomial and the characteristic polynomial of R_i.

Let I_{n_i} be the $n_i \times n_i$ unit matrix for $i = 1, 2, \ldots, k$ and $N = n_1 n_2 \cdots n_k$. Set

$$\sigma_i = I_{n_1} \otimes \cdots \otimes I_{n_i - 1} \otimes R_i \otimes I_{n_i + 1} \otimes \cdots \otimes I_{n_k}, \tag{4}$$

where $\otimes$ is a Kronecker product of matrices.

Definition 1. An $N \times N$ maxtrix A over $\mathbb{F}$ is callled a level-k scaled factor circulant matrix if there exists a polynomial

$$\begin{gathered} f\left(x_1, \ldots, x_k\right) = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \cdots \sum_{i_k=0}^{n_k-1} a_{i_1\cdots i_k} x_1^{i_1} \cdots x_k^{i_k} \\ \in \mathbb{F}\left[x_1, \ldots, x_k\right] \end{gathered} \tag{5}$$

such that

$$\begin{aligned} A &= f\left(\sigma_1, \ldots, \sigma_k\right) \\ &= \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \cdots \sum_{i_k=0}^{n_k-1} a_{i_1\cdots i_k} \sigma_1^{i_1} \cdots \sigma_k^{i_k}, \end{aligned} \tag{6}$$

where $f(x_1, \ldots, x_k)$ will be called the representer of a level-k scaled factor circulant matrix A.

Obviously, when field $\mathbb{F}$ is the complex field and $k = 1$, this kind of matrix is same as in [25], and when the field $\mathbb{F}$ is the complex field, this kind of matrix is the same as in [13], and if $D_i = \text{diag}\,(1, 1, \ldots, 1, r_i)$, $i = 1, 2, \ldots, k$, this kind of matrix is as in [14, 18, 22], and if $D_i = I_{n_i}$, $i = 1, 2, \ldots, k$, then we obtain the multilevel circulant matrix [9–12, 15, 19–22].

From the property of the Kronecker product of matrices, the level-k scaled factor circulant matrix A can also be expressed as

$$A = \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_k=0}^{n_k-1} a_{i_1\cdots i_k} R_1^{i_1} \otimes R_2^{i_2} \otimes \cdots \otimes R_k^{i_k}. \tag{7}$$

For a matrix A over $\mathbb{F}$, A is a level-k scaled factor circulant matrix if and only if A commutes with $(R_1 \otimes R_2 \otimes \cdots \otimes R_k)$; that is,

$$A\left(R_1 \otimes R_2 \otimes \cdots \otimes R_k\right) = \left(R_1 \otimes R_2 \otimes \cdots \otimes R_k\right) A. \tag{8}$$

In addition to the algebraic properties that can be easily derived from representation (6), we mention that level-k scaled factor circulant matrices have very nice structure. The product of two level-k scaled factor circulant matrices is also a level-k scaled factor circulant matrix. Furthermore, level-k scaled factor circulant matrices commute under multiplication and A^{-1} is also a level-k scaled factor circulant matrix.

3. Minimal Polynomials of Level-k Scaled Factor Circulant Matrices

Let $\mathbb{F}[\sigma_1, \ldots, \sigma_k] = \{A \mid A = f(\sigma_1, \ldots, \sigma_k), f(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]\}$. It is a routine to prove that $\mathbb{F}[\sigma_1, \ldots, \sigma_k]$ is a commutative ring with the matrix addition and multiplication.

Theorem 2. *Consider* $\mathbb{F}[x_1, \ldots, x_k]/\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle \cong \mathbb{F}[\sigma_1, \ldots, \sigma_k]$.

Proof. Consider the following F-algebra homomorphism:

$$\begin{aligned} \varphi : \mathbb{F}\left[x_1, \ldots, x_k\right] &\longrightarrow \mathbb{F}\left[\sigma_1, \ldots, \sigma_k\right] \\ f\left(x_1, \ldots, x_k\right) &\longmapsto A = f\left(\sigma_1, \ldots, \sigma_k\right) \end{aligned} \tag{9}$$

for $f(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$. It is clear that φ is an F-algebra epimorphism. So, we have

$$\frac{\mathbb{F}\left[x_1, \ldots, x_k\right]}{\ker\varphi \cong \mathbb{F}\left[\sigma_1, \ldots, \sigma_k\right]}. \tag{10}$$

We can prove that

$$\ker\varphi = \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle. \tag{11}$$

In fact, for $i = 1, 2, \ldots, k$, $x_i^{n_i} - \prod_{j_i=1}^{n_i} d_{ij_i} \in \ker\varphi$ because $\sigma_i^{n_i} - \prod_{j_i=1}^{n_i} d_{ij_i} I_{n_i} = 0$. Hence, $\ker\varphi \supseteq \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle$.

Conversely, for any $f(x_1, \ldots, x_k) \in \ker\varphi$, we have $A = f(\sigma_1, \ldots, \sigma_k) = 0$. Fix the lexicographical order on $\mathbb{F}[x_1, \ldots, x_k]$ with $x_1 > x_2 > \cdots > x_k$. Consider $x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}$ dividing $f(x_1, \ldots, x_k)$, and there exist

$$u_1\left(x_1, \ldots, x_k\right), v_1\left(x_1, \ldots, x_k\right) \in \mathbb{F}\left[x_1, \ldots, x_k\right] \tag{12}$$

such that

$$\begin{aligned} f\left(x_1, \ldots, x_k\right) = u_1\left(x_1, \ldots, x_k\right)\left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}\right) \\ + v_1\left(x_1, \ldots, x_k\right), \end{aligned} \tag{13}$$

where $v_1(x_1, \ldots, x_k) = 0$ or the largest degree of x_1 in $v_1(x_1, \ldots, x_k)$ is less than n_1. If $v_1(x_1, \ldots, x_k) = 0$, then $f(x_1, \ldots, x_k) \in \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle$. Otherwise, $x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}$ dividing $v_1(x_1, \ldots, x_k)$, and there exist $u_2(x_1, \ldots, x_k), v_2(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$, such that

$$\begin{aligned} v_1\left(x_1, \ldots, x_k\right) = u_2\left(x_1, \ldots, x_k\right)\left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}\right) \\ + v_2\left(x_1, \ldots, x_k\right), \end{aligned} \tag{14}$$

where $v_2(x_1, \ldots, x_k) = 0$ or the largest degree of x_2 in $v_2(x_1, \ldots, x_k)$ is less than n_2. If $v_2(x_1, \ldots, x_k) = 0$, then $f(x_1, \ldots, x_k) \in \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle$. Otherwise, the largest degree of x_1 in $v_2(x_1, \ldots, x_k)$ is less than n_1 because x_1 does not appear in $x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}$. Continuing this procedure, there exist $u_1(x_1, \ldots, x_k), \ldots, u_k(x_1, \ldots, x_k)$, and $v_k(x_1, \ldots, x_k) \in \mathbb{F}[x_1, \ldots, x_k]$, such that $f(x_1, \ldots, x_k) = u_1(x_1, \ldots, x_k)(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}) + \cdots + u_k(x_1, \ldots, x_k)(x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}) + v_k(x_1, \ldots, x_k)$, where $v_k(x_1, \ldots, x_k) = 0$ or the degrees of $x_1, x_2, \ldots, x_k$ in $v_k(x_1, \ldots, x_k)$ are less than $n_1, n_2, \ldots, n_k$, respectively. Since $f(\sigma_1, \ldots, \sigma_k) = 0$, $\sigma_i^{n_i} - \prod_{j_i=1}^{n_i} d_{ij_i} I_{n_i} = 0$. For $i = 1, 2, \ldots, k$, $u_k(\sigma_1, \ldots, \sigma_k) = 0$. The coefficients of all terms in $v_k(x_1, \ldots, x_k)$ are the entries of the matrix $v_k(\sigma_1, \ldots, \sigma_k)$ because the degrees of $x_1, x_2, \ldots, x_k$ in $v_k(x_1, \ldots, x_k)$ are less than $n_1, n_2, \ldots, n_k$, respectively. Therefore, the coefficient of each term in $v_k(x_1, \ldots, x_k)$ is 0; that is, $v_k(x_1, \ldots, x_k) = 0$. Thus,

$$f\left(x_1, \ldots, x_k\right) \in \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle. \tag{15}$$

□

Definition 3. Let $\mathbf{I}$ be a nonzero ideal of the polynomial ring $\mathbb{F}[y_1, \ldots, y_t]$. Then, $\mathbf{I}$ is called an annihilation ideal of square matrices $A_1, \ldots, A_t$, denoted by $\mathbf{I}(A_1, \ldots, A_t)$, if $f(A_1, \ldots, A_t) = 0$ for all $f(y_1, \ldots, y_t) \in \mathbf{I}$.

Definition 4. Suppose that $A_1, \ldots, A_t \in \mathbb{F}[\sigma_1, \ldots, \sigma_k]$ are not all zero matrices. The unique monic polynomial $g(x)$ of minimum degree that simultaneously annihilates $A_1, \ldots, A_t$ is called the common minimal polynomial of $A_1, \ldots, A_t$.

We give the special case of Theorem 2.4.10 [26] here for the convenience of applications.

Lemma 5. *Let* $\mathbf{I}$ *be an ideal of* $\mathbb{F}[x_1, \dots, x_k]$*. Given* $f_1, \dots, f_m \in \mathbb{F}[x_1, \dots, x_k]$*, consider the following F-algebra homomorphism:*

$$\begin{aligned} \phi : \mathbb{F}[y_1, \dots, y_m] &\longrightarrow \frac{\mathbb{F}[x_1, \dots, x_k]}{\mathbf{I}} \\ y_1 &\longmapsto f_1 + \mathbf{I} \\ &\vdots \\ y_m &\longmapsto f_m + \mathbf{I}. \end{aligned} \tag{16}$$

Let $\mathbf{E} = \langle \mathbf{I}, y_1 - f_1, \dots, y_m - f_m \rangle$ *be an ideal of* $\mathbb{F}[x_1, \dots, x_k, y_1, \dots, y_m]$ *generated by* $\mathbf{I}, y_1 - f_1, \dots, y_m - f_m$*. Then,* $\ker \phi = \mathbf{E} \cap \mathbb{F}[y_1, \dots, y_m]$*.*

The following lemma is well known [27].

Lemma 6. *Let* A *be a nonzero matrix over field* $\mathbb{F}$*. If the minimal polynomial of* A *is*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_0 \neq 0, \tag{17}$$

then

$$A^{-1} = \frac{1}{a_0} \left(-a_n A^{n-1} - a_{n-1} A^{n-2} - \cdots - a_1 \right). \tag{18}$$

The following lemma is the Exercise 2.38 of [26].

Lemma 7. *Let* $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m$ *be ideals of* $\mathbb{F}[x_1, \dots, x_k]$ *and let* $\mathbf{J} = \langle 1 - \sum_{i=1}^m \omega_i, \omega_1 \mathbf{L}_1, \omega_2 \mathbf{L}_2, \dots, \omega_m \mathbf{L}_m \rangle$ *be an ideal of* $\mathbb{F}[x_1, \dots, x_k, \omega_1, \dots, \omega_m]$ *generated by* $1 - \sum_{i=1}^m \omega_i, \omega_1 \mathbf{L}_1, \omega_2 \mathbf{L}_2, \dots, \omega_m \mathbf{L}_m$*. Then,* $\bigcap_{i=1}^m \mathbf{L}_i = \mathbf{J} \cap \mathbb{F}[x_1, x_2, \dots, x_k]$*.*

By Theorem 2 and Lemma 5, we can prove the following theorem.

Theorem 8. *The minimal polynomial of the level-k scaled factor circulant matrix* $A \in \mathbb{F}[\sigma_1, \dots, \sigma_k]$ *is the monic polynomial that generates the ideal*

$$\begin{aligned} \Bigg\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, \\ y - f(x_1, x_2, \dots, x_k) \Bigg\rangle > \cap \mathbb{F}[y], \end{aligned} \tag{19}$$

where the polynomial $f(x_1, x_2, \dots, x_k)$ *is the representer of* A*.*

Proof. Consider the following F-algebra homomorphism:

$$\begin{aligned} \phi : \mathbb{F}[y] &\longrightarrow \frac{\mathbb{F}[x_1, \dots, x_k]}{\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \rangle} \\ &>\longrightarrow \mathbb{F}[\sigma_1, \dots, \sigma_k], \\ y &\longmapsto f(x_1, \dots, x_k) \\ &\quad + \Bigg\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \Bigg\rangle \\ &>\longmapsto A = f(\sigma_1, \dots, \sigma_k). \end{aligned} \tag{20}$$

It is clear that $q(y) \in \ker \phi$ if and only if $q(A) = 0$. In view of Lemma 5, we have

$$\begin{aligned} \ker \phi = \Bigg\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, \\ y - f(x_1, x_2, \dots, x_k) \Bigg\rangle > \cap \mathbb{F}[y]. \end{aligned} \tag{21}$$

□

We know from Theorem 8 and Lemma 6 that the minimal polynomial and the inverse of a level-k scaled factor circulant matrix $A \in \mathbb{F}[\sigma_1, \dots, \sigma_k]$ is calculated by a Gröbner basis for a kernel of an F-algebra homomorphism. Therefore, we have the following algorithm to calculate the minimal polynomial and the inverse of a level-k scaled factor circulant matrix $A = f(\sigma_1, \dots, \sigma_k)$.

Step 1. Calculate the reduced Gröbner basis $\mathbf{G}$ for the ideal

$$\begin{aligned} \Bigg\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, \\ y - f(x_1, x_2, \dots, x_k) \Bigg\rangle > \cap \mathbb{F}[y] \end{aligned} \tag{22}$$

by CoCoA 4.0, using an elimination order with $x_1 > x_2 > \cdots > x_k > y$.

Step 2. Find the polynomial in $\mathbf{G}$ in which the variables $x_1, x_2, \dots, x_k$ do not appear. This polynomial $p(x)$ is the minimal polynomial of A.

Step 3. By Step 2, if a_0 in the minimal polynomial of A,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{23}$$

is zero; stop. Otherwise, calculate

$$A^{-1} = \frac{1}{a_0} \left(-a_n A^{n-1} - a_{n-1} A^{n-2} - \cdots - a_1 \right). \tag{24}$$

Example 9. Let $A = f(\sigma_1, \sigma_2)$ be a level-2 scaled factor circulant matrix, where

$$\begin{aligned} f(x,y) &= x^3y^2 + 3x^3y + 4x^2y^2 \\ &\quad + 2x^3 + 7x^2y + x^2 + xy^2 \\ &\quad + 2y^2 + 7xy + 2x + 5y + 8, \\ \sigma_1 &= R_1 \otimes I_3, \qquad \sigma_2 = I_4 \otimes R_2, \\ R_1 &= \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 3 \\ -4 & 0 & 0 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & -2 \\ 5 & 0 & 0 \end{pmatrix}, \\ I_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{25}$$

We now calculate the minimal polynomial and the inverse of A with entries in field $\mathbb{Z}_{11}$.

In fact, the reduced Gröbner basis for the ideal

$$\left\langle x^4 - \frac{18}{5}, y^3 + \frac{10}{3}, z - f(x,y) \right\rangle \tag{26}$$

is

$$\begin{aligned} \mathbf{G} = \{ & z^{10} - 5z^9 - z^8 + 2z^7 + 2z^6 \\ & + 5z^5 + z^4 - 4z^3 - z^2 - 5z - 1, x \\ & - 5yz - 5y - 2z^9 + 5z^8 - 3z^7 \\ & + 5z^6 + 3z^5 + z^3 + 4z + 2, y^2 \\ & + 5y - 5z^9 - 4z^8 - 4z^7 + 3z^6 \\ & + 5z^5 + 3z^3 + 3z^2 - 5z + 1, yz^2 \\ & - 3yz + 3y + 3z^9 - 5z^8 - 5z^7 \\ & + 5z^6 - 2z^5 + 3z^4 + 3z^3 - z^2 - z \}. \end{aligned} \tag{27}$$

So, the minimal polynomial of A is

$$\begin{aligned} & z^{10} - 5z^9 - z^8 + 2z^7 + 2z^6 + 5z^5 \\ & \quad + z^4 - 4z^3 - z^2 - 5z - 1, \end{aligned} \tag{28}$$

and the inverse of A is

$$\begin{aligned} A^{-1} &= A^9 - 5A^8 - A^7 + 2A^6 \\ &\quad + 2A^5 + 5A^4 + A^3 - 4A^2 - A - 5I. \end{aligned} \tag{29}$$

Theorem 10. *The annihilation ideal of the level-k scaled factor circulant matrices $A_1, \ldots, A_t \in \mathbb{F}[\sigma_1, \ldots, \sigma_k]$ is*

$$\begin{aligned} & \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, \right. \\ & \qquad \left. y_1 - f_1(x_1, \ldots, x_k), \ldots, y_t - f_t(x_1, \ldots, x_k) \right\rangle \\ & \quad > \cap \mathbb{F}[y_1, \ldots, y_t], \end{aligned} \tag{30}$$

where the polynomial $f_i(x_1, \ldots, x_k)$ is the representer of A_i, $i = 1, 2, \ldots, t$.

Proof. Consider the following F-algebra homomorphism:

$$\begin{aligned} & \frac{\phi : \mathbb{F}[y_1, \ldots, y_t] \longrightarrow \mathbb{F}[x_1, \ldots, x_k]}{\left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle} \\ & \quad > \longrightarrow \mathbb{F}[\sigma_1, \ldots, \sigma_k], \\ & y_1 \longmapsto f_1(x_1, \ldots, x_k) \\ & \qquad + \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle \\ & \quad > \longmapsto A_1 = f_1(\sigma_1, \ldots, \sigma_k), \ldots, \\ & y_t \longmapsto f_t(x_1, \ldots, x_k) \\ & \qquad + \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle \\ & \quad > \longmapsto A_t = f_t(\sigma_1, \ldots, \sigma_k). \end{aligned} \tag{31}$$

It is clear that $\phi(g(y_1, \ldots, y_t)) = 0$ if and only if $g(A_1, \ldots, A_t) = 0$. Hence, by Lemma 5

$$\mathbf{I}(A_1, \ldots, A_t) = \ker \phi = \mathbf{J} \cap \mathbb{F}[y_1, \ldots, y_t]. \tag{32}$$

□

According to Theorem 10, we give the following algorithm for the annihilation ideal of the level-k scaled factor circulant matrices $A_1, \ldots, A_t \in \mathbb{F}[\sigma_1, \ldots, \sigma_k]$.

Step 4. Calculate the reduced Gröbner basis $\mathbf{G}$ for the ideal

$$\begin{aligned} & \left\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, \right. \\ & \qquad \left. y_1 - f_1(x_1, \ldots, x_k), \ldots, y_t - f_t(x_1, \ldots, x_k) \right\rangle \end{aligned} \tag{33}$$

by CoCoA 4.0, using an elimination order with $x_1 > \cdots > x_k > y_1 > \cdots > y_k$.

Step 5. Find the polynomial in $\mathbf{G}$ in which the variables $x_1, x_2, \ldots, x_k$ do not appear. Then, the ideal generated by these polynomials is the annihilation ideal of $A_1, \ldots, A_t$.

Example 11. Let $A_1 = f_1(\sigma_1, \sigma_2)$ and $A_2 = f_2(\sigma_1, \sigma_2)$ be both level-2 scaled circulant factor matrices, where

$$\begin{gathered} f_1(x, y) = 7x^2y^2 + 5x^2y + 3x^2 \\ + xy^2 + 8xy + 4x + 9y^2 + 2y + 9, \\ f_2(x, y) = 10x^2y^2 + 4x^2y + 7x^2 \\ + xy^2 + 3xy + 9x + 4y^2 + 6y + 1, \\ \sigma_1 = R_1 \otimes I_3, \qquad \sigma_2 = I_3 \otimes R_2, \\ R_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 6 \\ -3 & 0 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -9 \\ \frac{1}{3} & 0 & 0 \end{pmatrix}, \\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{gathered} \tag{34}$$

We calculate the annihilation ideal of A_1 and A_2 over field $\mathbb{Z}_{11}$ as follows.

By CoCoA 4.0, we obtain that the reduced Gröbner basis for the ideal

$$\left\langle x^3 + 9, y^3 - 3, z - f_1(x, y), u - f_2(x, y) \right\rangle \tag{35}$$

is

$$\begin{aligned} \mathbf{G} = \{ & u^8 + 4u^7 + u^6 + 5u^5 - 4u^3 + 3u^2 + 4u - 1, \\ & + 3z + 5u^7 + 3u^6 - 4u^5 \\ & - 4u^4 + 4u^3 - 3u^2 + u - 3, \\ & - 4z - 4u^7 - 4u^6 - 3u^5 - u^4 \\ & - u^3 + 3u^2 + 5u + 2, \\ & - 3z - 5u^7 + 3u^5 - 2u^4 + 5u^3 - 4u^2 \\ & - u + 3, zu - 2z - u^7 - 5u^5 \\ & - 5u^3 + u^2 - 4u + 2 \}. \end{aligned} \tag{36}$$

So, the annihilation ideal of A_1 and A_2 is

$$\begin{aligned} \langle & u^8 + 4u^7 + u^6 + 5u^5 - 4u^3 + 3u^2 \\ & + 4u - 1, z^2 - 3z - 5u^7 + 3u^5 - 2u^4 \\ & + 5u^3 - 4u^2 - u + 3, zu - 2z - u^7 - 5u^5 \\ & -5u^3 + u^2 - 4u + 2 \rangle. \end{aligned} \tag{37}$$

To calculate the common minimal polynomial of $A_1, \ldots, A_t$, we first prove the following theorem.

Theorem 12. *Let $h(x)$ be the least common multiple of $p_1(x)$, $p_2(x), \ldots, p_k(x)$. Then,*

$$\bigcap_{i=1}^{k} \langle p_i(x) \rangle = \langle h(x) \rangle. \tag{38}$$

Proof. For any $f(x) \in \bigcap_{i=1}^{k} \langle p_i(x) \rangle$, we have $p_i(x) \mid f(x)$ for $i = 1, 2, \ldots, k$. Since $h(x)$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_k(x), h(x) \mid f(x)$. So $f(x) \in \langle h(x) \rangle$. Hence

$$\bigcap_{i=1}^{k} \langle p_i(x) \rangle \subseteq \langle h(x) \rangle. \tag{39}$$

Conversely, $p_i(x) \mid f(x)$ for $i = 1, 2, \ldots, k$ because $h(x)$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_k(x)$. Therefore,

$$\bigcap_{i=1}^{k} \langle p_i(x) \rangle \supseteq \langle h(x) \rangle. \tag{40}$$

□

Let $A_i \in \mathbb{F}[\sigma_1, \ldots, \sigma_k]$ be level-k scaled factor circulant matrix for $i = 1, 2, \ldots, t$. If the minimal polynomial of A_i is $p_i(x)$ for $i = 1, 2, \ldots, t$, then the common minimal polynomial of $A_1, \ldots, A_t$ is the least common multiple of $p_1(x), p_2(x), \ldots, p_t(x)$. By Theorem 12 and Lemma 7, we have the following algorithm for finding the common minimal polynomial of level-k scaled factor circulant matrices $A_i = f_i(\sigma_1, \ldots, \sigma_k)$ for $i = 1, 2, \ldots, t$.

Step 6. Calculate the Gröbner basis $\mathbf{G}_i$ for the ideal $\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}, y - f_i(x_1, x_2, \ldots, x_k) \rangle$ by CoCoA 4.0 for each $i = 1, 2, \ldots, t$, using an elimination order with $x_1 > \cdots > x_k > y$.

Step 7. Find out the polynomial $g_i(y)$ in $\mathbf{G}_i$ in which the variables $x_1, \ldots, x_k$ do not appear for each $i = 1, 2, \ldots, t$.

Step 8. Calculate the Gröbner basis $\mathbf{G}$ for the ideal

$$\left\langle 1 - \sum_{i=1}^{t} \omega_i, \omega_1 g_1(y), \ldots, \omega_t g_t(y) \right\rangle \tag{41}$$

by CoCoA 4.0, using elimination with $\omega_1 > \cdots > \omega_t > y$.

Step 9. Find out the polynomial $g(y)$ in $\mathbf{G}$ in which the variables $\omega_1, \ldots, \omega_t$ do not appear. Then, the polynomial $g(y)$ is the common minimal polynomial of $A_i = f_i(\sigma_1, \ldots, \sigma_k)$ for $i = 1, 2, \ldots, t$.

Example 13. We now calculate the common minimal polynomial of A_1 and A_2 of Example 11 over field $\mathbb{Z}_{11}$ as follows.

By CoCoA 4.0, we obtain that the reduced Gröbner basis for the ideal

$$\left\langle x^3 + 9, y^3 - 3, z - f_1(x, y) \right\rangle \tag{42}$$

is

$$\begin{aligned}\mathbf{G}_1 = \{ & z^7 - 4z^6 - 3z^5 + z^4 - 3z^2 \\ & + 4z + 3, x + 2y - 5z^6 - 2z^4 \\ & + 2z^3 + 4z^2 + 3z + 3, y^2 - 5yz \\ & + y + 5z^6 + 5z^5 + 4z^4 + z^3 \\ & + z^2 - 3z, yz^2 + y + 5z^6 - 4z^5 \\ & + 4z^4 + 2z^3 - 5z + 1\}.\end{aligned} \tag{43}$$

So, the minimal polynomial $p_1(z)$ of A_1 is

$$z^7 - 4z^6 - 3z^5 + z^4 - 3z^2 + 4z + 3. \tag{44}$$

Similarly, we get that the reduced Gröbner basis for the ideal

$$\langle x^3 + 9, y^3 - 3, z - f_2(x, y)\rangle \tag{45}$$

is

$$\begin{aligned}\mathbf{G}_2 = \{ & z^8 + 4z^7 + z^6 + 5z^5 - 4z^3 \\ & + 3z^2 + 4z - 1, x - 2y + 2z^7 \\ & + 2z^5 - 2z^4 - 5z^3 + 2z^2 \\ & + 2z + 4, y^2 - 2y - 2z^7 - z^6 \\ & - 3z^5 - 2z^4 + z^3 - 3z^2 - 5z \\ & + 4, yz - 2y + 5z^7 - 2z^6 + 5z^5 \\ & + z^4 + 2z^3 + 4z^2 + 3z\}.\end{aligned} \tag{46}$$

Thus, the minimal polynomial $p_2(z)$ of A_2 is

$$z^8 + 4z^7 + z^6 + 5z^5 - 4z^3 + 3z^2 + 4z - 1. \tag{47}$$

In addition, we obtain that the reduced Gröbner basis for the ideal

$$\langle 1 - u - v, up_1(z), vp_2(z)\rangle \tag{48}$$

is

$$\begin{aligned}\mathbf{G} = \{ & u + v - 1, vz - 2v - 4z^{13} \\ & + 5z^{12} - 3z^{11} + 2z^{10} + 3z^9 - 2z^8 + 2z^7 \\ & + 5z^6 - 3z^5 - 2z^3 - 3z^2 + z + 1, \\ & z^{14} + 2z^{13} - 3z^{12} - 5z^{11} + 4z^{10} - 2z^9 \\ & + z^8 + 4z^6 + 4z^5 - 2z^4 + 3z^3 + 5z^2 + 5z - 4\}.\end{aligned} \tag{49}$$

So, the common minimal polynomial $p(z)$ of A_1 and A_2 is

$$\begin{aligned}& z^{14} + 2z^{13} - 3z^{12} - 5z^{11} + 4z^{10} - 2z^9 \\ & + z^8 + 4z^6 + 4z^5 - 2z^4 + 3z^3 + 5z^2 + 5z - 4.\end{aligned} \tag{50}$$

4. Inverses of Level-k Scaled Factor Circulant Matrices

In this section, we discuss the nonsingularity and the inverse of a level-k scaled factor circulant matrix.

Theorem 14. *Let $A \in \mathbb{F}[\sigma_1, \dots, \sigma_k]$ be an $N \times N$ level-k scaled factor circulant matrix. Then, A is nonsingular if and only if*

$$\begin{aligned}1 \in \Big\langle & f(x_1, \dots, x_k), x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \\ & x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\Big\rangle,\end{aligned} \tag{51}$$

where the polynomial $f(x_1, \dots, x_k)$ is the representer of A.

Proof. A is nonsingular if and only if

$$\begin{aligned}f(x_1, \dots, x_k) + \Big\langle & x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \\ & x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\Big\rangle\end{aligned} \tag{52}$$

is an invertible element in

$$\frac{\mathbb{F}[x_1, \dots, x_k]}{\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle}. \tag{53}$$

By Theorem 2, if and only if there exists $h(x_1, \dots, x_k) + \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle \in \mathbb{F}[x_1, \dots, x_k]/\langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle$ such that $f(x_1, \dots, x_k) h(x_1, \dots, x_k) + \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle \equiv 1 + \langle x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\rangle$ if and only if there exist $h, u_1, \dots, u_k \in \mathbb{F}[x_1, \dots, x_k]$ such that

$$\begin{aligned}hf & + u_1\left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}\right) + u_2\left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}\right) \\ & + \cdots + u_k\left(x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\right) = 1\end{aligned} \tag{54}$$

if and only if

$$\begin{aligned}1 \in \Big\langle & f(x_1, \dots, x_k), x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \\ & x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \dots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k}\Big\rangle.\end{aligned} \tag{55}$$

□

Let $A \in \mathbb{F}[\sigma_1, \ldots, \sigma_k]$ be an $N \times N$ level-k scaled factor circulant matrix. By Theorem 14, we have the following algorithm which can find the inverse of the matrix A.

Step 10. Calculate the reduced Gröbner basis $\mathbf{G}$ for the ideal

$$\left\langle f(x_1, \ldots, x_k), x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \right. \\ \left. x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle, \tag{56}$$

where the polynomial $f(x_1, \ldots, x_k)$ is the representer of A, by CoCoA 4.0, using a given term order with $x_1 > \cdots > x_k$. If $\mathbf{G} \neq \{1\}$, then A is singular. Stop. Otherwise, go to Step 11.

Step 11. Using Buchberger's algorithm for computing Gröbner bases, by keeping track of linear combinations that give rise to the new polynomials in the generating set, we get $h, u_1, \ldots, u_k \in \mathbb{F}[x_1, \ldots, x_k]$ such that

$$hf + u_1 \left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1} \right) + u_2 \left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2} \right) \\ + \cdots + u_k \left(x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right) = 1. \tag{57}$$

Step 12. The variables $x_1, \ldots, x_k$ in formula (57) are replaced by $\sigma_1, \ldots, \sigma_k$, respectively. We have

$$A^{-1} = h(\sigma_1, \ldots, \sigma_k). \tag{58}$$

5. Inverse of Partitioned Matrix with Level-k Scaled Factor Circulant Matrix Blocks

Let A_1, A_2, A_3, and A_4 be level-k scaled factor circulant matrices with the representers f_1, f_2, f_3, and f_4, respectively. If A_1 is nonsingular, let

$$\Sigma = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} I & 0 \\ -A_3 A_1^{-1} & I \end{pmatrix}, \\ \Pi_2 = \begin{pmatrix} I & -A_1^{-1} A_2 \\ 0 & I \end{pmatrix}. \tag{59}$$

Then,

$$\Pi_1 \Sigma \Pi_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 - A_3 A_1^{-1} A_2 \end{pmatrix}. \tag{60}$$

So, Σ is nonsingular if and only if $A_4 - A_3 A_1^{-1} A_2$ is nonsingular. Since A_1, A_2, A_3, and A_4 are all level-k scaled factor circulant matrices, then A_i commutes with A_j if $i \neq j$. Thus,

$$A_1 \left(A_4 - A_3 A_1^{-1} A_2 \right) = A_1 A_4 - A_2 A_3. \tag{61}$$

From (61), we conclude that Σ is nonsingular if and only if $A_1 A_4 - A_2 A_3$ is nonsingular. Since $f_1 f_4 - f_2 f_3$ is the representer of $A_1 A_4 - A_2 A_3$, then Σ is nonsingular if and only if

$$1 \in \left\langle f_1 f_4 - f_2 f_3, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \right. \\ \left. x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle. \tag{62}$$

Furthermore, if Σ is nonsingular, by (60), we have

$$\Sigma^{-1} = \begin{pmatrix} I & -A_1^{-1} A_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & \left(A_4 - A_3 A_1^{-1} A_2 \right)^{-1} \end{pmatrix} \\ \times \begin{pmatrix} I & 0 \\ -A_3 A_1^{-1} & I \end{pmatrix} \\ = \begin{pmatrix} A_1^{-1} + \Delta_1^{-1} A_2 A_3 A_1^{-1} & -\Delta_1^{-1} A_2 \\ -\Delta_1^{-1} A_3 & \Delta_1^{-1} A_1 \end{pmatrix}, \tag{63}$$

where $\Delta_1 = A_1 A_4 - A_2 A_3$.

We summarize our discussion as the following.

Theorem 15. *Let*

$$\Sigma = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{64}$$

where A_1, A_2, A_3, and A_4 are all level-k scaled factor circulant matrices with the representers f_1, f_2, f_3, and f_4, respectively. If A_1 is nonsingular, then Σ is nonsingular if and only if

$$1 \in \left\langle f_1 f_4 - f_2 f_3, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \right. \\ \left. x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle. \tag{65}$$

Moreover, if Σ is nonsingular, then

$$\Sigma^{-1} = \begin{pmatrix} A_1^{-1} + \Delta_1^{-1} A_2 A_3 A_1^{-1} & -\Delta_1^{-1} A_2 \\ -\Delta_1^{-1} A_3 & \Delta_1^{-1} A_1 \end{pmatrix}, \tag{66}$$

where $\Delta_1 = A_1 A_4 - A_2 A_3$.

Theorem 16. *Let*

$$\Sigma = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{67}$$

where A_1, A_2, A_3, and A_4 are all level-k scaled factor circulant matrices with the representers f_1, f_2, f_3, and f_4, respectively. If A_4 is nonsingular, then Σ is nonsingular if and only if

$$1 \in \left\langle f_1 f_4 - f_2 f_3, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \right. \\ \left. x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle. \tag{68}$$

In addition, if Σ *is nonsingular, then*

$$\Sigma^{-1} = \begin{pmatrix} \Delta_1^{-1} A_4 & -\Delta_1^{-1} A_2 \\ -\Delta_1^{-1} A_3 & A_4^{-1} + \Delta_1^{-1} A_2 A_3 A_4^{-1} \end{pmatrix}, \quad (69)$$

where $\Delta_1 = A_1 A_4 - A_2 A_3$.

Proof. Since A_4 is nonsingular, then

$$\begin{aligned} &\begin{pmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{pmatrix} \Sigma \begin{pmatrix} I & 0 \\ -A_4^{-1} A_3 & I \end{pmatrix} \\ &\quad = \begin{pmatrix} A_1 - A_2 A_4^{-1} A_3 & 0 \\ 0 & A_4 \end{pmatrix}. \end{aligned} \quad (70)$$

So. Σ is nonsingular if and only if $A_1 - A_2 A_4^{-1} A_3$ is nonsingular. Since A_1, A_2, A_3, and A_4 are all level-k scaled factor circulant matrices, then A_i commutes with A_j if $i \neq j$. Thus,

$$A_4 \left(A_1 - A_2 A_4^{-1} A_3 \right) = A_1 A_4 - A_2 A_3. \quad (71)$$

By (71), we conclude that Σ is nonsingular if and only if $A_1 A_4 - A_2 A_3$ is nonsingular. Since $f_1 f_4 - f_2 f_3$ is the representer of $A_1 A_4 - A_2 A_3$, then Σ is nonsingular if and only if

$$\begin{aligned} 1 \in \Bigg\langle & f_1 f_4 - f_2 f_3, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \\ & x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \Bigg\rangle. \end{aligned} \quad (72)$$

If Σ is nonsingular, by (70), we have

$$\begin{aligned} \Sigma^{-1} &= \begin{pmatrix} I & 0 \\ -A_4^{-1} A_3 & I \end{pmatrix} \begin{pmatrix} \left(A_1 - A_2 A_4^{-1} A_3 \right)^{-1} & 0 \\ 0 & A_4^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \Delta_1^{-1} A_4 & -\Delta_1^{-1} A_2 \\ -\Delta_1^{-1} A_3 & A_4^{-1} + \Delta_1^{-1} A_2 A_3 A_4 \end{pmatrix}, \end{aligned} \quad (73)$$

where $\Delta_1 = A_1 A_4 - A_2 A_3$. □

We have the following algorithm for determining the nonsingularity and computing the inverse of Σ if it is nonsingular.

Step 13. Calculate the bases $\mathbf{G}_1, \mathbf{G}_4$ for the ideals

$$\begin{aligned} &\left\langle f_1, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle, \\ &\left\langle f_4, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right\rangle, \end{aligned} \quad (74)$$

respectively. If $\mathbf{G}_1 \neq \{1\}, \mathbf{G}_4 \neq \{1\}$ Stop. Otherwise, go to Step 14.

Step 14. If $\mathbf{G}_1 = \{1\}$, find $u_1, u_2, \ldots, u_k, h_1 \in \mathbb{F}[x_1, x_2, \ldots, x_k]$ such that

$$\begin{aligned} &h_1 f_1 + u_1 \left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1} \right) + u_2 \left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2} \right) \\ &\quad + \cdots + u_k \left(x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right) = 1. \end{aligned} \quad (75)$$

Then, h_1 is the representer of A_1^{-1}, and go to Step 16. Otherwise, go to Step 15.

Step 15. If $\mathbf{G}_4 = \{1\}$, find $u_1', u_2', \ldots, u_k', h_4 \in \mathbb{F}[x_1, x_2, \ldots, x_k]$ such that

$$\begin{aligned} &h_4 f_4 + u_1' \left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1} \right) + u_2' \left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2} \right) \\ &\quad + \cdots + u_k' \left(x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \right) = 1. \end{aligned} \quad (76)$$

Then, h_4 is the representer of A_4^{-1}, and go to Step 16.

Step 16. Calculate the Gröbner bases $\mathbf{G}$ for the ideal

$$\begin{aligned} \Bigg\langle & f_1 f_4 - f_2 f_3, x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1}, \\ & x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2}, \ldots, x_k^{n_k} - \prod_{j_k=1}^{n_k} d_{kj_k} \Bigg\rangle. \end{aligned} \quad (77)$$

If $\mathbf{G} \neq \{1\}$, then $A_1 A_4 - A_2 A_3$ is singular, Stop. Otherwise, go to Step 17.

Step 17. Find $v_1, v_2, \ldots, v_k, h \in \mathbb{F}[x_1, x_2, \ldots, x_k]$ such that

$$\begin{aligned} &h \left(f_1 f_4 - f_2 f_3 \right) + v_1 \left(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1j_1} \right) + v_2 \left(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2j_2} \right) \\ &\quad + \cdots + v_k \left(x_k^{n_k} - \prod_{j_k}^{n_k} d_{kj_k} \right) = 1. \end{aligned} \quad (78)$$

Then, h is the representer of $(A_1 A_4 - A_2 A_3)^{-1}$. Thus, we obtain that

if A_1 is nonsingular, then

$$\Sigma^{-1} = \begin{pmatrix} \mu_1 & -h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_2 \\ -h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_3 & h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_1 \end{pmatrix}, \quad (79)$$

where $\mu_1 = h_1(\sigma_1, \sigma_2, \ldots, \sigma_k)[I + h(\sigma_1, \sigma_2, \ldots, \sigma_k) \times A_2 A_3]$.

If A_4 is nonsingular, then

$$\Sigma^{-1} = \begin{pmatrix} h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_4 & -h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_2 \\ -h(\sigma_1, \sigma_2, \ldots, \sigma_k) A_3 & \mu_2 \end{pmatrix}, \tag{80}$$

where $\mu_2 = h_4(\sigma_1, \sigma_2, \ldots, \sigma_k)[I + h(\sigma_1, \sigma_2, \ldots, \sigma_k) \times A_2 A_3]$.

6. Conclusion

Algorithms for finding the minimal polynomial of the level-k scaled factor circulant matrices over any field are presented. And two algorithms for finding the inverses of such matrices are also presented. Finally, an algorithm for computing the inverse of partitioned matrix with level-k scaled factor circulant matrix blocks over any field is given. In the future, we will investigate the application in solving various differential equations based on multilevel circulant matrices.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. T. Lee, J. Liu, and H. W. Sun, “Combined compact difference scheme for linear second-order partial differential equations with mixed derivative,” *Journal of Computational and Applied Mathematics*, vol. 264, pp. 23–37, 2014.

[2] B. Karasözen and G. Şimşek, “Energy preserving integration of bi-Hamiltonian partial differential equations,” *Applied Mathematics Letters*, vol. 26, pp. 1125–1133, 2013.

[3] S. J. Guo, Y. M. Chen, and J. H. Wu, “Equivariant normal forms for parameterized delay differential equations with applications to bifurcation theory,” *Acta Mathematica Sinica*, vol. 28, no. 4, pp. 825–856, 2012.

[4] W. F. Trench, “On nonautonomous linear systems of differential and difference equations with R-symmetric coefficient matrices,” *Linear Algebra and Its Applications*, vol. 431, no. 11, pp. 2109–2117, 2009.

[5] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, “On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems,” *Physics Letters A*, vol. 358, no. 1, pp. 1–4, 2006.

[6] X. Q. Jin, S. L. Lei, and Y. M. Wei, “Circulant preconditioners for solving singular perturbation delay differential equations,” *Numerical Linear Algebra with Applications*, vol. 12, no. 2-3, pp. 327–336, 2005.

[7] J. C. R. Claeyssen and L. A. D. S. Leal, “Diagonalization and spectral decomposition of factor block circulant matrices,” *Linear Algebra and Its Applications*, vol. 99, pp. 41–61, 1988.

[8] A. C. Wilde, “Differential equations involving circulant matrices,” *Rocky Mountain Journal of Mathematics*, vol. 13, no. 1, pp. 1–13, 1983.

[9] R. M. Mallery, *Reverse Engineering Gene Networks with Microarray Data*, http://www.caam.rice.edu/~cox/genenets.pdf.

[10] S. S. Capizzano and E. Tyrtyshnikov, “Multigrid methods for multilevel circulant matrices,” *SIAM Journal on Scientific Computing*, vol. 26, no. 1, pp. 55–85, 2004.

[11] S. S. Capizzano and E. Tyrtyshnikov, “Any circulant-like preconditioner for multilevel matrices is not superlinear,” *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 2, pp. 431–439, 1999.

[12] G. Song and Y. Xu, “Approximation of high-dimensional kernel matrices by multilevel circulant matrices,” *Journal of Complexity*, vol. 26, no. 4, pp. 375–405, 2010.

[13] Z. L. Jiang and S. Y. Liu, “Level-m scaled circulant factor matrices over the complex number field and the quaternion division algebra,” *Journal of Applied Mathematics and Computing*, vol. 14, no. 1-2, pp. 81–96, 2004.

[14] S. G. Zhang, Z. L. Jiang, and S. Y. Liu, “An application of the Gröbner basis in computation for the minimal polynomials and inverses of block circulant matrices,” *Linear Algebra and Its Applications*, vol. 347, pp. 101–114, 2002.

[15] M. Morhac and V. Matousek, “Exact algorithm of multidimensional circulant deconvolution,” *Applied Mathematics and Computation*, vol. 164, no. 1, pp. 155–166, 2005.

[16] M. Rezghi and L. Elden, “Diagonalization of tensors with circulant structure,” *Linear Algebra and Its Applications*, vol. 435, no. 3, pp. 422–447, 2011.

[17] S. Georgiou and C. Koukouvinos, “Multi-level k-circulant supersaturated designs,” *Metrika*, vol. 64, no. 2, pp. 209–220, 2006.

[18] W. F. Trench, “Properties of multilevel block α-circulants,” *Linear Algebra and Its Applications*, vol. 431, no. 10, pp. 1833–1847, 2009.

[19] W. F. Trench, “Properties of unilevel block circulants,” *Linear Algebra and Its Applications*, vol. 430, no. 8-9, pp. 2012–2025, 2009.

[20] A. J. H. Block, *Circulants of Level-k*, Division of Applied Mathematics, Brown University, Providence, RI, USA, 1976.

[21] J. Baker, F. Hiergeist, and G. E. Trapp, “The structure of multiblock circulants,” *Kyungpook Mathematical Journal*, vol. 25, no. 1, pp. 71–75, 1985.

[22] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University, Chengdu, China, 1999.

[23] R. L. Smith, “Moore-Penrose inverses of block circulant and block k-circulant matrices,” *Linear Algebra and Its Applications*, vol. 16, no. 3, pp. 237–245, 1977.

[24] S. Rjasanow, “Effective algorithms with circulant-block matrices,” *Linear Algebra and Its Applications*, vol. 202, pp. 55–69, 1994.

[25] J. L. Stuart, “Diagonally scaled permutations and circulant matrices,” *Linear Algebra and Its Applications*, vol. 212-213, pp. 397–411, 1994.

[26] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, American Mathematical Society, Providence, RI, USA, 1994.

[27] D. Greenspan, “Methods of matrix inversion,” *The American Mathematical Monthly*, vol. 62, pp. 303–318, 1955.