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*Research Article*

# Impulsive Problems for Fractional Differential Equations with Nonlocal Boundary Value Conditions

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We investigate the nonlocal boundary value problems of impulsive fractional differential equations. By Banach's contraction mapping principle, Schaefer's fixed point theorem, and the nonlinear alternative of Leray-Schauder type, some related new existence results are established via a new special hybrid singular type Gronwall inequality. At last, some examples are also given to illustrate the results.

## 1. Introduction

Fractional differential equations have recently proved to be strong tools in the modeling of many physical phenomena. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media, and fluid dynamic traffic model. For more details on fractional calculus theory, one can see the monographs of Diethelm [1], Kilbas et al. [2], Lakshmikantham et al. [3], Miller and Ross [4], Podlubny [5], and Tarasov [6]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions (see, e.g., [7–13]).

The impulsive differential equations arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in biology, physics, engineering, and so forth. Due to their significance, many authors have established the solvability of impulsive differential equations. For the general theory and applications of such equations we refer the interested readers to see the papers [14–17] and references therein.

As one of the important topics in the research of differential equations, the boundary value problems have attained a great deal of attention from many researchers; see [18–23] and the references therein. As pointed out in [24], the nonlocal boundary condition can be more useful than the standard condition to describe some physical phenomena. But there are very few papers (see, e.g., [24–26]) dealing with the nonlocal boundary value problems of fractional differential equations. And even in [24–26], the impulsive effect has not been considered. In [27], the author considered the following problems:

$$
\begin{aligned}
&{}^{c}D^{q}u(t) = f\left(t, u(t), u'(t)\right), \quad 1 < q \le 2,\\
&\qquad t \in J_1 = [0,1] \setminus \left\{t_1, t_2, \ldots, t_p\right\},\\
&\Delta u(t_k) = I_k\left(u(t_k^-)\right), \qquad \Delta u'(t_k) = J_k\left(u(t_k^-)\right),\\
&\qquad t_k \in (1,0), \ k = 1,2,\ldots,p,\\
&\alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u),
\end{aligned}
\tag{1}
$$

where $J = [0,1]$, $f : J \times R \times R \to R$ is a continuous function, and $I_k, J_k : R \to R$ are continuous functions, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h\to 0^-} u(t_k + h)$, $k = 1,\ldots,p$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $\alpha > 0$, $\beta \ge 0$, and $g_1, g_2 : PC(J,R) \to R$ are two continuous functions, $PC(J,R) = \{x : J \to R; x \in C((t_k, t_{k+1}], R), k = 0,1,\ldots,p+1, x(t_k^+)$, and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k), k = 1,\ldots,p.\}$.

In [27], by a fixed point theorem due to O'Regan, the authors established sufficient conditions for the existence of at least one solution for the problem (1).

In [28], the authors considered the following problem:

$$
\begin{gathered}
{}^{c}D_{0,t}^{q}u(t) := {}^{c}D_{t}^{q}u(t) = f(t,u(t)),\\
t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, \quad J := [0,T],\\
\Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \quad k = 1,2,\ldots,m,\\
\alpha u(0) + b u(T) = c,
\end{gathered}
\tag{2}
$$

where ${}^{c}D_{t}^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $f : J \times R \to R$ is jointly continuous, $t_k$ satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$, $I_k \in C(R,R)$, and $a, b, c$ are real constants with $a + b \neq 0$.

In [29], the authors studied the following problem:

$$
\begin{gathered}
{}^{c}D_{t}^{q}u(t) = f(t,u(t)), \qquad t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\},\\
J := [0,1],\\
\Delta u(t_k) = I_k, \qquad \Delta u'(t_k) = J_k, \ k = 1,2,\ldots,m,\\
\alpha u(0) + b u(1) = 0, \qquad \alpha u'(0) + b u'(1) = 0,
\end{gathered}
\tag{3}
$$

where ${}^{c}D_{t}^{q}$ is the Caputo fractional derivative of order $q \in (1,2)$ with the lower limit zero, $a \geq b > 0$, $f : J \times R \to R$ is jointly continuous, $I_k, J_k \in R$, and $t_k$ satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ representing the right and left limits of $u(t)$ at $t = t_k$. In [29], the authors obtained the sufficient condition of the existence of at least one solution for problem (3).

Motivated by the work mentioned above, we consider the following impulsive fractional differential equation with nonlocal boundary value conditions:

$$
\begin{gathered}
{}^{c}D_{t}^{q}u(t) = f(t,u(t)), \quad 1 < q \leq 2,\\
t \in J_1 = [0,1] \setminus \{t_1, t_2, \ldots, t_p\},\\
\Delta u(t_k) = I_k(u(t_k^-)), \quad \Delta u'(t_k) = J_k(u(t_k^-)),\\
t_k \in (0,1), \ k = 1,2,\ldots,p,\\
a u(0) + b u(1) = g_1(u), \qquad a u'(0) + b u'(1) = g_2(u),
\end{gathered}
\tag{4}
$$

where $J = [0,1]$, $f : J \times R \to R$ is a continuous function, and $I_k, J_k : R \to R$ are continuous functions; $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ with $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$, $u'(t_k^+) = \lim_{h \to 0^+} u'(t_k + h)$, $u'(t_k^-) = \lim_{h \to 0^-} u'(t_k + h)$, $k = 1,2,\ldots,p$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, and $a > 0$, $b \geq 0$ and $g_1, g_2 : PC(J,R) \to R$ are two continuous functions, $PC(J,R) = \{x : J \to R;\ x \in C((t_k, t_{k+1}], R), k = 0,1,\ldots,p,$ and $x(t_k^+), x(t_k^-)$ exist with $x(t_k^-) = x(t_k), k = 1,\ldots,p.\}$. The main methods in our paper are Banach's contraction mapping principle, Schaefer's fixed point theorem, and the nonlinear alternative of Leray-Schauder type.

Obviously, the problems in our paper are different from those in [27], and we generalized the methods and results in [27]. Problems in our paper are more universal than problems in [28, 29]. It should also be noted that the basic space in our paper is $PC^1(J,R) = \{x \in PC(J,R) : x'(t) \in C((t_k, t_{k+1}], R), k = 0,1,\ldots,p, x'(t_k^+), x'(t_k^-)$ exist, and $x'(t)$ is left continuous at $t_k$, $k = 1,\ldots,p.\}$, which is a Banach space with the norm $\|x\| = \sup_{t \in J}\{\|x\|_{PC}, \|x'\|_{PC}\}$, where $\|x\|_{PC} = \sup_{t \in J}|x(t)|$, $\|x'\|_{PC} = \sup_{t \in J}|x'(t)|$. The basic space in [29] is $PC(J,R) = \{u : J \to R : u \in C((t_k, t_{k+1}], R), k = 0,1,\ldots,m,$ and $x(t_k^+), x(t_k^-)$ exist with $x(t_k^-) = x(t_k), k = 1,\ldots,m,$ with the norm $\|u\|_{PC} = \sup_{t \in J}|u(t)|\}$, which is unreasonable for the order $q \in (1,2)$ because ${}^{c}D_{t}^{q}u(t)$, $\Delta u'(t_k)$ may not exist, for $u \in PC[J,R]$. So the problem (3) in [29] are not well defined, and Definition 4.1 is also unreasonable and should be modified.

The rest of this paper is organized as follows. In Section 2, we will give some lemmas which are essential to prove our main results. In Section 3, we give the main results. The first result is based on the Banach contraction principle, the second result is based on Schaefer's fixed point theorem via a generalized hybrid singular Gronwall inequality, and the third result is based on a nonlinear alternative of Leray-Schauder type. In Section 4, some examples are offered to demonstrate the application of our main results.

## 2. Preliminaries

At first, we present the necessary definitions for the fractional calculus theory.

*Definition 1* (see [2, 5]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$
I_{0_+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,
\tag{5}
$$

where the right side is pointwise defined on $(0,+\infty)$.

*Definition 2* (see [2, 5]). The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$
{}^{c}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds,
\tag{6}
$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number $\alpha$, and the right side is pointwise defined on $(0,+\infty)$.

**Lemma 3** (see [2, 5]). *Let $\alpha > 0$; then the fractional differential equation ${}^{c}D^{\alpha}u(t) = 0$ has solutions*

$$
u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\tag{7}
$$

*where $c_i \in R$, $i = 0,1,\ldots,n-1$, and $n = [\alpha] + 1$.*

**Lemma 4** (see [2, 5]). *Let $\alpha > 0$. Then one has*

$$I_{0_+}^{\alpha}\, {}^{c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \quad (8)$$

*where $c_i \in R$, $i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.*

**Lemma 5** (see [29, Lemma 2.9]). *Let $y \in C(J, R)$ satisfy the following inequality:*

$$\begin{aligned} |y(t)| \le\ & p_1 + p_2 \int_0^t (t-s)^{q-1} |y(s)|^{\lambda} ds \\ & + p_3 \int_0^1 (1-s)^{q-1} |y(s)|^{\lambda} ds + p_4 \\ & \times \int_0^1 (1-s)^{q-2} |y(s)|^{\lambda} ds, \end{aligned} \quad (9)$$

*where $q \in (1, 2)$, $\lambda \in [0, 1 - (1/p_0))$; for some $1 < p_0 < 1/(2-q)$, $p_1, p_2, p_3, p_4 \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that*

$$|y(t)| \le M^*. \quad (10)$$

**Lemma 6** (Schaefer's fixed point theorem). *Let $X$ be a Banach space and let $F : X \to X$ be a completely continuous operator. If the set*

$$E(F) = \{y \in X : y = \lambda F y \ \text{for some}\ \lambda \in [0, 1]\} \quad (11)$$

*is bounded, then $F$ has at least a fixed point.*

**Lemma 7** (nonlinear alternative of Leray-Schauder type). *Let $C$ be a nonempty convex subset of $X$. Let $U$ be a nonempty open subset of $C$ with $0 \in U$ and let $F : \overline{U} \to C$ be a compact and continuous operator. Then either*

(i) *$F$ has fixed points, or*

(ii) *there exist $y \in \partial U$ and $\lambda^* \in [0, 1]$ with $y = \lambda^* F(y)$.*

*We define*

$PC(J, R) = \{x : J \to R; x \in C((t_k, t_{k+1}], R),$ $k = 0, 1, \ldots, p$, *and* $x(t_k^+), x(t_k^-)$ *exist with* $x(t_k^-) = x(t_k), k = 1, \ldots, p\}$.

$PC^1(J, R) = \{x \in PC(J, R);\ x'(t) \in C((t_k, t_{k+1}], R),$ $k = 0, 1, \ldots, p,$ $x'(t_k^+)$, $x'(t_k^-)$ *exist and $x'$ is left continuous at $t_k$, $k = 1, \ldots, p$}.

*Obviously, $PC^1(J, R)$ is a Banach space with the norm* $\|x\| = \sup_{t\in J}\{\|x\|_{PC}, \|x'\|_{PC}\}$, *where* $\|x\|_{PC} = \sup_{t\in J}|x(t)|$, $\|x'\|_{PC} = \sup_{t\in J}|x'(t)|$.

Then we can define the solution for the problem (4).

*Definition 8.* A function $u \in PC^1(J, R)$ with its Caputo derivative of order $1 < q \le 2$ existing on $J_1$ is a solution of the problem (4) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k' \in C([0, t_{k+1}], R)$ satisfies ${}^{c}D_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$, the restriction of $u_{k+1}(t)$ on $[0, t_{k+1})$ is just $u_k(t)$, and the conditions $\Delta u(t_k) = I_k(u(t_k^-))$, $\Delta u'(t_k) = J_k(u(t_k^-))$, $t_k \in (0, 1)$, $k = 1, 2, \ldots, p$ with $au(0) + bu(1) = g_1(u)$, $au'(0) + bu'(1) = g_2(u)$.

**Lemma 9.** *For any $h \in C[0, 1]$, a function $u$ is a solution of the nonlocal impulsive problem*

$$\begin{aligned} & {}^{c}D_t^q u(t) = h(t), \quad 1 < q \le 2, \\ & t \in J_1 = [0, 1] \setminus \{t_1, t_2, \ldots, t_p\} \\ & \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = J_k(u(t_k^-)), \\ & t_k \in (0, 1), \ k = 1, 2, \ldots, p, \\ & au(0) + bu(1) = g_1(u), \qquad au'(0) + bu'(1) = g_2(u) \end{aligned} \quad (12)$$

*if and only if $u$ is a solution of the fractional integral equation*

$$\begin{aligned} u(t) = & \int_{t_i}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\, ds \\ & + \sum_{0<t_k<t} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}}{\Gamma(q)} h(s)\, ds + I_k(u(t_k^-)) \right) \\ & + \sum_{0<t_k<t} (t - t_k) \left( \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}}{\Gamma(q-1)} h(s)\, ds + J_k(u(t_k^-)) \right) \\ & - c_0 - c_1 t, \end{aligned} \quad (13)$$

*with*

$$\begin{aligned} u'(t) = & \int_{t_i}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)\, ds \\ & + \sum_{0<t_k<t} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}}{\Gamma(q-1)} h(s)\, ds + J_k(u(t_k^-)) \right) \\ & - \frac{1}{a+b} \left[ b \left( \int_{t_p}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\, ds \right.\right. \\ & \left. + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}}{\Gamma(q-1)} h(s)\, ds \right.\right. \\ & \left.\left. + J_k(u(t_k^-)) \right) \right) - g_2(u) \Bigg], \end{aligned} \quad (14)$$

*where*

$$\begin{aligned} c_0 = \frac{b}{a+b} \Bigg\{ & \Bigg[ \int_{t_p}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\, ds \\ & + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}}{\Gamma(q)} h(s)\, ds \right. \\ & \left. + I_k(u(t_k^-)) \right) \end{aligned}$$

$$+\sum_{0<t_k<1}(1-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_k\left(u\left(t_k^-\right)\right)\right)\Bigg]$$
$$-\frac{1}{(a+b)}\Bigg[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_k\left(u\left(t_k^-\right)\right)\right)\right)$$
$$-g_2(u)\Bigg]\Bigg\}-\frac{g_1(u)}{a+b},$$
$$c_1=\frac{1}{a+b}\Bigg\{b\Bigg[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_k\left(u\left(t_k^-\right)\right)\right)\Bigg]-g_2(u)\Bigg\}. \tag{15}$$

*Proof.* By Lemmas 3 and 4, the solution of (12) can be written as

$$u(t)=I_+^q h(t)-c_0-c_1t=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds-c_0-c_1t,\quad t\in[0,t_1], \tag{16}$$

where $c_0,c_1\in R$. Taking the derivative of $u(t)$ gives

$$u'(t)=\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds-c_1,\quad t\in[0,t_1]. \tag{17}$$

If $t\in(t_1,t_2]$, then we have

$$u(t)=\int_{t_1}^t\frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds-d_0-d_1(t-t_1),$$
$$u'(t)=\int_{t_1}^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds-d_1, \tag{18}$$

where $d_0,d_1\in R$. In view of the impulse conditions

$$\Delta u(t_1)=u\left(t_1^+\right)-u\left(t_1^-\right)=I_1\left(u\left(t_1^-\right)\right),$$
$$\Delta u'(t_1)=u'\left(t_1^+\right)-u'\left(t_1^-\right)=J_1\left(u\left(t_1^-\right)\right), \tag{19}$$

and (16)–(18), we have

$$-d_0=\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}h(s)\,ds-c_0-c_1t_1+I_1\left(u\left(t_1^-\right)\right)$$
$$-d_1=\int_0^{t_1}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds-c_1+J_1\left(u\left(t_1^-\right)\right). \tag{20}$$

Taking (20) into (18), we can get

$$u(t)=\int_{t_1}^t\frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds+\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}h(s)\,ds-c_0-c_1t+I_1\left(u\left(t_1^-\right)\right)+\left(\int_0^{t_1}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_1\left(u\left(t_1^-\right)\right)\right)(t-t_1),\quad t\in(t_1,t_2]. \tag{21}$$

Repeating the process in this way, the solution $u(t)$ for $t\in(t_k,t_{k+1}]$ can be written as

$$u(t)=\int_{t_k}^t\frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds-c_0-c_1t+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}h(s)\,ds+I_k\left(u\left(t_k^-\right)\right)\right)+\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_k\left(u\left(t_k^-\right)\right)\right),\quad t\in(t_k,t_{k+1}],\ k=0,1,\ldots,p. \tag{22}$$

By taking the derivative of (22), we have

$$u'(t)=\int_{t_k}^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds-c_1+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds+J_k\left(u\left(t_k^-\right)\right)\right),\quad t\in(t_k,t_{k+1}],\quad k=0,1,\ldots,p. \tag{23}$$

Taking (16), (17), (22), and (23) to the boundary value conditions

$$au(0)+bu(1)=g_1(u),\qquad au'(0)+bu'(1)=g_2(u), \tag{24}$$

we can get

$$c_0 = \frac{b}{a+b}\left\{\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}h(s)\,ds + I_k\left(u\left(t_k^-\right)\right)\right) + \sum_{0<t_k<1}(1-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right] - \frac{1}{(a+b)} \times \left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right) - g_2(u)\right]\right\} - \frac{g_1(u)}{a+b},$$

$$c_1 = \frac{1}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right) - g_2(u)\right]. \tag{25}$$

Then the solution of (12) is (22), where $c_0, c_1$ are given by (25). Taking derivative of (13), we can get

$$u'(t) = \int_{t_i}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + J_k\left(u\left(t_k^-\right)\right)\right) - \frac{1}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right) - g_2(u)\right]. \tag{26}$$

Conversely, taking (13) and (14) into (12), we can easily get the equation

$$^{c}D_t^q u(t) = h(t), \quad 1 < q \le 2, \quad t \in J_1 = [0,1] \setminus \left\{t_1, t_2, \ldots, t_p\right\} \tag{27}$$

and all the impulse conditions and boundary value conditions are satisfied. So we complete the proof of Lemma 9. □

Consider the operator $F : PC^1(J,R) \to PC^1(J,R)$ defined by

$$(Fu)(t) = \int_{t_i}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,u(s))\,ds + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s,u(s))\,ds + I_k\left(u\left(t_k^-\right)\right)\right) + \sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right) - c_0 - c_1 t, \tag{28}$$

where

$$c_0 = \frac{1}{a+b}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,u(s))\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s,u(s))\,ds + I_k\left(u\left(t_k^-\right)\right)\right) + \sum_{0<t_k<1}(1-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right] - g_1(u)\right\} - \frac{b}{(a+b)^2}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right] - g_2(u)\right\},$$

$$c_1 = \frac{1}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right) - g_2(u)\right]. \tag{29}$$

Then we have

$$\begin{aligned}(Fu)'(t) = {} & \int_{t_i}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds \\ & + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right) \\ & - \frac{1}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))\,ds + J_k\left(u\left(t_k^-\right)\right)\right)\right) - g_2(u)\right].\end{aligned} \tag{30}$$

Clearly, $F$ is well defined.

## 3. Main Results

This section deals with the existence of solutions for problem (4). Before stating and proving the main results, we make the following hypotheses.

- $(H_1)$ $f: J\times R \to R$ is jointly continuous.
- $(H_2)$ $g_1, g_2 : R \to R$ are continuous functions and there exists $l_1(t), l_2(t) \in C[J, R_+]$ such that $|g_1(u_1(t)) - g_1(u_2(t))| \le l_1(t)\|u_1 - u_2\|$, $|g_2(u_1(t)) - g_2(u_2(t))| \le l_2(t)\|u_1 - u_2\|$, for $\forall u_1, u_2 \in PC^1(J,R)$, $t \in J$.
- $(H_3)$ There exist real functions $h_1(\cdot), h_2(\cdot) \in C(J, R_+)$ such that $|f(t,u)| \le h_1(t)$, $|f(t,u_1(t)) - f(t,u_2(t))| \le h_2(t)\|u_1 - u_2\|$, for $\forall u_1, u_2 \in PC^1(J,R)$, $t \in J$.
- $(H_4)$ $I_k, J_k : R \to R$ are continuous functions and there exist positive constants $M_k$, $m_k$ such that $|I_k(u_1) - I_k(u_2)| \le M_k\|u_1 - u_2\|$, $|J_k(u_1) - J_k(u_2)| \le m_k\|u_1 - u_2\|$, $\forall u_1, u_2 \in PC^1(J,R)$ and $k = 1, 2, \ldots, p$.

Let

$$\begin{aligned} W &= \sup_{t\in J} l_1(t), & w &= \sup_{t\in J} l_2(t), \\ E &= \sup_{t\in J} h_1(t) & e &= \sup_{t\in J} h_2(t). \end{aligned} \tag{31}$$

**Theorem 10.** *Assume that $(H_1)$–$(H_4)$ hold and $< 1$; then problem (4) has a unique solution, where*

$$n = \max\left\{\left[\frac{2e(1+p)}{\Gamma(q+1)} + \frac{2e(1+2p)}{\Gamma(q)} + W + 2w + \sum_{k=1}^{p}(2M_k + 4m_k)\right], \left(\frac{2e(1+p)}{\Gamma(q-1)} + w + 2\sum_{k=1}^{p} m_k\right)\right\}. \tag{32}$$

*Proof.*

*Step 1.* We show that $Fu \in PC^1(J,R)$, for all $u \in PC^1(J,R)$.

For all $u \in PC^1(J,R)$, $s_1, s_2 \in [0, t_1]$, or $s_1, s_2 \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, p$, $s_2 > s_1$, by (30) and the continuity of $g_2$, $u$, we have

$$\begin{aligned} & \left|(Fu)'(s_2) - (Fu)'(s_1)\right| \\ & \le \int_{t_i}^{s_1}\frac{\left[(s_2-s)^{q-2} - (s_1-s)^{q-2}\right]}{\Gamma(q-1)}\left|f(s,u(s))\right|ds + \int_{s_1}^{s_2}\frac{(s_2-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))\right|ds + \frac{1}{a+b}\left|(g_2(u(s_2)) - g_2(u(s_1)))\right| \\ & \le \frac{E}{\Gamma(q-1)}\left[\int_0^{s_1}\left[(s_2-s)^{q-2} - (s_1-s)^{q-2}\right]ds + \int_{s_1}^{s_2}(s_2-s)^{q-2}ds\right] + \frac{1}{a+b}\left|(g_2(u(s_2)) - g_2(u(s_1)))\right| \longrightarrow 0, \\ & \qquad \text{as } s_1 \longrightarrow s_2. \end{aligned} \tag{33}$$

So we know $(Fu)'(t) \in C((t_k, t_{k+1}], R)$, $k = 0, 1, \ldots, p+1$. It is easy to see that $(Fu)'(t_k^+)$, $(Fu)'(t_k^-)$ exist and $(Fu)'(t)$ is left continuous at $t_k, k = 1, \ldots, p$. So, for $\forall u \in PC^1(J,R)$, $Fu \in PC^1(J,R)$.

*Step 2.* We show that $F$ is a contraction operator on $PC^1(J,R)$. Consider

$$\begin{aligned} & |(Fu)(t) - (Fv)(t)| \\ & \le \int_{t_i}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f(s,u(s)) - f(s,v(s))\right|ds + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\left|f(s,u(s)) - f(s,v(s))\right|ds + \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right|\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\right.\\
&\qquad\times\left|f(s,u(s))-f(s,v(s))\right|ds\\
&\qquad\left.+\left|J_k\left(u\left(t_k^-\right)\right)-J_k\left(v\left(t_k^-\right)\right)\right|\right)\\
&+\frac{1}{a+b}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\right.\\
&\qquad\qquad\times\left|f(s,u(s))-f(s,v(s))\right|ds\\
&\qquad\qquad\left.+\left|I_k\left(u\left(t_k^-\right)\right)-I_k\left(v\left(t_k^-\right)\right)\right|\right)\\
&\qquad+\sum_{0<t_k<1}(1-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\right.\\
&\qquad\qquad\times\left|f(s,u(s))\right.\\
&\qquad\qquad\left.\left.-f(s,v(s))\right|ds\right.\\
&\qquad\qquad+\left|J_k\left(u\left(t_k^-\right)\right)\right.\\
&\qquad\qquad\left.\left.-J_k\left(v\left(t_k^-\right)\right)\right|\right)\Bigg]\\
&\qquad+\left|g_1(u)-g_1(u)\right|\Bigg\}\\
&+\frac{b}{(a+b)^2}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\right.\\
&\qquad\qquad\times\left|f(s,u(s))-f(s,v(s))\right|ds\\
&\qquad\qquad\left.+\left|J_k\left(u\left(t_k^-\right)\right)-J_k\left(v\left(t_k^-\right)\right)\right|\right)\Bigg]\\
&\qquad+\left|g_2(u)-g_2(v)\right|\Bigg\}\\
&+\frac{1}{a+b}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))\right.\right.\\
&\qquad\qquad\left.\left.-f(s,v(s))\right|ds\right.\\
&\qquad\qquad\left.+\left|J_k\left(u\left(t_k^-\right)\right)-J_k\left(v\left(t_k^-\right)\right)\right|\right)\Bigg]\\
&\qquad+\left|g_2(u)-g_2(v)\right|\Bigg\}\\
&\le\left[\frac{2e(1+p)}{\Gamma(q+1)}+\frac{2e(1+2p)}{\Gamma(q)}+W+2w\right.\\
&\qquad\left.+\sum_{k=1}^{p}(2M_k+4m_k)\right]\|u-v\|\\
&\le\|u-v\|,\\
&\left|(Fu)'(t)-(Fv)'(t)\right|\\
&\le\int_{t_i}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\\
&\quad+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.\\
&\qquad\left.+\left|J_k\left(u\left(t_k^-\right)\right)-J_k\left(v\left(t_k^-\right)\right)\right|\right)\\
&\quad+\frac{1}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))\right.\right.\right.\\
&\qquad\left.\left.-f(s,v(s))\right|ds\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\left|f(s,u(s))\right.\right.\\
&\qquad\qquad\left.\left.-f(s,v(s))\right|ds\right.\\
&\qquad\left.\left.+\left|J_k\left(u\left(t_k^-\right)\right)-J_k\left(v\left(t_k^-\right)\right)\right|\right)\right)\\
&\qquad\left.+\left|g_2(u)-g_2(v)\right|\right]\\
&\le\left(\frac{2e(1+p)}{\Gamma(q)}+w+2\sum_{k=1}^{p}m_k\right)\|u-v\|\\
&\le\|u-v\|.
\end{aligned}
\tag{34}
$$

Hence $\|F(u)-F(v)\|\le\|u-v\|$, that is, $F$ is a contraction operator on $PC^1(J,R)$. By applying the well-known Banach's contraction mapping principle, we know that the operator $F$ has a unique fixed point on $PC^1(J,R)$. Therefore, the problem (4) has a unique solution.

In order to get the second main result, we replace $(H_2)'$ with $(H_2)$.

$(H_2)$ $g_1, g_2 : R \to R$ are continuous functions and there exist positive constants $r_1, r_2$ and $l_1(t), l_2(t) \in C[J, R_+]$ such that $|g_1(u)| \le r_1$, $|g_2(u)| \le r_2$, $|g_1(u_1)-g_1(u_2)| \le l_1(t)\|u_1-u_2\|$, $|g_2(u_1)-g_2(u_2)| \le l_2(t)\|u_1-u_2\|$, for all $u_1, u_2 \in PC^1(J,R)$.

Next, we modify $(H_3)$ to the following linear growth condition $(H_3)'$:

$(H_3)'$ There exist constants $\overline{\lambda} \in [0,1)$ and $L > 0$ such that $|f(t,u(t))| \le L(1+|u(t)|^{\overline{\lambda}})$, $\forall t \in J, u(t) \in R$.

□

**Theorem 11.** *Assume that* $(H_1)$, $(H_2)'$, *and* $(H_3)'$ *hold; then the problem* (4) *has at least one solution.*

*Proof.* According to Lemma 6, if we want to get the solution of problem (4), we only need to consider the fixed point of operator $F$, which is defined by (28). We divide the proof into four steps.

*Step 1.* $F$ is continuous.

Let $\{u_n\}$ be a sequence such that $\{u_n\} \to u_0$ in $PC^1(J,R)$. $\forall t \in J$, we have

$$\begin{aligned}
&\left|(Fu_n)(t) - (Fu_0)(t)\right| \\
&\le \int_{t_i}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \\
&\quad + \sum_{0<t_k<t} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \right. \\
&\qquad \left. + \left|I_k(u_n(t_k^-)) - I_k(u_0(t_k^-))\right| \right) \\
&\quad + \sum_{0<t_k<t} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \right. \\
&\qquad \left. + \left|J_k(u_n(t_k^-)) - J_k(u_0(t_k^-))\right| \right) \\
&\quad + \frac{1}{a+b} \left\{ b \left[ \int_{t_p}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \right. \right. \\
&\qquad + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \right. \\
&\qquad\quad \left. \times \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \right. \\
&\qquad\quad \left. + \left|I_k(u_n(t_k^-)) - I_k(u_0(t_k^-))\right| \right) \\
&\qquad + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u_n(s)) \right.\right. \\
&\qquad\quad \left.\left. - f(s,u_0(s))\right| ds \right. \\
&\qquad\quad \left. \left. + \left|J_k(u_n(t_k^-)) - J_k(u_0(t_k^-))\right| \right) \right] \\
&\qquad \left. + \left|g_1(u) - g_1(u)\right| \right\} \\
&\quad + \frac{b}{(a+b)^2} \left\{ b \left[ \int_{t_p}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u(s)) - f(s,v(s))\right| ds \right. \right. \\
&\qquad + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u_n(s)) \right.\right. \\
&\qquad\quad \left.\left. - f(s,u_0(s))\right| ds \right. \\
&\qquad\quad \left. \left. + \left|J_k(u_n(t_k^-)) - J_k(u_0(t_k^-))\right| \right) \right] \\
&\qquad \left. + \left|g_2(u_n) - g_2(u_0)\right| \right\} + \frac{1}{a+b} \\
&\quad \times \left\{ b \left[ \int_{t_p}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u_n(s)) - f(s,u_0(s))\right| ds \right. \right. \\
&\qquad + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u_n(s)) \right.\right. \\
&\qquad\quad \left.\left. - f(s,u_0(s))\right| ds \right. \\
&\qquad\quad \left. \left. + \left|J_k(u_n(t_k^-)) - J_k(u_0(t_k^-))\right| \right) \right] \\
&\qquad \left. + \left|g_2(u_n) - g_2(u_0)\right| \right\}.
\end{aligned} \tag{35}$$

From $(H_1)$ and $(H_2)'$, we know $f$ is jointly continuous and $g_1, g_2$ are also continuous. Together with the continuity of $I_k, J_k$, we can also easily draw that $|(Fu_n)(t) - (Fu_0)(t)| \to 0$, as $u_n \to u_0$.

Similarly, we can obtain $|(Fu_n)'(t) - (Fu_0)'(t)| \to 0$, as $u_n \to u_0$, $\forall t \in J$. Then for $\{u_n\} \to u_0$, we have $\|Fu_n - Fu_0\| \to 0$, as $u_n \to u_0$, which implies that $F : PC^1(J,R) \to PC^1(J,R)$ is continuous.

*Step 2.* $F$ maps bounded sets into bounded sets in $PC^1(J,R)$.

Set $B_\mu = \{u \in PC^1(J,R) : \|u\| \le \mu\}$. For $u \in B_\mu, t \in J_1$, by the continuity of $I_k$, $J_k$, $g_1(u)$, $g_2(u)$, $\forall u \in B_\mu$, we know that $|I_k(u(t_k^-))| \le e_1'$, $|J_k(u(t_k^-))| \le e_2'$, where $e_1', e_2'$ are nonnegative constants.

For all $t \in J$, $u \in B_\mu$, we have

$$\begin{aligned}
|(Fu)(t)| \le{}& \int_{t_i}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left|f(s,u(s))\right| ds \\
&+ \sum_{0<t_k<t} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \left|f(s,u(s))\right| ds \right. \\
&\qquad \left. + \left|I_k(u(t_k^-))\right| \right) \\
&+ \sum_{0<t_k<t} (t-t_k) \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \left|f(s,u(s))\right| ds \right. \\
&\qquad \left. + \left|J_k(u(t_k^-))\right| \right) + \frac{1}{a+b} \\
&\times \left\{ b \left[ \int_{t_p}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \left|f(s,u(s))\right| ds \right. \right. \\
&\qquad + \sum_{0<t_k<1} \left( \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \left|f(s,u(s))\right| ds \right. \\
&\qquad\quad \left. \left. + \left|I_k(u(t_k^-))\right| \right) \right]
\end{aligned}$$

$$+\sum_{0<t_k<1}(1-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\times|f(s,u(s))|\,ds+|J_k(u(t_k^-))|\right)\Bigg]+|g_1(u)|\Bigg\}$$

$$+\frac{b}{(a+b)^2}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s))|\,ds+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\times|f(s,u(s))|\,ds+|J_k(u(t_k^-))|\right)\right]+|g_2(u)|\right\}$$

$$+\frac{1}{a+b}\left\{b\left[\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s))|\,ds+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\times|f(s,u(s))|\,ds+|J_k(u(t_k^-))|\right)\right]+|g_2(u)|\right\}$$

$$\le\frac{2L(1+p)\left(1+\|\mu\|^{\overline{\lambda}}\right)}{\Gamma(q+1)}+2pe_1'+\frac{L(4p+2)\left(1+\|\mu\|^{\overline{\lambda}}\right)}{\Gamma(q)}+4pe_2'+r_1+2r_2,$$

$$|(Fu)'(t)|\le\int_{t_i}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s))|\,ds+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s))|\,ds+|J_k(u(t_k^-))|\right)$$

$$+\frac{b}{a+b}\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s))|\,ds+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\times|f(s,u(s))|\,ds+|J_k(u(t_k^-))|\right)\right)+|g_2(u)|\right]$$

$$\le\frac{2L(p+1)\left(1+\|\mu\|^{\overline{\lambda}}\right)}{\Gamma(q)}+2pe_2'+r_2. \tag{36}$$

Then we can obtain $\|Fu\|\le\eta$, where

$$\eta=\max\left\{\frac{2L(1+p)\left(1+\mu^{\overline{\lambda}}\right)}{\Gamma(q+1)}+2pe_1'+\frac{L(4p+2)\left(1+\mu^{\overline{\lambda}}\right)}{\Gamma(q)}+4pe_2'+r_1+2r_2,\frac{2L(p+1)\left(1+\mu^{\overline{\lambda}}\right)}{\Gamma(q)}+2pe_2'+r_2\right\}. \tag{37}$$

If $0<\mu<\infty$, that is, $B_\mu$ is bounded, then $0<\eta<\infty$. Hence $F$ maps bounded sets into bounded sets in $PC^1(J,R)$.

*Step 3.* $F$ maps bounded sets into equicontinuous sets of $PC^1(J,R)$.

Consider $\forall u\in B_\mu=\{u\in PC^1(J,R):\|u\|\le\mu\}$, $\forall 0\le s_1<s_2\le t_1$; we have

$$|Fu(s_2)-Fu(s_1)|\le\int_0^{s_1}\frac{\left[(s_2-s)^{q-1}-(s_1-s)^{q-1}\right]}{\Gamma(q)}|f(s,u(s))|\,ds+\int_{s_1}^{s_2}\frac{(s_2-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|\,ds+\frac{b}{(a+b)^2}|g_2(u(s_2))-g_2(u(s_1))|$$

$$
\begin{aligned}
&+\frac{\left|g_1\left(u\left(s_2\right)\right)-g_1\left(u\left(s_1\right)\right)\right|}{a+b}+\frac{1}{a+b}\\
&\times\left[b\left(\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left|f\left(s,u\left(s\right)\right)\right|ds\right.\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{\left(t_k-s\right)^{q-2}}{\Gamma(q-1)}\left|f\left(s,u\left(s\right)\right)\right|ds\right.\\
&\qquad\left.\left.\left.+J_k\left(u\left(t_k^-\right)\right)\right)\right)\right]\\
&\times\left(s_2-s_1\right)+\frac{\left|s_2g_2\left(u\left(s_2\right)\right)-s_1g_2\left(u\left(s_1\right)\right)\right|}{a+b}\\
&\le\frac{L\left(1+\mu^{\overline{\lambda}}\right)}{\Gamma\left(q\right)}\left\{\int_0^{s_1}\left[\left(s_2-s\right)^{q-1}-\left(s_1-s\right)^{q-1}\right]ds\right.\\
&\qquad\left.+\int_{s_1}^{s_2}\left(s_2-s\right)^{q-1}ds\right\}\\
&+\left\{\frac{b}{a+b}\left[\frac{2L\left(1+\mu^{\overline{\lambda}}\right)\left(1+p\right)}{\Gamma\left(q\right)}+pe_2'\right]\right.\\
&\qquad\left.+\mu\frac{bw+(a+b)\left(W+w\right)}{(a+b)^2}+\frac{h_2'}{a+b}\right\}\left(s_2-s_1\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|(Fu)'\left(s_2\right)-(Fu)'\left(s_1\right)\right|\\
&\le\int_0^{s_1}\frac{\left[\left(s_2-s\right)^{q-2}-\left(s_1-s\right)^{q-2}\right]}{\Gamma\left(q-1\right)}\left|f\left(s,u\left(s\right)\right)\right|ds\\
&+\int_{s_1}^{s_2}\frac{\left(s_2-s\right)^{q-2}}{\Gamma\left(q-1\right)}\left|f\left(s,u\left(s\right)\right)\right|ds\\
&+\frac{b}{a+b}\left|g_2\left(u\left(s_2\right)\right)-g_2\left(u\left(s_1\right)\right)\right|\\
&\le\frac{L\left(1+\mu^{\overline{\lambda}}\right)}{\Gamma\left(q-1\right)}\left\{\int_0^{s_1}\left[\left(s_2-s\right)^{q-1}-\left(s_1-s\right)^{q-1}\right]ds\right.\\
&\qquad\left.+\int_{s_1}^{s_2}\left(s_2-s\right)^{q-1}ds\right\}+\frac{bh_2'\mu}{a+b}\left|s_2-s_1\right|.
\end{aligned}
\tag{38}
$$

Obviously, $|Fu(s_2)-Fu(s_1)|\to 0$, $|(Fu)'(s_2)-(Fu)'(s_1)|\to 0$, as $s_1\to s_2$. Hence $F$ is equicontinuous on interval $[0,t_1]$.

Similarly, we can prove $F$ is equicontinuous on interval $(t_k,t_{k+1}]$, $k=1,2,\ldots,p$.

As a consequence of Steps 1–3 together with the $PC$-type Arzela-Ascoli theorem, we know that $F: PC^1(J,R)\to PC^1(J,R)$ is continuous and completely continuous.

*Step 4*. There exists a priori bound.

Next we show that the set $E(F)=\{u\in PC^1(J,R): u=\lambda Fu, \text{for some } \lambda\in(0,1]\}$, is bounded.

Consider $\forall t\in J, u\in E(F)$; we have

$$
\begin{aligned}
|u(t)|&=|\lambda Fu(t)|\\
&\le\frac{L}{\Gamma\left(q\right)}+\frac{L}{\Gamma\left(q\right)}\int_0^t(t-s)^{q-1}|u(s)|^{\overline{\lambda}}ds\\
&+\left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}\left|I_k\left(u\left(t_k^-\right)\right)\right|\\
&+\frac{L}{\Gamma\left(q\right)}\left[2p+\frac{b}{a+b}\left(2+3p\right)+\frac{b^2}{(a+b)^2}\left(1+p\right)\right]\\
&+\frac{L}{\Gamma\left(q\right)}\left[p+\frac{b}{a+b}\left(1+p\right)\right]\int_0^1(1-s)^{q-1}|u(s)|^{\overline{\lambda}}ds\\
&+\frac{L}{\Gamma\left(q-1\right)}\left[p+\frac{b}{a+b}\left(1+2p\right)+\frac{b^2}{(a+b)^2}\left(1+p\right)\right]\\
&\times\int_0^1(1-s)^{q-2}|u(s)|^{\overline{\lambda}}ds\\
&+\left(1+\frac{2b}{a+b}+\frac{b^2}{(a+b)^2}\right)\sum_{0<t_k<1}\left|J_k\left(u\left(t_k^-\right)\right)\right|\\
&+\frac{r_1}{a+b}+\frac{a+2b}{(a+b)^2}r_2.
\end{aligned}
\tag{39}
$$

Then by Lemma 5, we know there exists a constant $M^*>0$ such that

$$
\left|y\left(t\right)\right|\le M^*. \tag{40}
$$

For all $t\in J$, $\forall u\in E(F)$, we also have

$$
\begin{aligned}
\left|u'\left(t\right)\right|&=\left|\lambda(Fu)'\left(t\right)\right|\\
&\le\frac{L}{\Gamma\left(q-1\right)}\left(1+p\right)\left(1+\frac{b}{a+b}\right)\\
&\times\int_0^1(1-s)^{q-2}|u(s)|^{\overline{\lambda}}ds\\
&+\frac{L}{\Gamma\left(q\right)}\left(1+p\right)\left(1+\frac{b}{a+b}\right)+\left(1+\frac{b}{a+b}\right)\\
&\times\sum_{0<t_k<1}\left|J_k\left(u\left(t_k^-\right)\right)\right|+\frac{r_2}{a+b}.
\end{aligned}
\tag{41}
$$

Also by Lemma 5, we can get that there exists a constant $M^{**}>0$ such that

$$
\left|y'\left(t\right)\right|\le M^{**}. \tag{42}
$$

So, for all $u\in E(F)$, we have $\|u\|=\sup_{t\in J}\{\|u\|_{PC},\|u'\|_{PC}\}\le M^{***}$, where $M^{***}=\max\{M^*,M^{**}\}$.

As a consequence of Schaefer's fixed point theorem (Lemma 6), we deduce that $F$ has at least one fixed point which means that the problem (4) has at least one solution.

Next we apply the nonlinear alternative of Leray-Schauder type to get Theorem 12. We give the following hypothesis $(H_3)''$.

$(H_3)''$ There exist a real valued function $\phi \in L^2(J,R)$, a $L^1$-integrable and nondecreasing function $\psi : [0,+\infty) \to [0,+\infty)$, and a positive constant $r$ such that

$$|f(t,u)| \le \phi(t)\psi(u), \quad \text{for } \forall t \in J,\ \forall u \in PC^1(J,R),$$
$$r\left(\frac{\psi(r)}{\Gamma(q)}\left\{\int_0^t (t-s)^{q-1}\phi(s)\,ds + T_1 \times \int_0^1 (1-s)^{q-1}\phi(s)\,ds\right\} + T_2\right)^{-1} > 1, \tag{43}$$

where

$$T_1 = p(1+q) + \frac{b}{a+b}(1+q+p+2pq) + \frac{b^2q}{(a+b)^2}(1+p),$$
$$T_2 = \left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}|I_k(u(t_k^-))| + \left(1+\frac{2b}{a+b}+\frac{b^2}{(a+b)^2}\right) \times \sum_{0<t_k<1}|J_k(u(t_k^-))| + \frac{br_1}{a+b} + \frac{a+2b}{(a+b)^2}r_2. \tag{44}$$

□

**Theorem 12.** *Assume that $(H_1)$, $(H_2)'$, and $(H_3)''$ hold; then the problem (4) has at least one solution.*

*Proof.* We consider the operator $F$ defined by (28). Let $y = \lambda Fy$, for $\lambda \in [0,1]$; then we have

$$\begin{aligned}|u(t)| = |\lambda Fu(t)| &\le \frac{\psi(\|u\|)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)\,ds + \left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}|I_k(u(t_k^-))| \\ &+ \frac{\psi(\|u\|)}{\Gamma(q)}\left[p(1+q) + \frac{b}{a+b}(1+q+p+2pq) + \frac{b^2q}{(a+b)^2}(1+p)\right] \\ &\times \int_0^1 (1-s)^{q-1}\phi(s)\,ds + \left(1+\frac{2b}{a+b}+\frac{b^2}{(a+b)^2}\right) \\ &\times \sum_{0<t_k<1}|J_k(u(t_k^-))| + \frac{r_1}{a+b} + \frac{a+2b}{(a+b)^2}r_2,\end{aligned}$$

$$\begin{aligned}|u'(t)| = |\lambda(Fu)'(t)| &\le \frac{\psi(\|u\|)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)\,ds \\ &+ \frac{\psi(\|u\|)}{\Gamma(q)}\left[pq + \frac{qb}{a+b}(1+p)\right]\int_0^1 (1-s)^{q-1}\phi(s)\,ds \\ &+ \left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}|J_k(u(t_k^-))| + \frac{r_2}{a+b}.\end{aligned} \tag{45}$$

Then we can obtain

$$\begin{aligned}\|u\| &\le \frac{\psi(\|u\|)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)\,ds + \left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}|I_k(u(t_k^-))| \\ &+ \frac{\psi(\|u\|)}{\Gamma(q)}\left[p(1+q) + \frac{b}{a+b}(1+q+p+2pq) + \frac{b^2q}{(a+b)^2}(1+p)\right]\int_0^1 (1-s)^{q-1}\phi(s)\,ds \\ &+ \left(1+\frac{2b}{a+b}+\frac{b^2}{(a+b)^2}\right)\sum_{0<t_k<1}|J_k(u(t_k^-))| + \frac{br_1}{a+b} + \frac{a+2b}{(a+b)^2}r_2,\end{aligned} \tag{46}$$

which implies

$$\|u\|\left(\frac{\psi(\|u\|)}{\Gamma(q)}\left\{\int_0^t (t-s)^{q-1}\phi(s)\,ds + T_1 \times \int_0^1 (1-s)^{q-1}\phi(s)\,ds\right\} + T_2\right)^{-1} \le 1, \tag{47}$$

where

$$T_1 = p(1+q) + \frac{b}{a+b}(1+q+p+2pq) + \frac{b^2q}{(a+b)^2}(1+p),$$
$$T_2 = \left(1+\frac{b}{a+b}\right)\sum_{0<t_k<1}|I_k(u(t_k^-))| + \left(1+\frac{2b}{a+b}+\frac{b^2}{(a+b)^2}\right) \times \sum_{0<t_k<1}|J_k(u(t_k^-))| + \frac{br_1}{a+b} + \frac{a+2b}{(a+b)^2}r_2. \tag{48}$$

By $(H_3)''$, there exists a $r$, such that $\|y\| \neq r$. Let $U = \{u \in PC^1[J,R] : \|u\| < r\}$. Then, as the proof of Steps 1–3,

we can easily get that $F : \overline{U} \to PC^1[J,R]$ is continuous and completely continuous. From the definition of $U$, we can know there exist no $y \in \partial U$, $\lambda \in [0,1]$ such that $y = \lambda(Fy)$. Otherwise, there exists at least one $y_0 \in \partial U$ such that $y_0 = \lambda(Ay_0)$. From the proof above, we know $\|y_0\| \neq r$. However, for $y_0 \in \partial U$, $\|y_0\| = r$, which is a contradiction. Therefore, there does not exist $y \in \partial U$, $\lambda \in [0,1]$ such that $y = \lambda(Fy)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $F$ has a fixed point $u \in U$, which implies that the problem (4) has at least one solution $u \in PC^1[J,R]$. □

## 4. Examples

In this section we give an example to illustrate the usefulness of our main result.

*Example 13.* Let us consider the following fractional impulsive problem:

$$
\begin{aligned}
{}^cD^{3/2}u(t) &= \frac{u(t)\sin t^2}{16\left(1+e^{t^2}\right)\left(e^t+|u(t)|\right)}, \\
&\qquad t \in J_1 = [0,1] \setminus \left\{\frac{1}{2}\right\}, \\
\Delta u\left(\frac{1}{2}\right) &= 1, \qquad \Delta u'\left(\frac{1}{2}\right) = 1, \\
au(0) + bu(1) &= \frac{u(t)}{18\left(1+|u(t)|\right)}, \\
au'(0) + bu'(1) &= \frac{\sin u(t)}{18e^t}.
\end{aligned}
\tag{49}
$$

First, we prove that Example 13 satisfies all the assumptions of Theorem 10.

In Example 13, it is easy to see that $f(t,u) = (u(t)\sin t^2/(1+e^{t^2})(e^t+|u(t)|)) \in C([0,1]\times R,R)$; so $(H_1)$ holds.

For $t \in [0,1]$, $u \in PC^1(J,R)$, we have

$$
\begin{aligned}
\left|f(t,u(t))\right| &= \left|\frac{u(t)\sin t^2}{16\left(1+e^{t^2}\right)\left(e^t+|u(t)|\right)}\right| \le \frac{1}{32}, \\
\left|f(t,u) - f(t,v)\right| &\le \frac{1}{32}\|u-v\|.
\end{aligned}
\tag{50}
$$

So $(H_3)$ is satisfied, where $h_1(t) = h_2(t) = 1/32$.

For $t \in [0,1]$, $u \in PC^1(J,R)$, $g_1(u) = (u(t)/18(1+|u(t)|))$, $g_2(u) = (1/18e^t)\sin u(t)$; then we know $|g_1(u) - g_1(v)| \le (1/18)\|u-v\|$, $|g_2(u) - g_2(v)| \le (\|u-v\|/18)$ with $l_1(t) = (1/18)$, $l_2(t) = (1/18)$, so $(H_2)$ is also satisfied.

For $t \in [0,1]$, $u \in PC^1(J,R)$, $I_k = J_k = 1$; then $|I_k(u) - I_k(v)| = 0$, $|J_k(u) - J_k(v)| = 0$, so $(H_4)$ holds with $M_k = m_k = 0$.

From Example 13, we also have $p = 1$, $q = 3/2$; then

$$
\begin{aligned}
n = \max\ &\left\{\left[\frac{1}{8\Gamma(5/2)} + \frac{3}{16\Gamma(3/2)} + \frac{1}{6}\right], \right. \\
&\left.\left(\frac{1}{8\Gamma(1/2)} + \frac{1}{18}\right)\right\} \le \frac{2}{3},
\end{aligned}
\tag{51}
$$

so all the conditions of Theorem 10 are satisfied; as a consequence of Theorem 10, Example 13 has a unique solution.

Second, we verify that all the assumptions of Theorem 11 are satisfied.

Obviously, $|g_1(u)| = |u(t)/18(1+|u(t)|)| \le (1/18)$, $|g_2(u)| = |(1/18e^t)\sin u(t)| \le (1/18)$; then $g_1(u)$, $g_2(u)$ are bounded. The other conditions of $(H_2)'$ in theorem can be verified as the condition of $(H_2)$ in Theorem 10.

For $t \in [0,1]$, $u \in PC^1(J,R)$, we have

$$
\left|f(t,u(t))\right| = \left|\frac{u(t)\sin u^2(t)}{16\left(1+e^{t^2}\right)\left(e^t+|u(t)|\right)}\right| \le \frac{1}{64}\left(1+|u|^0\right).
\tag{52}
$$

Thus, all the assumptions in Theorem 11 are satisfied; our results can be applied to Example 13; that is, Example 13 has at least one solution.

*Example 14.* Let us consider the following fractional impulsive problem:

$$
\begin{aligned}
{}^cD^{3/2}u(t) &= \frac{u(t)}{175\left(1+e^{t^2}\right)}, \quad t \in J_1 = [0,1] \setminus \left\{\frac{1}{2}\right\}, \\
\Delta u\left(\frac{1}{2}\right) &= 1, \qquad \Delta u'\left(\frac{1}{2}\right) = 1, \\
u(0) + u(1) &= \frac{u(t)}{18\left(1+|u(t)|\right)}, \\
u'(0) + u'(1) &= \frac{\sin u(t)}{18e^t}.
\end{aligned}
\tag{53}
$$

It is easy to check $(H_1)$ is satisfied. Similar to the proof in Example 13, we can also verify $(H_2)'$ holds for Example 14.

In Example 14, we have $f(t) = (u(t)/175(1+e^{t^2})) \le \psi(u)\phi(t)$ with $\psi(u) = \|u\|/175$, $\phi(t) = 1/2$. Obviously, $\psi(u)$ is $L^1$-integrable and nondecreasing function, $\phi \in L^2(J,R)$. And for $\forall r > 0$,

$$
\begin{aligned}
&r\left(\frac{\psi(r)}{\Gamma(q)}\left\{\int_0^t (t-s)^{q-1}\phi(s)\,ds + T_1 \right.\right. \\
&\qquad \left.\left. \times \int_0^1 (1-s)^{q-1}\phi(s)\,ds\right\} + T_2\right)^{-1} \\
&= 175\Gamma\left(\frac{3}{2}\right)\left(\frac{1}{2}\int_0^t (t-s)^{1/2}ds + \frac{13}{4} \right. \\
&\qquad \left. \times \int_0^1 (1-s)^{1/2}ds + \frac{275}{72}\right)^{-1} > 1.
\end{aligned}
\tag{54}
$$

Then all the conditions of $(H_3)''$ are satisfied. As a consequence of Theorem 12, then Example 14 has at least one solution.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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