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Research Article

Existence of Solutions for Fractional q-Integrodifference Equations with Nonlocal Fractional q-Integral Conditions

Suphawat Asawasamrit,[1] Jessada Tariboon,[1] and Sotiris K. Ntouyas[2]

[1] *Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*
[2] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Correspondence should be addressed to Jessada Tariboon; jessadat@kmutnb.ac.th

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We study a class of fractional q-integrodifference equations with nonlocal fractional q-integral boundary conditions which have different quantum numbers. By applying the Banach contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative, the existence and uniqueness of solutions are obtained. In addition, some examples to illustrate our results are given.

1. Introduction

In this paper, we deal with the following nonlocal fractional q-integral boundary value problem of nonlinear fractional q-integrodifference equation:

$$\begin{gathered} D_q^{\alpha} x(t) = f\left(t, x(t), I_z^{\delta} x(t)\right), \quad t \in (0, T), \\ x(0) = 0, \qquad \lambda I_p^{\beta} x(\eta) = I_r^{\gamma} x(\xi), \end{gathered} \tag{1}$$

where $0 < p, q, r, z < 1$, $1 < \alpha \le 2$, $\beta, \gamma, \delta > 0$, $\lambda \in \mathbb{R}$ are given constants, D_q^{α} is the fractional q-derivative of Riemann-Liouville type of order α, I_{ϕ}^{ψ} is the fractional ϕ-integral of order ψ with $\phi = p, r, z$, and $\psi = \beta, \gamma, \delta$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The early work on q-difference calculus or *quantum calculus* dates back to Jackson's paper [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in ([5–17]) and the references cited therein, whereas the background material on q-fractional calculus can be found in a recent book [18].

In this paper, we will study the existence and uniqueness of solutions of a class of boundary value problems for fractional q-integrodifference equations with nonlocal fractional q-integral conditions which have different quantum numbers. So, the novelty of this paper lies in the fact that there are *four different quantum numbers*. In addition, the boundary condition of (1) does not contain the value of unknown function x at the right side of boundary point $t = T$. *One may interpret the q-integral boundary condition in* (1) *as the q-integrals with different quantum numbers are related through a real number λ.*

The paper is organized as follows. In Section 2, for the convenience of the reader, we cite some definitions and fundamental results on q-calculus as well as the fractional q-calculus. Some auxiliary lemmas, needed in the proofs of our main results, are presented in Section 3. Section 4 contains the existence and uniqueness results for problem (1) which are shown by applying Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder's nonlinear alternative. Finally, some examples illustrating the applicability of our results are presented in Section 5.

2. Preliminaries

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [6, 18, 19].

For $q \in (0,1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2}$$

The q-analogue of the power function $(1-b)^k$ with $k \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is

$$(1-b)^{(0)} = 1, \qquad (1-b)^{(k)} = \prod_{i=0}^{k-1}\left(1-bq^i\right), \quad k \in \mathbb{N}, \quad b \in \mathbb{R}. \tag{3}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(1-b)^{(\gamma)} = \prod_{i=0}^{\infty}\frac{1-bq^i}{1-bq^{\gamma+i}}. \tag{4}$$

We use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}. \tag{5}$$

Obviously, $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function h is defined by

$$\left(D_q h\right)(x) = \frac{h(x)-h(qx)}{(1-q)x} \quad \text{for } x \neq 0, \qquad \left(D_q h\right)(0) = \lim_{x\to 0}\left(D_q h\right)(x), \tag{6}$$

and q-derivatives of higher order are given by

$$\left(D_q^0 h\right)(x) = h(x), \qquad \left(D_q^k h\right)(x) = D_q\left(D_q^{k-1}h\right)(x), \quad k \in \mathbb{N}. \tag{7}$$

The q-integral of a function h defined on the interval $[0,b]$ is given by

$$\left(I_q h\right)(x) = \int_0^x h(s)\,d_q s = x(1-q)\sum_{i=0}^{\infty}h\left(xq^i\right)q^i, \quad x \in [0,b]. \tag{8}$$

If $a \in [0,b]$ and h is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b h(s)\,d_q s = \int_0^b h(s)\,d_q s - \int_0^a h(s)\,d_q s. \tag{9}$$

Similar to derivatives, an operator I_q^k is given by

$$\left(I_q^0 h\right)(x) = h(x), \qquad \left(I_q^k h\right)(x) = I_q\left(I_q^{k-1}h\right)(x), \quad k \in \mathbb{N}. \tag{10}$$

The fundamental theorem of calculus applies to operators D_q and I_q; that is,

$$\left(D_q I_q h\right)(x) = h(x), \tag{11}$$

and if h is continuous at $x = 0$. Then

$$\left(I_q D_q h\right)(x) = h(x) - h(0). \tag{12}$$

Definition 1. Let $\nu \geq 0$ and h be a function defined on $[0,T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and

$$\left(I_q^{\nu} h\right)(x) = \frac{1}{\Gamma_q(\nu)}\int_0^x (x-qs)^{(\nu-1)}h(s)\,d_q s, \quad \nu > 0, \quad x \in [0,T]. \tag{13}$$

Definition 2. The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and

$$\left(D_q^{\nu} h\right)(x) = \left(D_q^l I_q^{l-\nu} h\right)(x), \quad \nu > 0, \tag{14}$$

where l is the smallest integer greater than or equal to ν.

Definition 3. For any $m, n > 0$,

$$B_q(m,n) = \int_0^1 u^{(m-1)}(1-qu)^{(n-1)}d_q u \tag{15}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(m,n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}. \tag{16}$$

Lemma 4 (see [4]). *Let $\alpha, \beta \geq 0$, and f be a function defined in $[0,T]$. Then, the following formulas hold:*

(1) $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$;

(2) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 5 (see [6]). *Let $\alpha > 0$ and ν be a positive integer. Then, the following equality holds:*

$$\left(I_q^{\alpha} D_q^{\nu} f\right)(x) = \left(D_q^{\nu} I_q^{\alpha} f\right)(x) - \sum_{k=0}^{\nu-1}\frac{x^{\alpha-\nu+k}}{\Gamma_q(\alpha+k-\nu+1)}\left(D_q^k f\right)(0). \tag{17}$$

3. Some Auxiliary Lemmas

Lemma 6. *Let $\alpha, \beta > 0$, and $0 < q < 1$. Then one has*

$$\int_0^{\eta}(\eta-qs)^{(\alpha-1)}s^{\beta}d_q s = \eta^{\alpha+\beta}B_q(\alpha,\beta+1). \tag{18}$$

Proof. Using the definitions of q-analogue of power function and q-beta function, we have

$$\begin{aligned}
&\int_0^\eta (\eta - qs)^{(\alpha-1)} s^\beta d_q s \\
&= (1-q)\,\eta \sum_{n=0}^{\infty} q^n (\eta - q\eta q^n)^{(\alpha-1)} (\eta q^n)^\beta \\
&= (1-q)\,\eta \sum_{n=0}^{\infty} q^n \eta^{\alpha-1} (1 - q q^n)^{(\alpha-1)} \eta^\beta q^{n\beta} \\
&= (1-q)\,\eta^{\alpha+\beta} \sum_{n=0}^{\infty} q^n (1 - q q^n)^{(\alpha-1)} q^{n\beta} \\
&= \eta^{\alpha+\beta} \int_0^1 (1-qs)^{(\alpha-1)} s^{(\beta)} d_q s \\
&= \eta^{\alpha+\beta} B_q(\alpha, \beta+1).
\end{aligned} \tag{19}$$

The proof is complete. □

Lemma 7. *Let* $\alpha, \beta, \gamma > 0$, *and* $0 < p, q, r < 1$. *Then one has*

$$\begin{aligned}
&\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} (y - rz)^{(\gamma-1)} d_r z \, d_q y \, d_p x \\
&\quad = \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1) \eta^{\alpha+\beta+\gamma}.
\end{aligned} \tag{20}$$

Proof. Taking into account Lemma 6, we have

$$\begin{aligned}
&\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} (y - rz)^{(\gamma-1)} d_r z \, d_q y \, d_p x \\
&\quad = \frac{1}{[\gamma]_r} \int_0^\eta \int_0^x (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} y^{(\gamma)} d_q y \, d_p x \\
&\quad = \frac{1}{[\gamma]_r} \int_0^\eta (\eta - px)^{(\alpha-1)} \int_0^x (x - qy)^{(\beta-1)} y^{(\gamma)} d_q y \, d_p x \\
&\quad = \frac{1}{[\gamma]_r} B_q(\beta, \gamma+1) \int_0^\eta (\eta - px)^{(\alpha-1)} x^{(\beta+\gamma)} d_p x \\
&\quad = \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1) \eta^{\alpha+\beta+\gamma}.
\end{aligned} \tag{21}$$

This completes the proof. □

Lemma 8. *Let* $\beta, \gamma > 0$, $\lambda \in \mathbb{R}$, *and* $0 < p, q, r < 1$. *Then, for* $y \in C([0,T], \mathbb{R})$, *the unique solution of boundary value problem,*

$$D_q^\alpha x(t) = y(t), \quad t \in (0,T), \ 1 < \alpha \le 2, \tag{22}$$

subject to the nonlocal fractional condition,

$$x(0) = 0, \qquad \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \tag{23}$$

is given by

$$\begin{aligned}
x(t) = {} & \frac{\lambda t^{\alpha-1}}{\Omega \Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s \\
& - \frac{t^{\alpha-1}}{\Omega \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
& \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_r s \\
& + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) \, d_q s,
\end{aligned} \tag{24}$$

where

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)} \xi^{\alpha+\gamma-1} - \lambda \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} \eta^{\alpha+\beta-1} \neq 0. \tag{25}$$

Proof. From $1 < \alpha \le 2$, we let $n = 2$. Using the Definition 2 and Lemma 4, (22) can be expressed as

$$\left(I_q^\alpha D_q^2 I_q^{2-\alpha} x\right)(t) = \left(I_q^\alpha y\right)(t). \tag{26}$$

From Lemma 5, we have

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) \, d_q s \tag{27}$$

for some constants $k_1, k_2 \in \mathbb{R}$. It follows from the first condition of (23) that $k_2 = 0$. Applying the Riemann-Liouville fractional p-integral of order $\beta > 0$ for (27) with $k_2 = 0$ and taking into account of Lemma 6, we have

$$\begin{aligned}
I_p^\beta x(t) = {} & \int_0^t \frac{(t - ps)^{(\beta-1)}}{\Gamma_p(\beta)} \\
& \times \left(k_1 s^{\alpha-1} + \int_0^s \frac{(s - qu)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(u) \, d_q u \right) d_p s \\
= {} & \frac{k_1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} s^{\alpha-1} d_p s + \frac{1}{\Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^t \int_0^s (t - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s \\
= {} & k_1 \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{1}{\Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^t \int_0^s (t - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s.
\end{aligned} \tag{28}$$

In particular, we have

$$I_p^{\beta}x(\eta)=k_1\frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)}\eta^{\alpha+\beta-1}+\frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)}\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)\,d_qu\,d_ps. \quad (29)$$

Using the Riemann-Liouville fractional r-integral of order $\gamma>0$ and repeating the above process, we get

$$I_r^{\gamma}x(\xi)=k_1\frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)}\xi^{\alpha+\gamma-1}+\frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)}\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)\,d_qu\,d_rs. \quad (30)$$

The second nonlocal condition of (23) implies

$$k_1=\frac{\lambda}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)}\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)\,d_qu\,d_ps-\frac{1}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)}\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)\,d_qu\,d_rs. \quad (31)$$

Substituting the values of k_1 and k_2 in (27), we get the desired result in (24). □

4. Main Results

In this section, we denote $\mathscr{C}=C([0,T],\mathbb{R})$ as the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup_{t\in[0,T]}|x(t)|$. In view of Lemma 8, we define an operator $\mathcal{Q}:\mathscr{C}\to\mathscr{C}$ by

$$\begin{aligned}(\mathcal{Q}x)(t)&=\frac{\lambda t^{\alpha-1}}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)}\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\&\quad\times f\left(u,x(u),I_z^{\delta}x(u)\right)d_qu\,d_ps\\&\quad-\frac{t^{\alpha-1}}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)}\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\&\quad\times f\left(u,x(u),I_z^{\delta}x(u)\right)d_qu\,d_rs\\&\quad+\int_0^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f\left(s,x(s),I_z^{\delta}x(s)\right)d_qs,\end{aligned} \quad (32)$$

where $\Omega\neq0$ is defined by (25). It should be noticed that problem (1) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

For the sake of convenience of proving the results, we set

$$\begin{aligned}\Lambda&=\frac{|\lambda|T^{\alpha-1}}{|\Omega|\Gamma_p(\beta)}\times\left[\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right]\\&\quad+\frac{T^{\alpha-1}}{|\Omega|\Gamma_r(\gamma)}\times\left[\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right]\\&\quad+\frac{T^{\alpha}}{\Gamma_q(\alpha+1)}L_1+\frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2,\end{aligned} \quad (33)$$

$$\Psi=\frac{|\lambda|T^{\alpha-1}\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha+1)}+\frac{T^{\alpha-1}\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha+1)}+\frac{T^{\alpha}}{\Gamma_q(\alpha+1)}. \quad (34)$$

The first result on the existence and uniqueness of solutions is based on the Banach contraction mapping principle.

Theorem 9. *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

(H_1) there exist constants $L_1,L_2>0$ such that

$$\left|f(t,w_1,w_2)-f(t,\overline{w}_1,\overline{w}_2)\right|\leq L_1\left|w_1-\overline{w}_1\right|+L_2\left|w_2-\overline{w}_2\right|, \quad (35)$$

for each $t\in[0,T]$ and $w_1,w_2,\overline{w}_1,\overline{w}_2\in\mathbb{R}$.

If

$$\Lambda\leq\theta<1, \quad (36)$$

where Λ is given by (33)*, then the boundary value problem* (1) *has a unique solution on $[0,T]$.*

Proof. We transform problem (1) into a fixed point problem, $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (32). By applying the Banach contraction mapping principle, we will show that $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1).

Setting $\sup_{t\in[0,T]}|f(t,0,0)| = M < \infty$ and choosing

$$r \geq \frac{\Psi M}{1-\varepsilon}, \tag{37}$$

where $\theta \leq \varepsilon < 1$, and the constant Ψ defined by (34), we will show that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t)| \\
&\leq \sup_{t\in[0,T]}\Bigg\{\frac{|\lambda|\, t^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \\
&\qquad\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\qquad\times\left|f\left(u,x(u),I_z^{\delta}x(u)\right)\right| d_qu\, d_ps \\
&\qquad+\frac{t^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \\
&\qquad\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\qquad\times\left|f\left(u,x(u),I_z^{\delta}x(u)\right)\right| d_qu\, d_rs \\
&\qquad+\int_0^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f\left(s,x(s),I_z^{\delta}x(s)\right)\right| d_qs\Bigg\}.
\end{aligned} \tag{38}$$

The assumption (H_1) implies that

$$\begin{aligned}
\left|f(t,w_1,w_2)\right| &\leq \left|f(t,w_1,w_2)-f(t,0,0)\right| + \left|f(t,0,0)\right| \\
&\leq L_1\left|w_1\right| + L_2\left|w_2\right| + M,
\end{aligned} \tag{39}$$

for all $t \in [0,T]$ and $w_1, w_2 \in \mathbb{R}$.

Then, by using Lemmas 6 and 7, we have

$$\begin{aligned}
&|\mathcal{Q}x(t)| \\
&\leq \frac{|\lambda|\, T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \\
&\quad\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\times\left|f\left(u,x(u),I_z^{\delta}x(u)\right)\right| d_qu\, d_ps \\
&\quad+\frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \\
&\quad\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\times\left|f\left(u,x(u),I_z^{\delta}x(u)\right)\right| d_qu\, d_rs \\
&\quad+\int_0^{T}\frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f\left(s,x(s),I_z^{\delta}x(s)\right)\right| d_qs \\
&\leq \frac{|\lambda|\, T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \\
&\quad\times\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\times\left(L_1r + L_2r\int_0^{u}\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv + M\right) d_qu\, d_ps \\
&\quad+\frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \\
&\quad\times\int_0^{\xi}\int_0^{s}(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\qquad\times\left(L_1r + L_2r\int_0^{u}\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv + M\right) d_qu\, d_rs \\
&\quad+\int_0^{T}\frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad\times\left(L_1r + L_2r\int_0^{s}\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv + M\right) d_qs \\
&= \frac{|\lambda|\, T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \\
&\quad\times\left(\frac{\eta^{\alpha+\beta}}{[\alpha]_q}\left(L_1r+M\right)B_p(\beta,\alpha+1)\right. \\
&\qquad\left.+\frac{L_2r\eta^{\alpha+\beta+\delta}}{\Gamma_z(\delta)\,[\delta]_z}B_q(\alpha,\delta+1)\,B_p(\beta,\alpha+\delta+1)\right) \\
&\quad+\frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \\
&\quad\times\left(\frac{\xi^{\alpha+\gamma}}{[\alpha]_q}\left(L_1r+M\right)B_r(\gamma,\alpha+1)\right. \\
&\qquad\left.+\frac{L_2r\xi^{\alpha+\gamma+\delta}}{\Gamma_z(\delta)\,[\delta]_z}B_q(\alpha,\delta+1)\,B_r(\gamma,\alpha+\delta+1)\right) \\
&\quad+\frac{T^{\alpha}}{\Gamma_q(\alpha)\,[\alpha]_q}\left(L_1r+M\right)+\frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta)\,[\delta]_z}L_2r \\
&= \Lambda r + \Psi M \leq r.
\end{aligned} \tag{40}$$

Then, we have $\|\mathcal{Q}x\| \leq r$ which yields $\mathcal{Q}B_r \subset B_r$.

Next, for any $x, y \in \mathscr{C}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&\left|\mathscr{Q}x(t) - \mathscr{Q}y(t)\right| \\
&\le \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(\left| f\left(u, x(u), I_z^{\delta} x(u)\right) - f\left(u, y(u), I_z^{\delta} y(u)\right)\right|\right) d_q u\, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(\left| f\left(u, x(u), I_z^{\delta} x(u)\right) - f\left(u, y(u), I_z^{\delta} y(u)\right)\right|\right) d_q u\, d_r s \\
&\quad + \int_0^{T} \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad \times \left(\left| f\left(s, x(s), I_z^{\delta} x(s)\right) - f\left(s, y(s), I_z^{\delta} y(s)\right)\right|\right) d_q s \\
&\le \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 \|x - y\| + L_2 \|x - y\| \int_0^{u} \frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q u\, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 \|x - y\| + L_2 \|x - y\| \times \int_0^{u} \frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q u\, d_r s \\
&\quad + \int_0^{T} \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad \times \left(L_1 \|x - y\| + L_2 \|x - y\| \times \int_0^{s} \frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q s \\
&= \Lambda \|x - y\|.
\end{aligned} \tag{41}$$

The above result implies that $\|\mathscr{Q}x - \mathscr{Q}y\| \le \Lambda \|x - y\|$. As $\Lambda < 1$, $\mathscr{Q}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathscr{Q}$ has a fixed point which is the unique solution of problem (1). □

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 10 (Krasnoselskii's fixed point theorem [20]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) *$Ax + By \in M$ whenever $x, y \in M$;* (b) *A is compact and continuous;* (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 11. *Assume that $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption (H_1). In addition one supposes that*

(H_2) $|f(t, w_1, w_2)| \le \kappa(t)$, for all $(t, w_1, w_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ and $\kappa \in C([0,T], \mathbb{R}^+)$.

If

$$\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} L_1 + \frac{T^{\alpha+\delta} B_q(\alpha, \delta+1)}{\Gamma_q(\alpha) \Gamma_z(\delta+1)} L_2 < 1, \tag{42}$$

then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof. Let us set $\sup_{t\in[0,T]} |\kappa(t)| = \|\kappa\|$ and choose a suitable constant ρ as

$$\rho \ge \|\kappa\| \Psi, \tag{43}$$

where Ψ is defined by (34). Now, we define the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ on the set $B_\rho = \{x \in \mathscr{C} : \|x\| \le \rho\}$ as

$$\begin{aligned}
&(\mathscr{Q}_1 x)(t) \\
&\quad = \frac{\lambda t^{\alpha-1}}{\Omega \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\qquad \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad\qquad \times f\left(u, x(u), I_z^{\delta} x(u)\right) d_q u\, d_p s \\
&\qquad - \frac{t^{\alpha-1}}{\Omega \Gamma_r(\gamma) \Gamma_q(\alpha)}
\end{aligned}$$

$$\times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta} x(u)\right) d_q u \, d_r s$$

$$(\mathcal{Q}_2 x)(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, x(s), I_z^{\delta} x(s)\right) d_q s. \tag{44}$$

Firstly, we will show that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy condition (a) of Lemma 10. For $x, y \in B_\rho$, we have

$$\begin{aligned}
&\left\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\right\| \\
&\leq \|\kappa\| \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} d_q u \, d_p s \\
&\quad + \|\kappa\| \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} d_q u \, d_r s \\
&\quad + \|\kappa\| \int_0^T \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&= \Psi \|\kappa\| \leq \rho.
\end{aligned} \tag{45}$$

Therefore $(\mathcal{Q}_1 x) + (\mathcal{Q}_2 y) \in B_\rho$. Further, condition (H_1) coupled with (42) implies that $\mathcal{Q}_2$ is contraction mapping. Therefore, condition (c) of Lemma 10 is satisfied.

Finally, we will show that $\mathcal{Q}_1$ is compact and continuous. Using the continuity of f and (H_2), we deduce that the operator $\mathcal{Q}_1$ is continuous and uniformly bounded on B_ρ. We define $\sup_{(t, w_1, w_2) \in [0,T] \times B_\rho^2} |f(t, w_1, w_2)| = N < \infty$. For $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $x \in B_\rho$, we have

$$\begin{aligned}
&\left|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)\right| \\
&\leq \frac{|\lambda| \left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^{\delta} x(u)\right)\right| d_q u \, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^{\delta} x(u)\right)\right| d_q u \, d_r s \\
&\leq \frac{|\lambda| \left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \times \int_0^{\eta} \int_0^{s} (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} d_q u \, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} d_q u \, d_r s \\
&\leq \frac{|\lambda| \left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega| \Gamma_p(\beta)} \left(\frac{\eta^{\alpha+\beta} B_p(\beta, \alpha+1)}{\Gamma_q(\alpha+1)}\right) \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega| \Gamma_r(\gamma)} \left(\frac{\xi^{\alpha+\gamma} B_r(\gamma, \alpha+1)}{\Gamma_q(\alpha+1)}\right).
\end{aligned} \tag{46}$$

Actually, as $t_1 - t_2 \to 0$ the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. Therefore, $\mathcal{Q}_1$ is relatively compact on B_ρ. Applying the Arzelá-Ascoli theorem, we get that $\mathcal{Q}_1$ is compact on B_ρ. Thus all assumptions of Lemma 10 are satisfied. Therefore, the boundary value problem (1) has at least one solution on $[0, T]$. The proof is complete. □

Using the Leray-Schauder nonlinear alternative, we give the third result.

Lemma 12 (nonlinear alternative for single-valued maps [21]). *Let E be a Banach space, let C be a closed, convex subset of E, let U be an open subset of C, and let $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

For the sake of convenience of proving the last result, we set

$$\Phi_1 = \frac{|\lambda| T^{\alpha-1} \eta^{\alpha+\beta} B_p(\beta, \alpha+1)}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha+1)} + \frac{T^{\alpha-1} \xi^{\alpha+\gamma} B_r(\gamma, \alpha+1)}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha+1)} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}, \tag{47}$$

$$\begin{aligned}
\Phi_2 &= \frac{|\lambda| T^{\alpha-1} \eta^{\alpha+\beta+\delta} B_q(\alpha, \delta+1) B_p(\beta, \alpha+\delta+1)}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha) \Gamma_z(\delta+1)} \\
&\quad + \frac{T^{\alpha+\delta} B_q(\alpha, \delta+1)}{\Gamma_q(\alpha) \Gamma_z(\delta+1)} \\
&\quad + \frac{T^{\alpha-1} \xi^{\alpha+\gamma+\delta} B_q(\alpha, \delta+1) B_r(\gamma, \alpha+\delta+1)}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha) \Gamma_z(\delta+1)}.
\end{aligned} \tag{48}$$

Theorem 13. *Assume that $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In addition one supposes that*

(H_3) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that

$$|f(t,w_1,w_2)| \le p(t)\,\psi(|w_1|) + |w_2| \quad \text{for each } (t,w_1,w_2) \in [0,T] \times \mathbb{R}^2; \tag{49}$$

(H_4) there exists a constant $K > 0$ such that

$$\frac{(1-\Phi_2)K}{\|p\|\,\psi(K)\,\Phi_1} > 1, \tag{50}$$

where Φ_1 and Φ_2 are defined by (47) *and* (48), *respectively, and*

$$\Phi_2 < 1. \tag{51}$$

Then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof. Firstly, we will show that the operator $\mathcal{Q}$, defined by (32), *maps bounded sets (balls) into bounded sets in $\mathcal{C}$*. For a positive number R, we set a bounded ball in $\mathcal{C}$ as $B_R = \{x \in \mathcal{C} : \|x\| \le R\}$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
|\mathcal{Q}x(t)| &\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \times \int_0^\eta \int_0^s (\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u,x(u),I_z^\delta x(u)\right)\right| d_qu\, d_ps \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \times \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u,x(u),I_z^\delta x(u)\right)\right| d_qu\, d_rs \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f\left(s,x(s),I_z^\delta x(s)\right)\right| d_qs \\
&\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \times \int_0^\eta \int_0^s (\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\,\psi(\|x\|) + \|x\| \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_zv\right) d_qu\, d_ps \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \times \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\,\psi(\|x\|) + \|x\| \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_zv\right) d_qu\, d_rs \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times \left(p(s)\,\psi(\|x\|) + \|x\| \int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_zv\right) d_qs \\
&\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)} \times \left(\frac{\eta^{\alpha+\beta} B_p(\beta,\alpha+1)\,\|p\|\,\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\eta^{\alpha+\beta+\delta} B_q(\alpha,\delta+1)\, B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)} \times \left(\frac{\xi^{\alpha+\gamma} B_r(\gamma,\alpha+1)\,\|p\|\,\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\xi^{\alpha+\gamma+\delta} B_q(\alpha,\delta+1)\, B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{\|p\|\,\psi(R)\,T^\alpha}{\Gamma_q(\alpha+1)} + \frac{RT^{\alpha+\delta} B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)} \\
&:= G.
\end{aligned} \tag{52}$$

Therefore, we conclude that $\|\mathcal{Q}x\| \le G$.

Secondly, we will show that *$\mathcal{Q}$ maps bounded sets into equicontinuous sets of $\mathscr{C}$*. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and B_R be a bounded set of $C([0,T],\mathbb{R})$ as in the previous step, and let $x \in B_R$. Then we have

$$\begin{aligned}
&\left|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\right| \\
&\le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_r s \\
&\quad + \left|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f\left(s, x(s), I_z^\delta x(s)\right)\right| d_q s - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f\left(s, x(s), I_z^\delta x(s)\right)\right| d_q s\right| \\
&\le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\psi(\|x\|) + \|x\| \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v\right) d_q u\, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\psi(\|x\|) + \|x\| \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v\right) d_q u\, d_r s \\
&\quad + \left|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times \left(p(s)\psi(\|x\|) + \|x\| \int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v\right) d_q s \right.\\
&\qquad \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times \left(p(s)\psi(\|x\|) + \|x\| \int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v\right) d_q s\right| \\
&\le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)} \times \left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)} \times \left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|\|p\|\psi(R)}{\Gamma_q(\alpha+1)} + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| R B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}.
\end{aligned} \tag{53}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_R$ as $t_1 \to t_2$. Therefore, by applying the Arzelá-Ascoli theorem, we deduce that $\mathcal{Q} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to the equation $x(t) = \omega(\mathcal{Q}x)(t)$ for some $0 < \omega < 1$. Let x be a solution. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
(\mathcal{Q}x)(t) = {} & \frac{\omega\lambda t^{\alpha-1}}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^\delta x(u)\right) d_q u\, d_p s \\
& - \frac{\omega t^{\alpha-1}}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)}
\end{aligned}$$

$$\times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta}x(u)\right) d_q u\, d_r s + \omega \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, x(s), I_z^{\delta}x(s)\right) d_q s. \quad (54)$$

As before, one can easily find that

$$\|x\| = \sup_{t\in[0,T]} |\omega(\mathcal{Q}x)(t)| \le \|p\|\,\psi(\|x\|)\,\Phi_1 + \|x\|\,\Phi_2, \quad (55)$$

which can alternatively be written as

$$\frac{(1-\Phi_2)\|x\|}{\|p\|\,\psi(\|x\|)\,\Phi_1} \le 1. \quad (56)$$

In view of (H_4), there exists K such that $\|x\| \ne K$. Let us set

$$\mathcal{U} = \{x \in C([0,T], \mathbb{R}) : \|x\| < K\}. \quad (57)$$

Note that the operator $\mathcal{Q} : \overline{\mathcal{U}} \to C(0,T,\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x = \omega\mathcal{Q}x$ for some $\omega \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 12), we deduce that $\mathcal{Q}$ has a fixed point $x \in \overline{\mathcal{U}}$ which is a solution of problem (1). This completes the proof. □

5. Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following nonlocal fractional q-integral boundary value problem:

$$D_{1/2}^{3/2}x(t) = \frac{2\sin\pi t}{(e^t+4)^2}\cdot\frac{|x(t)|}{2+|x(t)|} + \frac{e^{-t^2}}{(6+t)^2} I_{3/4}^{7/5}x(t) + \frac{1}{2}, \quad 0<t<3,$$
$$x(0) = 0, \qquad \frac{1}{5} I_{3/5}^{1/2} x\left(\frac{5}{2}\right) = I_{2/3}^{5/2} x\left(\frac{3}{2}\right). \quad (58)$$

Here $\alpha = 3/2$, $q = 1/2$, $\delta = 7/5$, $z = 3/4$, $\lambda = 1/5$, $\beta = 1/2$, $p = 3/5$, $\eta = 5/2$, $\gamma = 5/2$, $r = 2/3$, $\xi = 3/2$, $T = 3$, and $f(t,x,I_z^{\delta}x) = (2\sin\pi t/(e^t+4)^2)(|x|/(2+|x|)) + (e^{-t^2}/((6+t)^2))I_{3/4}^{7/5}x + 1/2$.

Since $|f(t,w_1,w_2) - f(t,\overline{w}_1,\overline{w}_2)| \le (1/25)|w_1 - \overline{w}_1| + (1/36)|w_2 - \overline{w}_2|$, then (H_1) is satisfied with $L_1 = 1/25$ and $L_2 = 1/36$. By using the Maple program, we find that

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)}\xi^{\alpha+\gamma-1} - \lambda\frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)}\eta^{\alpha+\beta-1} \approx 0.4141558,$$

$$\Lambda = \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)} \times \left[\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)L_1}{\Gamma_q(\alpha+1)} + \frac{\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right] + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)} \times \left[\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)L_1}{\Gamma_q(\alpha+1)} + \frac{\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right] + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}L_1 + \frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2 \approx 0.8514717 < 1. \quad (59)$$

Hence, by Theorem 9, the nonlocal boundary value problem (58) has a unique solution on $[0,3]$.

Example 2. Consider the following nonlocal fractional q-integral boundary value problem:

$$D_{2/3}^{9/5}x(t) = \frac{1}{4\pi^2+t^2}\tan^{-1}\left(\frac{\pi x}{2}\right) + \frac{1}{30\pi}(1+\sin(\pi t)) + I_{1/10}^{3/5}x(t), \quad 0<t<1,$$
$$x(0) = 0, \quad \frac{1}{50} I_{1/5}^{1/10} x\left(\frac{2}{3}\right) = I_{1/8}^{2/9} x\left(\frac{1}{2}\right). \quad (60)$$

Here $\alpha = 9/5$, $q = 2/3$, $\delta = 3/5$, $z = 1/10$, $\lambda = 1/50$, $\beta = 1/10$, $p = 1/5$, $\eta = 2/3$, $\gamma = 2/9$, $r = 1/8$, $\xi = 1/2$, $T = 1$, and $f(t,x,I_z^{\delta}x) = (\tan^{-1}(\pi x/2))/(4\pi^2+t^2) + (1+\sin(\pi t))/(30\pi) + I_{1/10}^{3/5}x$.

By using the Maple program, we find that

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)}\xi^{\alpha+\gamma-1} - \lambda\frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)}\eta^{\alpha+\beta-1} \approx 0.4691329,$$

$$\Phi_1 = \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha+1)} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \approx 1.0408909,$$

$$\Phi_2 = \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T^{\alpha-1}\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)\Gamma_z(\delta+1)} \approx 0.5751429 < 1. \tag{61}$$

Clearly,

$$\begin{aligned}|f(t,w_1,w_2)| &= \left|\frac{1}{4\pi^2+t^2}\tan^{-1}\left(\frac{\pi w_1}{2}\right) + \frac{1}{30\pi}(1+\sin(\pi t)) + w_2\right| \\ &\le \frac{1}{120\pi}(1+\sin\pi t)(15|w_1|+4) + |w_2|.\end{aligned} \tag{62}$$

Choosing $p(t) = 1 + \sin\pi t$ and $\psi(|w_1|) = (1/120\pi)(15|w_1| + 4)$, we can show that

$$\frac{(1-\Phi_2)K}{\|p\|\psi(K)\Phi_1} > 1 \tag{63}$$

which implies that $K > 0.0645811$. Hence, by Theorem 13, the nonlocal boundary value problem (60) has at least one solution on $[0, 1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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