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Research Article

An Application of Variant Fountain Theorems to a Class of Impulsive Differential Equations with Dirichlet Boundary Value Condition

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We consider the existence of infinitely many classical solutions to a class of impulsive differential equations with Dirichlet boundary value condition. Our main tools are based on variant fountain theorems and variational method. We study the case in which the nonlinearity is sublinear. Some recent results are extended and improved.

1. Introduction

Consider the following Dirichlet boundary value problem of impulsive differential equations:

$$\begin{gathered} -u''(t_0) + g(t)\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta\left(u'\left(t_j\right)\right) = I_j\left(u\left(t_j\right)\right), \quad j = 1, 2, \ldots, p, \\ u(0) = u(T) = 0, \end{gathered} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $\Delta(u'(t_j)) = u'(t_j^+) - u'(t_j^-) = \lim_{s \to t_j^+} u'(s) - \lim_{s \to t_j^-} u'(s)$, $g \in L^\infty[0, T]$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $1 \le j \le p$ are continuous.

Since impulsive differential equations can describe many evolution processes in which their states are changed abruptly at certain moments of time, they play an important role in applications, such as in control theory, optimization theory, biology, and some physics or mechanics problem; see [1–5]. For general theory of impulsive differential equations, we refer the readers to the monographs as [6, 7]. The existence and multiplicity of solutions to impulsive differential equations with boundary value condition have been obtained by using fixed point theorems and upper and lower solutions method; see [8–12] and references therein. Recently, some authors creatively applied variational method to deal with impulsive problems and obtained some new results; see [13–18]. For general theory of variational method, we refer the readers to the monographs as [19, 20]. More precisely, Nieto and O'Regan [13] studied Dirichlet problem as follows:

$$\begin{gathered} -\ddot{u}(t) + \lambda u(t) = f(t, u(t)), \quad t \neq t_j, \\ \Delta\dot{u}\left(t_j\right) = I_j\left(u\left(t_j\right)\right), \quad j = 1, 2, \ldots, p, \\ u(0) = u(T) = 0. \end{gathered} \tag{2}$$

For the sublinear case, they obtained the following result.

Lemma 1 (See [13]). *Assume that the following conditions are satisfied.*

(1) *There exist $a, b > 0$ and $\gamma \in [0, 1)$ such that*

$$\left|f(t, u)\right| \le a + b|u|^\gamma \quad \textit{for every } (t, u) \in [0, T] \times \mathbb{R}. \tag{3}$$

(2) *There exist $a_j, b_j > 0$ and $\gamma_j \in [0, 1)$ $(j = 1, 2, \ldots, p)$ such that*

$$\left|I_j(u)\right| \le a_j + b_j^{\gamma_j} \quad \textit{for every } u \in \mathbb{R}. \tag{4}$$

Then Problem (2) *has at least one solution.*

For superlinear case, Zhang and Yuan [15] obtained the existence of one and infinitely many solutions for Problem (2) under the well-known Ambrosetti-Rabinowitz condition; that is, there exists $\theta > 2$ such that

$$0 < \theta F(t,u) \leq f(t,u)u, \quad \forall u \in \mathbb{R} \setminus \{0\},\ t \in [0,T], \quad (5)$$

where F is a primitive function of f. Soon after, Zhou and Li [16] obtained the existence of infinitely many solutions for Problem (1) under the weaker condition; there exist $\theta > 2$ and $R > 0$ such that

$$0 < \theta F(t,u) \leq f(t,u)u, \quad |u| \geq R, \quad \forall t \in [0,T]. \quad (6)$$

Recently, Sun and Chen studied the existence of infinitely many solutions for Problem (1) with superlinear nonlinearity f which is not satisfied (5) or (6). In addition, they also studied the case where the nonlinearity is asymptotically linear.

Motivated by the above facts, in this paper, our aim is to study the existence of infinitely many solutions for Problem (1) with nonlinearity f which is sublinear. To the best of our knowledge, there are few papers concerned with this. For sublinear case, Nieto and O'Regan [13] only obtain the existence of at least one solution.

We make the following assumptions:

(H_1) I_j $(1 \leq j \leq p)$ are odd and satisfy

$$\frac{1}{2}I_j(u)u - \int_0^u I_j(s)\,ds \geq 0, \qquad \int_0^u I_j(s)\,ds \geq 0 \quad (7)$$

for all $u \in \mathbb{R}$.

(H_2) For any $j \in \{1,2,\ldots,p\}$, there exist constants $b_j > 0$ and $\gamma_j \geq 1$ such that

$$\left|I_j(u)\right| \leq b_j|u|^{\gamma_j} \quad (8)$$

for all $u \in \mathbb{R}$.

(H_3) There exist constants $R_0 > 0$, $d > 0$ and $\gamma \geq 1$ such that

$$f(t,u)u \geq 0, \qquad F(t,u) - \frac{1}{2}f(t,u)u \geq d|u|^{\gamma}, \quad (9)$$

for every $t \in [0,T]$ and $u \in \mathbb{R}$ with $|u| \geq R_0$, where $F(t,u) = \int_0^u f(t,s)ds$. Moreover, $F(t,u) \geq 0$ for all $t \in [0,T]$ and $\in \mathbb{R}$.

(H_4) There exist constants $\mu \in (1,2)$ and $C_1 > 0$ such that $|f(t,u)| \leq C_1(1 + |u|^{\mu-1})$. In what follows, C_i, $i = 1,2,\ldots$ denote positive constants.

(H_5) there exist constants $R_1 > 0$, $C_2 > 0$, $\delta \in [1,2)$ such that

$$F(t,u) \geq C_2|u|^{\delta} \quad (10)$$

for every $t \in [0,T]$ and $|u| \leq R_1$.

(H_6) $F(t,u)$ is even in u, that is, $F(t,-u) = F(t,u)$.

Theorem 2. *Assume that (H_1)–(H_6) are satisfied. Then Problem (1) has infinitely many classical solutions.*

Remark 3. For the definition of classical solution, we refer readers to paper [17, 18]. By (H_1) and (H_2), I_j $(j = 1,2,\ldots,p)$ are not sublinear as those in [13–18]. We note that there are functions I_j $(j = 1,2,\ldots,p)$ and f which satisfy the conditions of Theorem 2 but do not satisfy the conditions in references we mentioned above. For example, let

$$I_j(s) = s^3, \quad j = 1,2,\ldots,p,$$
$$F(t,u) = \begin{cases} |u|^{3/2}, & |u| < 1, \\ \dfrac{(3|u| - 1)}{2}, & |u| \geq 1, \end{cases}$$
$$f(t,u) = \begin{cases} \dfrac{3}{2}, & u \geq 1, \\ \dfrac{3}{2}|u|^{1/2}, & |u| < 1, \\ -\dfrac{3}{2}, & u \leq -1. \end{cases} \quad (11)$$

If we choose $\gamma_j = 3$, $R_0 = 1$, $d = 1/4$, $\delta = 3/2$, $\gamma = 1$, and $\mu = 3/2$, then it is easy to check that the conditions in Theorem 2 are satisfied.

The organization of this paper is as follows. In Section 2, we shall give some lemmas and some preliminary results. In Section 3, the proofs of the main results are given.

2. Preliminaries

In order to prove our main results, we recall the variant fountain theorem. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j=1}^{k} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Consider the following C^1-functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2]. \quad (12)$$

Lemma 4 (see [21]). *Suppose that the functional $I_\lambda(u)$ defined above satisfies the following.*

(C1) *I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$.*

(C2) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

(C3) *There exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) \quad (13)$$

for all $\lambda \in [1,2]$ and $d_k(\lambda) := \inf_{u \in Z_k \|u\| \leq \rho_k} I_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$. Then there exist $\lambda_n \to$

$1, u_{\lambda_n} \in Y_n$ *such that* $I'_{\lambda_n} \mid Y_n(u(\lambda_n)) = 0$, $I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ *as* $n \to \infty$. *In particular, if* $\{u(\lambda_n)\}$ *has a convergent subsequence for every* k, *then* I_1 *has infinitely many nontrivial critical points* $\{u_n\} \subset E \setminus \{0\}$ *satisfying* $I_1(u_k) \to 0^-$ *as* $k \to \infty$.

In the Sobolev space $E := H_0^1(0,T)$, consider the inner product

$$(u,v)_0 = \int_0^T u'(t)\, v'(t)\, dt, \tag{14}$$

inducing the norm

$$\|u\|_0 = \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2} \tag{15}$$

for any $u, v \in E$. Since E is compactly embedded in $L^s([0,T])$ with norm $|u|_s = (\int_0^T |u(t)|^s dt)^{1/2}$ for $s \in [2,+\infty]$, as in [18, 22], we know that the eigenvalues of operator $S = -(d^2/dt^2) + g$ with the Dirichlet boundary conditions are numbered by $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \to \infty$ (counted in their multiplicities) and a corresponding system of eigenfunctions $\{e_j\}$, which forms the completely orthogonal basis in $L^2([0,T])$. Assume $\lambda_1, \ldots, \lambda_{n^-} < 0$, $\lambda_{n^-+1} = \cdots = \lambda_{n^0} = 0$ and let $E^- = \operatorname{span}\{e_1, e_2, \ldots, e_{n^-}\}$, $E^0 = \operatorname{span}\{e_{n^-+1}, \ldots, e_{n^0}\}$, and $E^+ = \operatorname{span}\{e_{n^0+1}, \ldots, \}$. Then $E = E^- \oplus E^0 \oplus E^+$. We introduce on E the following product $(u,v) = (|S^{1/2}|u, |S^{1/2}|v)_{L^2} + (u^0, v^0)_{L^2}$ and norm $\|u\| = (u,u)^{1/2}$, where $u = u^- + u^0 + u^+$, $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$. Then $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. By the Soblev imbedding theorem, E is compactly embedded in $C[0,T]$, and there exists $C_3 > 0$ such that

$$\|u\|_\infty \le C_3 \|u\|, \tag{16}$$

where

$$\|u\|_\infty = \max_{t \in [0,T]} |u(t)|. \tag{17}$$

Define a functional φ on E by

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T \left(\left|u'\right|^2 + g(t)\, u^2\right) dt + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)\, ds \\ &\quad - \int_0^T F(t,u)\, dt \\ &= \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)\, ds \\ &\quad - \int_0^T F(t,u)\, dt,\end{aligned} \tag{18}$$

where $u = u^- + u^0 + u^+ \in E$ with $u^- \in E^-$, $u^0 \in E^0$, $u^+ \in E^+$. By the conditions of Theorem 2, we know that φ is continuously differentiable and

$$\begin{aligned}\left\langle \varphi'(u), v\right\rangle &= \int_0^T \left(u'v' + g(t)\, uv\right) dt + \sum_{j=1}^p I_j\left(u\left(t_j\right)\right) v\left(t_j\right) \\ &\quad - \int_0^T f(t,u)\, v dt \\ &= (u^+ - u^-, v) + \sum_{j=1}^p I_j\left(u\left(t_j\right)\right) v\left(t_j\right) - \int_0^T f(t,u)\, v dt,\end{aligned} \tag{19}$$

for any $u, v \in E$.

Like for Lemma 2.4 in [17], one can prove that the critical points of the functional φ are the classical solutions for Problem (1).

3. Proofs of the Main Results

Now we define a class of functionals on E by

$$\begin{aligned}\varphi_\lambda(u) &:= \frac{1}{2}\|u^+\|^2 + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)\, ds \\ &\quad - \lambda\left(\frac{1}{2}\|u^-\|^2 + \int_0^T F(t,u)\, dt\right) \\ &:= A(u) - \lambda B(u), \quad \lambda \in [1,2],\end{aligned} \tag{20}$$

where $u^- \in E^-$, $u^+ \in E^+$. Denote by $X_j := \operatorname{span}\{e_j\}$, $j \in \mathbb{N}$. Clearly, we see that $\varphi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1,2]$ and the critical points of φ_1 correspond to the solutions to Problem (1).

Lemma 5. *Let* (H_3) *be satisfied. Then* $B(u) \ge 0$. *Furthermore,* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* E.

Proof. Evidently, $B(u) \ge 0$ follows by the definition of the functional B and (H_3). Now we claim that for any finite dimensional subspace of $F \subset E$, there exists $\varepsilon > 0$ such that

$$\operatorname{meas}(\{t \in [0,T] : |u(t)| \ge \varepsilon \|u\|\}) \ge \varepsilon, \forall u \in F \setminus \{0\}, \tag{21}$$

where meas denotes the Lebesgue measure in $\mathbb{R}$.

Otherwise, for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$\operatorname{meas}\left(\left\{t \in [0,T] : \left|u_n(t)\right| \ge \frac{1}{n}\|u\|\right\}\right) < \frac{1}{n}. \tag{22}$$

Let $v_n = u_n/\|u_n\| \in F \setminus \{0\}$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ and

$$\operatorname{meas}\left(\left\{t \in [0,T] : \left|u_n(t)\right| \ge \frac{1}{n}\right\}\right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{23}$$

Since $\dim F < \infty$, it follows from the compactness of the unit sphere of F that there exists a subsequence, say $\{v_n\}$, such that $v_n \to v_0$ in F. Hence, we have $\|v_0\| = 1$. By the equivalence of the norms on the finite dimensional space F, we have $v_n \to v_0$ in $L^2[0,T]$, that is,

$$\int_0^T |v_n - v_0|^2 dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{24}$$

Thus there exist $\xi_1, \xi_2 > 0$ such that

$$\text{meas}\{t \in [0,T] : |v_0(t)| \geq \xi_1\} \geq \xi_2. \tag{25}$$

In fact, if not, we have

$$\text{meas}\left\{t \in [0,T] : |v_0(t)| \geq \frac{1}{n}\right\} = 0, \quad \forall n \in \mathbb{N}. \tag{26}$$

This implies that

$$0 < \int_0^T |v_0|^2 dt < \frac{1}{n^2} T \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{27}$$

which gives a contradiction.

Now let

$$\begin{aligned} \Omega_0 &= \{t \in [0,T] : |v_0(t)| \geq \xi_1\}, \\ \Omega_n &= \left\{t \in [0,T] : |v_n(t)| < \frac{1}{n}\right\}, \end{aligned} \tag{28}$$

and $\Omega_n^{\perp} = [0,T] \setminus \Omega_n$. We have

$$\text{meas}(\Omega_n \cap \Omega_0) \geq \text{meas}(\Omega_0) - \text{meas}(\Omega_n^{\perp} \cap \Omega_0) \geq \xi_2 - \frac{1}{n} \tag{29}$$

for all positive integer n. Let n be large enough such that $\xi_2 - (1/n) \geq (1/2)\xi_2$ and $\xi_1 - (1/n) \geq (1/2)\xi_1$. Then we have

$$|v_n(t) - v_0(t)|^2 \geq \left(\xi_1 - \frac{1}{n}\right)^2 \geq \frac{1}{4}\xi_1^2, \quad \forall t \in \Omega_n \cap \Omega_0. \tag{30}$$

This implies that

$$\begin{aligned} &\int_0^T |v_n(t) - v_0(t)|^2 dt \\ &\quad \geq \int_{\Omega_n \cap \Omega_0} |v_n(t) - v_0(t)|^2 dt \\ &\quad \geq \frac{1}{4}\xi_1^2 \,\text{meas}(\Omega_n \cap \Omega_0) \\ &\quad \geq \frac{1}{4}\xi_1^2 \left(\xi_2 - \frac{1}{n}\right) \geq \frac{1}{8}\xi_1^2 \xi_2 > 0 \end{aligned} \tag{31}$$

for all large n, which is a contradiction with (24). For the ε given in (21), let

$$\Lambda_u = \{t \in [0,T] : |u(t)| \geq \varepsilon \| u \|\}, \quad \forall u \in F \setminus \{0\}. \tag{32}$$

Then

$$\text{meas}(\Lambda_u) \geq \varepsilon, \quad \forall u \in F \setminus \{0\}. \tag{33}$$

Observing that for any $u \in F$ with $\|u\| \geq R_0/\varepsilon$, the following inequality holds

$$|u(t)| \geq R_0, \quad \forall t \in \Lambda_u. \tag{34}$$

Combining (34) and (H_3), for any $u \in F$ with $\|u\| \geq R_0/\varepsilon$, we have

$$\begin{aligned} B(u) &= \frac{1}{2}\|u^-\|^2 + \int_0^T F(t,u)\,dt \\ &\geq \int_{\Lambda_u} F(t,u)\,dt \\ &\geq \int_{\Lambda_u} d|u|^{\gamma} dt \\ &\geq d(\varepsilon\|u\|)^{\gamma}\,\text{meas}(\Lambda_u) \geq d\varepsilon^{\gamma+1}\|u\|^{\gamma}. \end{aligned} \tag{35}$$

This implies $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $F \subset E$. □

Lemma 6. *Assume that (H_1), (H_4) are satisfied. Then there exist a positive integer k_0 and a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0, \quad \forall k \geq k_0, \tag{36}$$

and $d_k(\lambda) := \inf_{u \in Z_k \|u\| \leq \rho_k} I_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$, where $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j} = \overline{\text{span}\{e_k,\ldots\}}$ for all $k \in \mathbb{N}$.

Proof. Note first that $Z_k \subset E^+$ for all $k \geq n^0 + 1$ by definition of E^+ in Section 2. Thus for any $k \geq n^0 + 1$, by (H_1), (H_4), we have

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\,ds - \lambda \int_0^T F(t,u)\,dt \\ &\geq \frac{1}{2}\|u\|^2 - 2C_1 \int_0^T (|u| + |u|^{\mu})\,dt \\ &\geq \frac{1}{2}\|u\|^2 - 2TC_1(\|u\|_\infty + \|u\|_\infty^{\mu}) \end{aligned} \tag{37}$$

for all $(\lambda, u) \in [1,2] \times Z_k$. Set $\beta_k := \sup_{u \in Z_k, \|u\|=1} \| u\|_\infty$. Then

$$\beta_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{38}$$

Indeed, it is clear that $0 < \beta_{k+1} \leq \beta_k$, so $\beta_k \to \overline{\beta} \geq 0$ as $k \to \infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_\infty > \beta_k/2$. By the definition of Z_k, $u_k \rightharpoonup 0$ in E. Then this implies that $u_k \to 0$ in $C[0,T]$. Thus we have proved that $\overline{\beta} = 0$. Therefore, for any $k \geq n^0 + 1$, the following inequality holds:

$$\varphi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2TC_1(\beta_k\|u\| + \beta_k^{\mu}\|u\|^{\mu}), \tag{39}$$

for all $(\lambda, u) \in [1,2] \times Z_k$. Let $\rho_k = 16TC_1\beta_k + [16TC_1\beta_k^{\mu}]^{1/(2-\mu)}$. Then by (38), we have $\rho_k \to 0^+$ as $k \to \infty$. Hence, by (39), straightforward computation shows that

$a_k(\lambda) \geq \rho_k^2/4 > 0$. Furthermore, for any $k \geq k_0$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\varphi_\lambda(u) \geq -2TC_1\left(\beta_k\|u\| + \beta_k^\mu\|u\|^\mu\right). \tag{40}$$

Clearly, we see that $F(t,0) = 0$ by the definition of F and $\varphi_\lambda(0) = 0$. Therefore,

$$0 \geq \inf_{u\in Z_k,\|u\|\leq\rho_k}\varphi_\lambda(u) \geq -2TC_1\left(\beta_k\|u\| + \beta_k^\mu\|u\|^\mu\right). \tag{41}$$

Combining (38) and (41), we have $d_k(\lambda) := \inf_{u\in Z_k\|u\|\leq\rho_k} I_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$. □

Lemma 7. *Let (H_2), (H_5) be satisfied. Then for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ obtained in Lemma 6, there exist $0 < r_k < \rho_k$ for all $k \in \mathbb{N}$ such that*

$$b_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} I_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{42}$$

where $Y_k = \bigoplus_{j=1}^k X_j = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$ for all $k \in \mathbb{N}$.

Proof. Let $u \in Y_k$ with $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. By (16), for any $u \in Y_k$ with $\|u\| \leq R_1/C_3$, one has $\|u\|_\infty \leq R_1$. By (H_5), we obtain

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{2}\|u^+\|^2 + \sum_{j=1}^p\int_0^{u(t_j)} I_j(s)\,ds \\ &\quad - \lambda\left(\frac{1}{2}\|u^-\|^2 + \int_0^T F(t,u)\,dt\right) \\ &\leq \frac{1}{2}\|u^+\|^2 + \sum_{j=1}^p b_j\left|u(t_j)\right|^{\gamma_j+1} - C_2\int_0^T |u|^\delta dt \\ &\leq \frac{1}{2}\|u^+\|^2 + \sum_{j=1}^p b_j C_3^{\gamma_j+1}\|u\|^{\gamma_j+1} - C_2C_4\|u\|^\delta\end{aligned} \tag{43}$$

for any $u \in Y_k$ with $\|u\| \leq R_1/C_3$, where the last inequality follows by the equivalence of the norms on the finite dimensional space Y_k. Since $\delta < 2$, $\gamma_j \geq 1$ ($j \in \{1,2,\ldots,p\}$), for $\|u\| = r_k$, are small enough, we can get $b_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} I_\lambda(u) < 0$, $\forall k \in \mathbb{N}$. □

Proof of Theorem 2. By (H_2), (H_4), φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Evidently, (H_1), (H_6) imply that $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$. Thus by Lemma 4, there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ such that $\varphi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, $\varphi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. For the sake of notational simplicity, in what follows we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$. By (H_1), (H_3), one has

$$\begin{aligned}-\varphi_{\lambda_n}(u_n) &= \frac{1}{2}\varphi'_{\lambda_n}|_{Y_n}(u_n)u_n - \varphi_{\lambda_n}(u_n) \\ &= \sum_{j=1}^p\left[\frac{1}{2}I_j\left(u_n(t_j)\right)u_n(t_j) - \int_0^{u_n(t_j)} I_j(s)\,ds\right] \\ &\quad + \lambda_n\int_0^T\left[F(t,u_n) - \frac{1}{2}f(t,u_n)u_n\right]dt \\ &\geq \lambda_n\int_{\Gamma_n}\left[F(t,u_n) - \frac{1}{2}f(t,u_n)u_n\right]dt - C_5 \\ &\geq d\lambda_n\int_{\Gamma_n}|u_n|^\gamma dt - C_5, \quad n \in \mathbb{N},\end{aligned} \tag{44}$$

where $\Gamma_n = \{t \in [0,T] : |u_n(t)| \geq R_0\}$ and $C_5 > 0$ is a constant. Hence, there exists a constant $C_6 > 0$ such that $\int_{\Gamma_n}|u_n|^\gamma dt \leq C_6$, $\forall n \in \mathbb{N}$. On the other hand, we can easily obtain that $\int_{[0,T]\setminus\Gamma_n}|u_n|^\gamma dt \leq TR_0^\gamma$, $\forall n \in \mathbb{N}$. Thus, we have $\int_0^T|u_n|^\gamma dt \leq C_7$. In view of the equivalence of any two norms on finite dimensional space $E^- \oplus E^0$ and (16), we obtain

$$\begin{aligned}\left|u_n^- + u_n^0\right|_2^2 &= \left(u_n^- + u_n^0, u_n\right)_2 \\ &= \left|u_n\right|_\gamma\left|u_n^- + u_n^0\right|_{\gamma'} \\ &\leq C_8\left|u_n^- + u_n^0\right|_2,\end{aligned} \tag{45}$$

where $\gamma' = (\gamma/(\gamma-1))$ ($\gamma' = \infty$ when $\gamma = 1$). Therefore, we have

$$\left|u_n^- + u_n^0\right|_2 \leq C_8. \tag{46}$$

In view of the equivalence of norms on $E^- \oplus E^0$, we obtain $\|u_n^- + u_n^0\| \leq C_9$, $\forall n \in \mathbb{N}$. Note that

$$\begin{aligned}\|u_n^+\|^2 &= 2\varphi_{\lambda_n}(u_n) + \lambda_n\|u_n^-\|^2 + 2\lambda_n\int_0^T F(t,u_n)\,dt \\ &\quad - \sum_{j=1}^p\int_0^{u_n(t_j)} I_j(s)\,ds.\end{aligned} \tag{47}$$

Thus

$$\begin{aligned}\|u_n\|^2 &= \|u_n^+\|^2 + \left\|u_n^- + u_n^0\right\|^2 \\ &\leq 2\varphi_{\lambda_n}(u_n) + \lambda_n\|u_n^-\|^2 + \left\|u_n^- + u_n^0\right\|^2 \\ &\quad + 2\lambda_n\int_0^T F(t,u_n)\,dt - \sum_{j=1}^p\int_0^{u_n(t_j)} I_j(s)\,ds \\ &\leq C_{10} + C_{11}\int_0^T\left[|u_n| + |u_n|^\mu\right] \\ &\leq C_{12} + C_{13}T\left[C_3\|u_n\| + C_3^\mu\|u_n\|^\mu\right].\end{aligned} \tag{48}$$

Since $\mu < 2$, we have $\|u_n\| \leq C_{14}$; that is, $\{u_n\}$ is bounded in E. By a standard argument, this yields a critical point u^k of φ such that $\varphi(u^k) \in [d_k(2), c_k(1)]$. Since $d_k(2) \to 0^-$ as $k \to \infty$, we can obtain infinitely many critical points. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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