

Research Article

On Strong Convergence of Halpern's Method Using Averaged Type Mappings

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Received 14 May 2014; Accepted 17 June 2014; Published 3 July 2014

Academic Editor: Luigi Muglia

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Under suitable hypotheses on control coefficients, we study Halpern's method to approximate strongly common fixed points of a nonexpansive mapping and of a nonspreading mapping or a fixed point of one of them. A crucial tool in our results is the regularization with the averaged type mappings.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|$.

Let C be a nonempty, closed, and convex subset of H . Let T be a nonlinear mapping of C into itself; we denote by $\text{Fix}(T)$ the set of fixed points of T , that is, $\text{Fix}(T) = \{z \in C : Tz = z\}$.

We recall that a mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1)$$

The problem of finding fixed points of nonexpansive mappings has been widely investigated by many authors.

Halpern [1] was the first to consider the following explicit method:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (2)$$

where $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1]$ and $u \in C$ is fixed.

Moreover, Halpern proved in [1] the following theorem on the convergence of (2) for a particular choice of $(\alpha_n)_{n \in \mathbb{N}}$.

Theorem 1. *Let C be a bounded, closed, and convex subset of Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. For any initialization $x_1 \in C$ and anchor $u \in C$, define a sequence $(x_n)_{n \in \mathbb{N}}$ in C by*

$$x_{n+1} = n^{-\theta} u + (1 - n^{-\theta})Tx_n, \quad \forall n \geq 1, \quad (3)$$

where $\theta \in (0, 1)$. Then $(x_n)_{n \in \mathbb{N}}$ converges strongly to the element of $\text{Fix}(T)$ nearest to u .

He also showed that the control conditions,

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(C2) \sum_{n=1}^{\infty} \alpha_n = \infty,$$

are necessary for the convergence of (2) to a fixed point of T .

Subsequently, several authors carefully studied the following problem: are the control conditions (C1) and (C2) sufficient for the convergence of (2)?

In this direction, C. E. Chidume and C. O. Chidume [2] and Suzuki [3], independently, proved that the conditions (C1) and (C2) are sufficient to assure the strong convergence to a fixed point of T of the following iterative sequence:

$$x_1, u \in C; \quad (4)$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \quad \forall n \geq 1.$$

Recently, in the setting of Banach spaces, Song and Chai [4], under the same conditions (C1) and (C2) but under stronger hypotheses on the mapping, obtained strong convergence of Halpern iterations (2). In particular, they assumed that E is a real reflexive Banach space with a uniformly Gateaux differentiable norm and with the fixed point property for nonexpansive self-mappings, and considered an important subclass of nonexpansive mappings which is the firmly type nonexpansive mappings.

Let T be a mapping with domain $D(T)$. T is said to be firmly type nonexpansive [4] if for all $x, y \in D(T)$, there exists $k \in (0, +\infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2. \quad (5)$$

A more general class of firmly type nonexpansive mappings is the class of the strongly nonexpansive mappings. Recall that a mapping $T : C \rightarrow C$ is said to be strongly nonexpansive if

- (1) T is nonexpansive;
- (2) $x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$, whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in C such that $(x_n - y_n)_{n \in \mathbb{N}}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$.

Saejung [5] proved the strong convergence of Halpern's iterations (2) for strongly nonexpansive mappings in a Banach space E such that one of the following conditions is satisfied:

- (i) E is uniformly smooth;
- (ii) E is reflexive, strictly convex with a uniformly Gateaux differentiable norm.

In the setting of Hilbert spaces, Kohsaka and Takahashi [6] defined $T : C \rightarrow C$ a nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \quad (6)$$

The following Lemma is a useful characterization of nonspreading mapping.

Lemma 2 (see [7]). *Let C be a nonempty closed subset of Hilbert space H . Then, a mapping $T : C \rightarrow C$ is nonspreading if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \quad (7)$$

Observe that if T is a nonspreading mapping from C into itself and $\text{Fix}(T) \neq \emptyset$, then T is quasi-nonexpansive; that is,

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \forall p \in \text{Fix}(T). \quad (8)$$

Further, the set of fixed points of a quasi-nonexpansive mapping is closed and convex [8].

Osilike and Isiogugu [9] studied Halpern's type for k -strictly pseudononspreading mappings T , which are a more general class of the nonspreading mappings.

To obtain the strong convergence of (2), they replaced the mapping T with the averaged type mapping T_δ , that is, with the mapping

$$T_\delta = (1 - \delta)I + \delta T, \quad \delta \in (0, 1). \quad (9)$$

Iemoto and Takahashi [7] approximated common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space using Moudafi's iterative scheme [10]. They obtained the following Theorem that states the weak convergence of their iterative method.

Theorem 3. *Let H be a Hilbert space and let C be a nonempty, closed, and convex subset of H . Assume that $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Define a sequence $(x_n)_{n \in \mathbb{N}}$ as follows:*

$$x_1 \in C \tag{10}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1 - \beta_n)Tx_n],$$

for all $n \in \mathbb{N}$, where $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}} \subset [0, 1]$. Then, the following hold.

- (i) If $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(S)$.
- (ii) If $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(T)$.
- (iii) If $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, then $(x_n)_{n \in \mathbb{N}}$ converges weakly to $p \in \text{Fix}(S) \cap \text{Fix}(T)$.

In this paper, inspired by Iemoto and Takahashi [7], we introduce an iterative method of Halpern's type involving the averaged type mappings T_δ and S_δ , where T is a nonexpansive mapping and S is a nonspreading mapping. The averaged type mappings T_δ and S_δ have a regularizing role in order to prove the strong convergence of our iterative scheme. In particular, we prove that the method strongly converges to the unique solution z of the variational inequality

$$\langle u - z, y - z \rangle \leq 0, \quad \forall y \in D, \tag{11}$$

where $u \in C$ is an anchor and, depending on the hypotheses on control coefficients, D is the set of fixed points of T , the set of fixed points of S , or the set of common fixed points of T and S .

Suitable tools in our proofs are Maingé's Lemma [11] and some techniques used by Maingé in [12] to study the strong convergence of the viscosity approximation method. However, Wongchan and Saejung [13] found a small mistake in Maingé's proof.

2. Preliminaries

To begin, we collected some lemmas which we will use in our proofs in the next section.

In the sequel, we denote by H a real Hilbert space and by C a nonempty closed convex subset of H .

Lemma 4. *The following known results hold:*

- (1) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, for all $x, y \in H$ and for all $t \in [0, 1]$,
- (2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, for all $x, y \in H$.

We recall that for every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{12}$$

Such P_C is called the metric projection of H onto C .

Lemma [14] characterizes the projection P_C .

Lemma 5. Let C be a closed and convex subset of a real Hilbert space and let P_C be the metric projection from H onto C . Given that $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the inequality

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \quad (13)$$

By Lemma 5, if $u \in C$ is fixed, $z = P_C u$ is the unique solution of the variational inequality (13).

To prove our main theorem, we need some fundamental properties of the involved mappings in the variational inequality.

The following result summarizes some significant properties of $I - T$ if T is a nonexpansive mapping ([15, 16]).

Lemma 6. Let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be nonexpansive. Then,

- (1) $I - T : C \rightarrow H$ is $(1/2)$ -inverse strongly monotone, that is,

$$\frac{1}{2} \|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y \rangle, \quad (14)$$

for all $x, y \in C$;

- (2) moreover, if $\text{Fix}(T) \neq \emptyset$, $I - T$ is demiclosed at 0; that is, for every sequence $(x_n)_{n \in \mathbb{N}}$ weakly convergent to p such that $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$, it follows $p \in \text{Fix}(T)$.

Iemoto and Takahashi showed the demiclosedness of $I - S$ at 0 and a suitable property of $I - S$. These results are summarized in the following two Lemmas.

Lemma 7 (see [7]). Let C be a nonempty, closed, and convex subset of H . Let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S) \neq \emptyset$. Then, $I - S$ is demiclosed at 0.

Lemma 8 (see [7]). Let C be a nonempty, closed, and convex subset of H . Let $S : C \rightarrow C$ be a nonspreading mapping. Then,

$$\begin{aligned} & \|(I - S)x - (I - S)y\|^2 \\ & \leq \langle x - y, (I - S)x - (I - S)y \rangle \\ & \quad + \frac{1}{2} (\|x - Sx\|^2 + \|y - Sy\|^2), \end{aligned} \quad (15)$$

for all $x, y \in C$.

If T is a nonexpansive mapping of C into itself, Byrne [15] defined the averaged mapping as follows:

$$T_\delta = (1 - \delta)I + \delta T = I - \delta(I - T), \quad (16)$$

where $\delta \in (0, 1)$.

Moreover, Byrne [15] and successively Moudafi [17] proved some properties of the averaged mappings; in particular, they showed that T_δ is a nonexpansive mapping. In this paper, inspired by [15, 17], we define the averaged type mapping V_δ as in (16) for a nonlinear mapping $V : C \rightarrow C$;

we notice that $\text{Fix}(V) = \text{Fix}(V_\delta)$. It is easy to verify that if S is a nonspreading mapping of C into itself and $\text{Fix}(S) \neq \emptyset$, the averaged type mapping S_δ is quasi-nonexpansive and consequently the set of fixed points of S_δ is closed and convex.

Actually, it follows from [9] that S_δ is quasi-firmly type nonexpansive mapping; that is, it is a firmly type nonexpansive mapping (5) on fixed points of S . For completeness we include the easy proof.

Proposition 9. Let C be a nonempty closed and convex subset of H and let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S)$ is nonempty. Then the averaged type mapping

$$S_\delta = (1 - \delta)I + \delta S \quad (17)$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1 - \delta) \in (0, 1)$.

Proof. We obtain

$$\begin{aligned} & \|S_\delta x - S_\delta y\|^2 \\ & = \|(1 - \delta)(x - y) + \delta(Sx - Sy)\|^2 \\ & \text{(by Lemma 4)} \\ & = (1 - \delta)\|x - y\|^2 + \delta\|Sx - Sy\|^2 \\ & \quad - \delta(1 - \delta)\|(x - Sx) - (y - Sy)\|^2 \\ & \text{(by (7))} \\ & \leq (1 - \delta)\|x - y\|^2 \\ & \quad + \delta [\|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle] \\ & \quad - \delta(1 - \delta)\|(x - Sx) - (y - Sy)\|^2 \\ & = \|x - y\|^2 + \frac{2}{\delta} \langle \delta(x - Sx), \delta(y - Sy) \rangle \\ & \quad - \frac{1 - \delta}{\delta} \|\delta(x - Sx) - \delta(y - Sy)\|^2 \\ & \text{(by (17))} \\ & = \|x - y\|^2 + \frac{2}{\delta} \langle x - S_\delta x, y - S_\delta y \rangle \\ & \quad - \frac{1 - \delta}{\delta} \|(x - S_\delta x) - (y - S_\delta y)\|^2 \\ & \leq \|x - y\|^2 + \frac{2}{\delta} \langle x - S_\delta x, y - S_\delta y \rangle \\ & \quad - (1 - \delta)\|(x - S_\delta x) - (y - S_\delta y)\|^2. \end{aligned} \quad (18)$$

Hence, we have

$$\begin{aligned} & \|S_\delta x - S_\delta y\|^2 \leq \|x - y\|^2 + \frac{2}{\delta} \langle x - S_\delta x, y - S_\delta y \rangle \\ & \quad - (1 - \delta)\|(x - S_\delta x) - (y - S_\delta y)\|^2. \end{aligned} \quad (19)$$

In particular, choosing in (19) $y = p$, where $p \in \text{Fix}(S) = \text{Fix}(S_\delta)$ we obtain

$$\|S_\delta x - p\|^2 \leq \|x - p\|^2 - (1 - \delta) \|x - S_\delta x\|^2. \quad (20)$$

□

The following lemma is useful in the proof of our main result.

Lemma 10. *Let C be a nonempty closed and convex subspace of H , $u \in C$ fixed, T a nonexpansive mapping from C into itself, and S a nonspreading mapping from C into itself such that $\text{Fix}(T) \cap \text{Fix}(S) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n \in \mathbb{N}} \subset C$. Then,*

(1) *if $\|y_n - Ty_n\| \rightarrow 0$, as $n \rightarrow \infty$, then*

$$\limsup_{n \rightarrow \infty} \langle u - \bar{p}, y_n - \bar{p} \rangle \leq 0, \quad (21)$$

where $\bar{p} = P_{\text{Fix}(T)}u$ is the unique point in $\text{Fix}(T)$ that satisfies the variational inequality

$$\langle u - \bar{p}, x - \bar{p} \rangle \leq 0, \quad \forall x \in \text{Fix}(T). \quad (22)$$

(2) *If $\|y_n - Sy_n\| \rightarrow 0$, as $n \rightarrow \infty$, then*

$$\limsup_{n \rightarrow \infty} \langle u - \hat{p}, y_n - \hat{p} \rangle \leq 0, \quad (23)$$

where $\hat{p} = P_{\text{Fix}(S)}u$ is the unique point in $\text{Fix}(S)$ that satisfies the variational inequality

$$\langle u - \hat{p}, x - \hat{p} \rangle \leq 0, \quad \forall x \in \text{Fix}(S). \quad (24)$$

(3) *If $\|y_n - Sy_n\| \rightarrow 0$ and $\|y_n - Ty_n\| \rightarrow 0$, as $n \rightarrow \infty$, then*

$$\limsup_{n \rightarrow \infty} \langle u - p_0, y_n - p_0 \rangle \leq 0, \quad (25)$$

where $p_0 = P_{\text{Fix}(T) \cap \text{Fix}(S)}u$ is the unique point in $\text{Fix}(T) \cap \text{Fix}(S)$ that satisfies the variational inequality

$$\langle u - p_0, x - p_0 \rangle \leq 0, \quad \forall x \in \text{Fix}(T) \cap \text{Fix}(S). \quad (26)$$

Proof. (1) Let \bar{p} satisfy (22). Let $(y_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(y_n)_{n \in \mathbb{N}}$ for which

$$\limsup_{n \rightarrow \infty} \langle \bar{p} - u, y_n - \bar{p} \rangle = \lim_{k \rightarrow \infty} \langle \bar{p} - u, y_{n_k} - \bar{p} \rangle. \quad (27)$$

Select a subsequence $(y_{n_{k_j}})_{j \in \mathbb{N}}$ of $(y_{n_k})_{k \in \mathbb{N}}$ such that $y_{n_{k_j}} \rightarrow v$ (this is possible by boundedness of $(y_n)_{n \in \mathbb{N}}$). By the hypothesis $\|y_n - Ty_n\| \rightarrow 0$, as $n \rightarrow \infty$, and by demiclosedness of $I - T$ at 0 we have $v \in \text{Fix}(T)$ and

$$\limsup_{n \rightarrow \infty} \langle \bar{p} - u, y_n - \bar{p} \rangle = \lim_{j \rightarrow \infty} \langle \bar{p} - u, y_{n_{k_j}} - \bar{p} \rangle = \langle \bar{p} - u, v - \bar{p} \rangle, \quad (28)$$

so the claim follows by (22).

(2) The proof is the same of (1) since also S is demiclosed in 0.

(3) Select a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ such that

$$\limsup_{n \rightarrow \infty} \langle p_0 - u, y_n - p_0 \rangle = \lim_{k \rightarrow \infty} \langle p_0 - u, y_{n_k} - p_0 \rangle, \quad (29)$$

where p_0 satisfies (26). Now select a subsequence $(y_{n_{k_j}})_{j \in \mathbb{N}}$ of $(y_{n_k})_{k \in \mathbb{N}}$ such that $y_{n_{k_j}} \rightarrow w$. Then by demiclosedness of $I - T$ and $I - S$ at 0 and by the hypotheses $\|y_n - Ty_n\| \rightarrow 0$ and $\|y_n - Sy_n\| \rightarrow 0$, as $n \rightarrow \infty$, we obtain that $w = Tw = Sw$, that is, $w \in \text{Fix}(T) \cap \text{Fix}(S)$. So,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle p_0 - u, y_n - p_0 \rangle &= \lim_{j \rightarrow \infty} \langle p_0 - u, y_{n_{k_j}} - p_0 \rangle \\ &= \langle p_0 - u, w - p_0 \rangle, \end{aligned} \quad (30)$$

so the claim follows by (26). □

A pertinent tool for us is the well-known lemma of Xu [18].

Lemma 11. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \sigma_n + \theta_n, \quad n \geq 0, \quad (31)$$

where,

- (i) $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$;
- (iii) $\theta_n \geq 0$, $\sum_{n=1}^{\infty} \theta_n < \infty$.

Then,

$$\lim_{n \rightarrow \infty} a_n = 0. \quad (32)$$

Finally, a crucial tool for our results is the following lemma proved by Maingé.

Lemma 12 (see [11]). *Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_j})_{j \in \mathbb{N}}$ of $(\gamma_n)_{n \in \mathbb{N}}$ such that $\gamma_{n_j} < \gamma_{n_{j+1}}$, for all $j \in \mathbb{N}$. Consider the sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ defined by*

$$\tau(n) := \max \{k \leq n : \gamma_k < \gamma_{k+1}\}. \quad (33)$$

Then, $(\tau(n))_{n \in \mathbb{N}}$ is a nondecreasing sequence for all $n \geq n_0$, satisfying

- (i) $\lim_{n \rightarrow \infty} \tau(n) = \infty$;
- (ii) $\gamma_{\tau(n)} < \gamma_{\tau(n)+1}$, $\forall n \geq n_0$;
- (iii) $\gamma_n < \gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Main Result

In all sections we denote by $T : C \rightarrow C$ a nonexpansive mapping, $S : C \rightarrow C$ a nonspreading mapping, and $T_\delta, S_\delta : C \rightarrow C$ the respectively averaged type mappings. Moreover,

$(\beta_n)_{n \in \mathbb{N}} \subset [0, 1]$ denotes a real sequence and $U_n : C \rightarrow C$ denotes the convex combination of T_δ and S_δ , that is,

$$U_n = \beta_n T_\delta x_n + (1 - \beta_n) S_\delta x_n, \quad n \in \mathbb{N}. \quad (34)$$

Further we assume that

- (i) $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$;
- (ii) $(\alpha_n)_{n \in \mathbb{N}} \subset (0, 1)$ a real sequence such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = 0$;
- (iii) $O(1)$ is any bounded real sequence.

We start with the following lemma:

Lemma 13. Let $u \in C$ be an anchor and let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) U_n x_n, \quad (35)$$

where

$$U_n = \beta_n T_\delta + (1 - \beta_n) S_\delta. \quad (36)$$

Then

- (1) U_n is quasi-nonexpansive for all $n \in \mathbb{N}$;
- (2) $(x_n)_{n \in \mathbb{N}}, (Sx_n)_{n \in \mathbb{N}}, (Tx_n)_{n \in \mathbb{N}}, (S_\delta x_n)_{n \in \mathbb{N}}, (T_\delta x_n)_{n \in \mathbb{N}}, (U_n x_n)_{n \in \mathbb{N}}$ are bounded sequences.

Proof. (1) Any convex combination of quasi-nonexpansive mappings is quasi-nonexpansive too. So is every U_n , since T_δ and S_δ are quasi-nonexpansive.

(2) The boundedness of $(x_n)_{n \in \mathbb{N}}$ follows the fact that U_n is quasi-nonexpansive. In fact, let $q \in \text{Fix}(T) \cap \text{Fix}(S)$. Then

$$\begin{aligned} & \|x_{n+1} - q\| \\ &= \|\alpha_n (u - q) + (1 - \alpha_n) (U_n x_n - q)\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \|U_n x_n - q\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n) \|x_n - q\|. \end{aligned} \quad (37)$$

Since

$$\|x_1 - q\| \leq \max \{ \|u - q\|, \|x_1 - q\| \}, \quad (38)$$

and by induction we assume that

$$\|x_n - q\| \leq \max \{ \|u - q\|, \|x_1 - q\| \}, \quad (39)$$

and then

$$\begin{aligned} & \|x_{n+1} - q\| \\ &\leq \alpha_n \|u - q\| \\ &\quad + (1 - \alpha_n) \max \{ \|u - q\|, \|x_1 - q\| \} \\ &\leq \alpha_n \max \{ \|u - q\|, \|x_1 - q\| \} \\ &\quad + (1 - \alpha_n) \max \{ \|u - q\|, \|x_1 - q\| \} \\ &= \max \{ \|u - q\|, \|x_1 - q\| \}. \end{aligned} \quad (40)$$

Thus $(x_n)_{n \in \mathbb{N}}$ is bounded. The boundedness of the other sequences follows by boundedness of $(x_n)_{n \in \mathbb{N}}$ and by the quasi-nonexpansivity of involved mappings. \square

Now, we prove our strong convergence theorem.

Theorem 14. Let H be a Hilbert space and let C be a nonempty closed and convex subset of H . Let $T : C \rightarrow C$ be a nonexpansive mapping and let $S : C \rightarrow C$ be a nonspreading mapping such that $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings. Suppose that $(\alpha_n)_{n \in \mathbb{N}}$ is a real sequence in $(0, 1)$ satisfying the conditions

- (1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (2) $\sum_{n=1}^\infty \alpha_n = \infty$.

If $(\beta_n)_{n \in \mathbb{N}}$ is a sequence in $[0, 1]$, we define a sequence $(x_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{aligned} & x_1 \in C \\ & x_{n+1} = \alpha_n u + (1 - \alpha_n) [\beta_n T_\delta x_n + (1 - \beta_n) S_\delta x_n], \quad n \in \mathbb{N}. \end{aligned} \quad (41)$$

Then, the following hold.

- (i) If $\sum_{n=1}^\infty (1 - \beta_n) < \infty$, then $(x_n)_{n \in \mathbb{N}}$ strongly converges to $\bar{p} = P_{\text{Fix}(T)} u$ which is the unique solution in $\text{Fix}(T)$ of the variational inequality $\langle u - \bar{p}, x - \bar{p} \rangle \leq 0$, for all $x \in \text{Fix}(T)$.
- (ii) If $\sum_{n=1}^\infty \beta_n < \infty$, then $(x_n)_{n \in \mathbb{N}}$ strongly converges to $\hat{p} = P_{\text{Fix}(S)} u$ which is the unique solution in $\text{Fix}(S)$ of the variational inequality $\langle u - \hat{p}, x - \hat{p} \rangle \leq 0$, for all $x \in \text{Fix}(S)$.
- (iii) If $\liminf_{n \rightarrow \infty} \beta_n (1 - \beta_n) > 0$, then $(x_n)_{n \in \mathbb{N}}$ strongly converges to $p_0 = P_{\text{Fix}(T) \cap \text{Fix}(S)} u$ which is the unique solution in $\text{Fix}(T) \cap \text{Fix}(S)$ of the variational inequality $\langle u - p_0, x - p_0 \rangle \leq 0$, for all $x \in \text{Fix}(T) \cap \text{Fix}(S)$.

Proof. (i) We rewrite the sequence $(x_{n+1})_{n \in \mathbb{N}}$ as

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\delta x_n + (1 - \beta_n) E_n, \quad (42)$$

where $E_n = (1 - \alpha_n)(S_\delta x_n - T_\delta x_n)$ is bounded, that is, $\|E_n\| \leq O(1)$.

We begin to prove that $\lim_{n \rightarrow \infty} \|x_n - T_\delta x_n\| = 0$.

Let $\bar{p} \in \text{Fix}(T) = \text{Fix}(T_\delta)$ the unique solution in $\text{Fix}(T)$ of the variational inequality

$$\langle u - \bar{p}, x - \bar{p} \rangle \leq 0, \quad \forall x \in \text{Fix}(T). \quad (43)$$

We have

$$\begin{aligned} & \|x_{n+1} - \bar{p}\|^2 \\ &= \|\alpha_n u + (1 - \alpha_n) (1 - \delta) x_n \\ &\quad + (1 - \alpha_n) \delta T x_n + (1 - \beta_n) E_n - \bar{p}\|^2 \\ &= \|[(1 - \alpha_n) \delta (T x_n - x_n) + x_n - \bar{p}] \\ &\quad + [\alpha_n (u - x_n) + (1 - \beta_n) E_n]\|^2 \end{aligned}$$

(by Lemma 4)

$$\begin{aligned}
&\leq \|(1 - \alpha_n) \delta (Tx_n - x_n) + x_n - \bar{p}\|^2 \\
&\quad + 2 \langle \alpha_n (u - x_n) + (1 - \beta_n) E_n, x_{n+1} - \bar{p} \rangle \\
&\leq \|(1 - \alpha_n) \delta (Tx_n - x_n) + x_n - \bar{p}\|^2 \\
&\quad + 2\alpha_n \langle u - x_n, x_{n+1} - \bar{p} \rangle + 2(1 - \beta_n) \langle E_n, x_{n+1} - \bar{p} \rangle \\
&\leq (1 - \alpha_n)^2 \delta^2 \|Tx_n - x_n\|^2 + \|x_n - \bar{p}\|^2 \\
&\quad - 2(1 - \alpha_n) \delta \langle x_n - \bar{p}, x_n - Tx_n \rangle \\
&\quad + 2\alpha_n \|u - x_n\| \|x_{n+1} - \bar{p}\| + 2(1 - \beta_n) \|E_n\| \|x_{n+1} - \bar{p}\| \\
&= (1 - \alpha_n)^2 \delta^2 \|x_n - Tx_n\|^2 + \|x_n - \bar{p}\|^2
\end{aligned}$$

$((I - T)\bar{p} = 0)$

$$\begin{aligned}
&- 2(1 - \alpha_n) \delta \langle x_n - \bar{p}, (I - T)x_n - (I - T)\bar{p} \rangle \\
&\quad + \alpha_n O(1) + (1 - \beta_n) O(1)
\end{aligned}$$

(by Lemma 6)

$$\begin{aligned}
&\leq \|x_n - \bar{p}\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - Tx_n\|^2 \\
&\quad - (1 - \alpha_n) \delta \|(I - T)x_n - (I - T)\bar{p}\|^2 \\
&\quad + \alpha_n O(1) + (1 - \beta_n) O(1) \\
&= \|x_n - \bar{p}\|^2 - (1 - \alpha_n) \delta [1 - \delta(1 - \alpha_n)] \|x_n - Tx_n\|^2 \\
&\quad + \alpha_n O(1) + (1 - \beta_n) O(1)
\end{aligned} \tag{44}$$

and hence

$$\begin{aligned}
0 &\leq (1 - \alpha_n) \delta [1 - \delta(1 - \alpha_n)] \|x_n - Tx_n\|^2 \\
&\leq \|x_n - \bar{p}\|^2 - \|x_{n+1} - \bar{p}\|^2 \\
&\quad + \alpha_n O(1) + (1 - \beta_n) O(1).
\end{aligned} \tag{45}$$

We turn our attention to the monotony of the sequence $(\|x_n - \bar{p}\|)_{n \in \mathbb{N}}$.

We consider the following two cases.

Case A. $\|x_{n+1} - \bar{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - \bar{p}\| < \|x_{n_{k+1}} - \bar{p}\| \quad \forall k \in \mathbb{N}. \tag{46}$$

Case A. Since $(\|x_n - \bar{p}\|)_{n \in \mathbb{N}}$ is definitively nonincreasing, $\lim_{n \rightarrow \infty} \|x_n - \bar{p}\|^2$ exists. From (45), $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, we have

$$\begin{aligned}
0 &\leq \limsup_{n \rightarrow \infty} ((1 - \alpha_n) \delta [1 - \delta(1 - \alpha_n)] \|x_n - Tx_n\|^2) \\
&\leq \limsup_{n \rightarrow \infty} (\|x_n - \bar{p}\|^2 - \|x_{n+1} - \bar{p}\|^2 \\
&\quad + \alpha_n O(1) + (1 - \beta_n) O(1)) = 0,
\end{aligned} \tag{47}$$

so, we can conclude that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0, \tag{48}$$

$$\lim_{n \rightarrow \infty} \|x_n - T_\delta x_n\| = \lim_{n \rightarrow \infty} \delta \|x_n - Tx_n\| = 0. \tag{49}$$

By Lemma 10, it follows that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{p}, x_n - \bar{p} \rangle \leq 0. \tag{50}$$

Finally, we prove that $(x_n)_{n \in \mathbb{N}}$ converges strongly to \bar{p} .

We compute that

$$\begin{aligned}
&\|x_{n+1} - \bar{p}\|^2 \\
&= \|\alpha_n (u - \bar{p}) + (1 - \alpha_n) (T_\delta x_n - \bar{p}) + (1 - \beta_n) E_n\|^2
\end{aligned}$$

(by Lemma 4)

$$\begin{aligned}
&\leq \|\alpha_n (u - \bar{p}) + (1 - \alpha_n) (T_\delta x_n - \bar{p})\|^2 \\
&\quad + 2(1 - \beta_n) \langle E_n, x_{n+1} - \bar{p} \rangle \\
&\leq \alpha_n^2 \|u - \bar{p}\|^2 + (1 - \alpha_n)^2 \|T_\delta x_n - \bar{p}\|^2 \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \bar{p}, T_\delta x_n - \bar{p} \rangle + (1 - \beta_n) O(1)
\end{aligned}$$

$(T_\delta$ nonexpansive)

$$\begin{aligned}
&\leq (1 - \alpha_n) \|x_n - \bar{p}\|^2 + \alpha_n^2 O(1) \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \bar{p}, T_\delta x_n - x_n \rangle \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \bar{p}, x_n - \bar{p} \rangle + (1 - \beta_n) O(1) \\
&\leq (1 - \alpha_n) \|x_n - \bar{p}\|^2 + \alpha_n^2 O(1) + \alpha_n O(1) \|T_\delta x_n - x_n\| \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \bar{p}, x_n - \bar{p} \rangle + (1 - \beta_n) O(1).
\end{aligned} \tag{51}$$

If we put $\sigma_n = \alpha_n O(1) + O(1) \|T_\delta x_n - x_n\| + 2(1 - \alpha_n) \langle u - \bar{p}, x_n - \bar{p} \rangle$ and $\theta_n = (1 - \beta_n) O(1)$, we have

$$\|x_{n+1} - \bar{p}\|^2 \leq (1 - \alpha_n) \|x_n - \bar{p}\|^2 + \alpha_n \sigma_n + \theta_n. \tag{52}$$

Hence, from assumption $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, from (49) and $\limsup_{n \rightarrow \infty} \langle u - \bar{p}, x_n - \bar{p} \rangle \leq 0$ we can apply Xu's Lemma 11.

Case B. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - \bar{p}\| < \|x_{n_{k+1}} - \bar{p}\| \quad \forall k \in \mathbb{N}. \tag{53}$$

Then by Maingé Lemma 12 there exists a sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ that satisfies

- (a) $(\tau(n))_{n \in \mathbb{N}}$ is nondecreasing;
- (b) $\lim_{n \rightarrow \infty} \tau(n) = \infty$;
- (c) $\|x_{\tau(n)} - \bar{p}\| < \|x_{\tau(n)+1} - \bar{p}\|$;
- (d) $\|x_n - \bar{p}\| < \|x_{\tau(n)+1} - \bar{p}\|$.

Consequently,

$$\begin{aligned}
 0 &\leq \liminf_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \bar{p}\| - \|x_{\tau(n)} - \bar{p}\|) \\
 &\leq \limsup_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \bar{p}\| - \|x_{\tau(n)} - \bar{p}\|) \\
 &\leq \limsup_{n \rightarrow \infty} (\|x_{n+1} - \bar{p}\| - \|x_n - \bar{p}\|) \\
 &\text{(by (42))} \\
 &= \limsup_{n \rightarrow \infty} (\|\alpha_n (u - T_\delta x_n) + T_\delta x_n - \bar{p} + (1 - \beta_n) E_n\| \\
 &\quad - \|x_n - \bar{p}\|) \\
 &\text{(} T_\delta \text{ nonexpansive)} \\
 &\leq \limsup_{n \rightarrow \infty} (\alpha_n O(1) + \|x_n - \bar{p}\| + (1 - \beta_n) O(1) - \|x_n - \bar{p}\|) \\
 &= 0,
 \end{aligned} \tag{54}$$

so

$$\lim_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \bar{p}\| - \|x_{\tau(n)} - \bar{p}\|) = 0. \tag{55}$$

By (45), we have

$$\begin{aligned}
 0 &\leq (1 - \alpha_{\tau(n)}) \delta [1 - \delta (1 - \alpha_{\tau(n)})] \|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\
 &\leq \|x_{\tau(n)} - \bar{p}\|^2 - \|x_{\tau(n)+1} - \bar{p}\|^2 \\
 &\quad + \alpha_n O(1) + (1 - \beta_{\tau(n)}) O(1),
 \end{aligned} \tag{56}$$

and from (55), $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$ we get

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{57}$$

By Lemma 10 and (57) we have

$$\limsup_{n \rightarrow \infty} \langle u - \bar{p}, x_{\tau(n)} - \bar{p} \rangle \leq 0. \tag{58}$$

Finally, we show that $(x_n)_{n \in \mathbb{N}}$ converges strongly to \bar{p} .

As in Case A, we can obtain

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - \bar{p}\| = 0; \tag{59}$$

then, from property (d) of Maingé Lemma 12 and (55) we can conclude

$$\lim_{n \rightarrow \infty} \|x_n - \bar{p}\| = 0. \tag{60}$$

□

Proof. (ii) Now, we rewrite the sequence $(x_{n+1})_{n \in \mathbb{N}}$ as

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_\delta x_n + \beta_n E_n, \tag{61}$$

where $E_n = (1 - \alpha_n)(T_\delta x_n - S_\delta x_n)$ is bounded, that is, $\|E_n\| \leq O(1)$.

We begin to prove that $\lim_{n \rightarrow \infty} \|x_n - S_\delta x_n\| = 0$.

Let $\hat{p} \in \text{Fix}(S) = \text{Fix}(S_\delta)$ the unique solution in $\text{Fix}(S)$ of the variational inequality $\langle u - \hat{p}, x - \hat{p} \rangle \leq 0$, for all $x \in \text{Fix}(S)$. We have

$$\begin{aligned}
 &\|x_{n+1} - \hat{p}\|^2 \\
 &= \|\alpha_n u + (1 - \alpha_n)(1 - \delta)x_n \\
 &\quad + (1 - \alpha_n)\delta Sx_n + \beta_n E_n - \hat{p}\|^2 \\
 &= \|[(1 - \alpha_n)\delta(Sx_n - x_n) + x_n - \hat{p}] \\
 &\quad + [\alpha_n(u - x_n) + \beta_n E_n]\|^2
 \end{aligned}$$

(by Lemma 4)

$$\begin{aligned}
 &\leq \|(1 - \alpha_n)\delta(Sx_n - x_n) + x_n - \hat{p}\|^2 \\
 &\quad + 2\langle \alpha_n(u - x_n) + \beta_n E_n, x_{n+1} - \hat{p} \rangle \\
 &\leq \|(1 - \alpha_n)\delta(Sx_n - x_n) + x_n - \hat{p}\|^2 \\
 &\quad + 2\alpha_n \langle u - x_n, x_{n+1} - \hat{p} \rangle + 2\beta_n \langle E_n, x_n - \hat{p} \rangle \\
 &\leq (1 - \alpha_n)^2 \delta^2 \|Sx_n - x_n\|^2 + \|x_n - \hat{p}\|^2 \\
 &\quad - 2(1 - \alpha_n)\delta \langle x_n - \hat{p}, x_n - Sx_n \rangle \\
 &\quad + \alpha_n O(1) + \beta_n O(1) \\
 &= (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 + \|x_n - \hat{p}\|^2
 \end{aligned}$$

$((I - S)\hat{p} = 0)$

$$\begin{aligned}
 &- 2(1 - \alpha_n)\delta \langle x_n - \hat{p}, (I - S)x_n - (I - S)\hat{p} \rangle \\
 &\quad + \alpha_n O(1) + \beta_n O(1)
 \end{aligned}$$

(by Lemma 8)

$$\begin{aligned}
 &\leq \|x_n - \hat{p}\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 \\
 &\quad - 2(1 - \alpha_n)\delta \left[\|(I - S)x_n - (I - S)\hat{p}\|^2 \right. \\
 &\quad \quad \left. - \frac{1}{2} (\|x_n - Sx_n\|^2 + \|p - Sp\|^2) \right] \\
 &\quad + \alpha_n O(1) + \beta_n O(1) \\
 &\leq \|x_n - \hat{p}\|^2 + (1 - \alpha_n)^2 \delta^2 \|x_n - Sx_n\|^2 \\
 &\quad - (1 - \alpha_n)\delta \|x_n - Sx_n\|^2 + \alpha_n O(1) + \beta_n O(1)
 \end{aligned}$$

$$\begin{aligned}
&= \|x_n - \widehat{p}\|^2 - (1 - \alpha_n) \delta [1 - \delta (1 - \alpha_n)] \|x_n - Sx_n\|^2 \\
&\quad + \alpha_n O(1) + \beta_n O(1)
\end{aligned} \tag{62}$$

and hence

$$\begin{aligned}
0 &\leq (1 - \alpha_n) \delta [1 - \delta (1 - \alpha_n)] \|x_n - Sx_n\|^2 \\
&\leq \|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 \\
&\quad + \alpha_n O(1) + \beta_n O(1).
\end{aligned} \tag{63}$$

Again, we turn our attention to the monotony of the sequence $(\|x_n - \widehat{p}\|)_{n \in \mathbb{N}}$. We consider the following two cases.

Case A. $\|x_{n+1} - \widehat{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_{k+1}} - \widehat{p}\| \quad \forall k \in \mathbb{N}. \tag{64}$$

Case A. Since $(\|x_n - \widehat{p}\|)_{n \in \mathbb{N}}$ is definitively nonincreasing, $\lim_{n \rightarrow \infty} \|x_n - \widehat{p}\|^2$ exists. From (45), $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$\begin{aligned}
0 &\leq \limsup_{n \rightarrow \infty} \left((1 - \alpha_n) \delta [1 - \delta (1 - \alpha_n)] \|x_n - Sx_n\|^2 \right) \\
&\leq \limsup_{n \rightarrow \infty} \left(\|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 \right) \\
&\quad + \alpha_n O(1) + \beta_n O(1) = 0,
\end{aligned} \tag{65}$$

and hence

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0, \tag{66}$$

$$\lim_{n \rightarrow \infty} \|x_n - S_\delta x_n\| = \lim_{n \rightarrow \infty} \delta \|x_n - Sx_n\| = 0. \tag{67}$$

By Lemma 10, it follows that

$$\limsup_{n \rightarrow \infty} \langle u - \widehat{p}, x_n - \widehat{p} \rangle \leq 0. \tag{68}$$

Finally, we prove that $(x_n)_{n \in \mathbb{N}}$ converges strongly to \widehat{p} .

We compute that

$$\begin{aligned}
&\|x_{n+1} - \widehat{p}\|^2 \\
&= \|\alpha_n (u - \widehat{p}) + (1 - \alpha_n) (S_\delta x_n - \widehat{p}) + \beta_n E_n\|^2
\end{aligned}$$

(by Lemma 4)

$$\begin{aligned}
&\leq \|\alpha_n (u - \widehat{p}) + (1 - \alpha_n) (S_\delta x_n - \widehat{p})\|^2 \\
&\quad + 2\beta_n \langle E_n, x_{n+1} - \widehat{p} \rangle \\
&\leq \alpha_n^2 \|u - \widehat{p}\|^2 + (1 - \alpha_n)^2 \|S_\delta x_n - \widehat{p}\|^2 \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \widehat{p}, S_\delta x_n - \widehat{p} \rangle + \beta_n O(1)
\end{aligned}$$

(S_δ quasi-nonexpansive)

$$\begin{aligned}
&\leq (1 - \alpha_n) \|x_n - \widehat{p}\|^2 + \alpha_n^2 O(1) \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \widehat{p}, S_\delta x_n - x_n \rangle \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \widehat{p}, x_n - \widehat{p} \rangle + \beta_n O(1) \\
&\leq (1 - \alpha_n) \|x_n - \widehat{p}\|^2 + \alpha_n^2 O(1) + \alpha_n O(1) \|S_\delta x_n - x_n\| \\
&\quad + 2\alpha_n (1 - \alpha_n) \langle u - \widehat{p}, x_n - \widehat{p} \rangle + \beta_n O(1).
\end{aligned} \tag{69}$$

If we put $\sigma_n = \alpha_n O(1) + O(1) \|S_\delta x_n - x_n\| + 2(1 - \alpha_n) \langle u - \widehat{p}, x_n - \widehat{p} \rangle$ and $\theta_n = \beta_n O(1)$, we have

$$\|x_{n+1} - \widehat{p}\|^2 \leq (1 - \alpha_n) \|x_n - \widehat{p}\|^2 + \alpha_n \sigma_n + \theta_n. \tag{70}$$

So, from assumption $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, from (67) and $\limsup_{n \rightarrow \infty} \langle \widehat{p} - u, x_n - \widehat{p} \rangle \geq 0$ we can apply Xu's Lemma 11.

Case B. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_{k+1}} - \widehat{p}\| \quad \forall k \in \mathbb{N}. \tag{71}$$

Then by Maingé Lemma there exists a sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ that satisfies

- (a) $(\tau(n))_{n \in \mathbb{N}}$ is nondecreasing;
- (b) $\lim_{n \rightarrow \infty} \tau(n) = \infty$;
- (c) $\|x_{\tau(n)} - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$;
- (d) $\|x_n - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$.

Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|) \\
&\leq \limsup_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|) \\
&\leq \limsup_{n \rightarrow \infty} (\|x_{n+1} - \widehat{p}\| - \|x_n - \widehat{p}\|) \\
&\text{(by (61))} \\
&= \limsup_{n \rightarrow \infty} (\|\alpha_n (u - S_\delta x_n) + S_\delta x_n - \widehat{p} + \beta_n E_n\| - \|x_n - \widehat{p}\|) \\
&\text{(S_δ quasi-nonexpansive)} \\
&\leq \limsup_{n \rightarrow \infty} (\alpha_n O(1) + \|x_n - \widehat{p}\| + \beta_n O(1) - \|x_n - \widehat{p}\|) = 0,
\end{aligned} \tag{72}$$

so

$$\lim_{n \rightarrow \infty} (\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|) = 0. \quad (73)$$

By (63), we obtain

$$\begin{aligned} 0 &\leq (1 - \alpha_{\tau(n)}) \delta [1 - \delta (1 - \alpha_{\tau(n)})] \|x_{\tau(n)} - Sx_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - \widehat{p}\|^2 - \|x_{\tau(n)+1} - \widehat{p}\|^2 + \alpha_n O(1) + \beta_{\tau(n)} O(1), \end{aligned} \quad (74)$$

and from (73), $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ we get

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - Sx_{\tau(n)}\| = 0. \quad (75)$$

By Lemma 10 and (75) we get

$$\limsup_{n \rightarrow \infty} \langle u - \widehat{p}, x_{\tau(n)} - \widehat{p} \rangle \leq 0. \quad (76)$$

Finally, we show that $(x_n)_{n \in \mathbb{N}}$ converges strongly to \bar{p} .

As in Case A, we obtain

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - \widehat{p}\| = 0, \quad (77)$$

and then from property (d) of Maingé Lemma and (73) we can conclude that

$$\lim_{n \rightarrow \infty} \|x_n - \widehat{p}\| = 0. \quad (78)$$

□

Proof. (iii) We recall that the sequence $(x_{n+1})_{n \in \mathbb{N}}$ is defined as

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) U_n x_n, \quad (79)$$

where $U_n = \beta_n T_\delta x_n + (1 - \beta_n) S_\delta x_n$.

We first show that $\lim_{n \rightarrow \infty} \|x_n - U_n x_n\| = 0$.

Let $p_0 \in \text{Fix}(T) \cap \text{Fix}(S)$ be the unique solution of the variational inequality $\langle u - p_0, x - p_0 \rangle \leq 0$, for all $x \in \text{Fix}(T) \cap \text{Fix}(S)$. We compute that

$$\begin{aligned} &\|U_n x_n - p_0\|^2 \\ &= \|\beta_n (T_\delta x_n - p_0) + (1 - \beta_n) (S_\delta x_n - p_0)\|^2 \\ &\text{(by Lemma 4)} \\ &= \beta_n \|T_\delta x_n - p_0\|^2 + (1 - \beta_n) \|S_\delta x_n - p_0\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2 \\ &\text{(} T_\delta \text{ nonexpansive and by (20))} \\ &\leq \beta_n \|x_n - p_0\|^2 + (1 - \beta_n) \|x_n - p_0\|^2 \\ &\quad - (1 - \beta_n) (1 - \delta) \|x_n - S_\delta x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2 \\ &= \|x_n - p_0\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - S_\delta x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2. \end{aligned} \quad (80)$$

So, we get

$$\begin{aligned} \|U_n x_n - p_0\|^2 &\leq \|x_n - p_0\|^2 \\ &\quad - (1 - \beta_n) (1 - \delta) \|x_n - S_\delta x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2. \end{aligned} \quad (81)$$

We have

$$\begin{aligned} &\|x_{n+1} - p_0\|^2 \\ &= \|U_n x_n - p_0 + \alpha_n (u - U_n x_n)\|^2 \\ &\leq \|U_n x_n - p_0\|^2 \\ &\quad + \alpha_n (\alpha_n \|u - U_n x_n\|^2 \\ &\quad + 2 \|U_n x_n - p_0\| \|u - U_n x_n\|) \end{aligned} \quad (82)$$

$$\leq \|U_n x_n - p_0\|^2 + \alpha_n O(1)$$

(by (81))

$$\begin{aligned} &\leq \|x_n - p_0\|^2 - (1 - \beta_n) (1 - \delta) \|x_n - S_\delta x_n\|^2 \\ &\quad - \beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2 + \alpha_n O(1). \end{aligned}$$

From (82), we derive

$$(1 - \beta_n) (1 - \delta) \|x_n - S_\delta x_n\|^2 \quad (83)$$

$$\leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1),$$

$$\beta_n (1 - \beta_n) \|T_\delta x_n - S_\delta x_n\|^2 \quad (84)$$

$$\leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1).$$

Now, also we consider two cases.

Case A. $\|x_{n+1} - \widehat{p}_0\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}_0\| < \|x_{n_{k+1}} - \widehat{p}_0\| \quad \forall k \in \mathbb{N}. \quad (85)$$

Case A. Since $(\|x_n - p_0\|)_{n \in \mathbb{N}}$ is definitively nonincreasing, $\lim_{n \rightarrow \infty} \|x_n - p_0\|^2$ exists. From (83) and $\lim_{n \rightarrow \infty} \alpha_n = 0$ and since $\liminf_{n \rightarrow \infty} \beta_n (1 - \beta_n) > 0$ we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - S_\delta x_n\| = \lim_{n \rightarrow \infty} \delta \|x_n - Sx_n\| = 0. \quad (86)$$

Furthermore, from (84) we have

$$\lim_{n \rightarrow \infty} \|S_\delta x_n - T_\delta x_n\| = \lim_{n \rightarrow \infty} \delta \|Sx_n - Tx_n\| = 0; \quad (87)$$

since

$$\|x_n - Tx_n\| \leq \|x_n - Sx_n\| + \|Sx_n - Tx_n\|, \quad (88)$$

by (86) and (87) we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_\delta x_n\| = \lim_{n \rightarrow \infty} \delta \|x_n - Tx_n\| = 0. \quad (89)$$

Then, since

$$\|U_n x_n - x_n\| \leq \beta_n \|T_\delta x_n - x_n\| + (1 - \beta_n) \|S_\delta x_n - x_n\|, \tag{90}$$

by (86) and (89) we get

$$\lim_{n \rightarrow \infty} \|x_n - U_n x_n\| = 0. \tag{91}$$

By Lemma 10, we have

$$\limsup_{n \rightarrow \infty} \langle u - p_0, x_n - p_0 \rangle \leq 0. \tag{92}$$

Finally, $(x_n)_{n \in \mathbb{N}}$ converges strongly to p_0 .

We compute that

$$\begin{aligned} & \|x_{n+1} - p_0\|^2 \\ & \leq \|(1 - \alpha_n)(U_n x_n - p_0) + \alpha_n(u - p_0)\|^2 \\ & = (1 - \alpha_n)^2 \|U_n x_n - p_0\|^2 + \alpha_n^2 \|u - p_0\|^2 \\ & \quad + 2\alpha_n(1 - \alpha_n) \langle U_n x_n - p_0, u - p_0 \rangle \\ & \text{(U_n quasi-nonexpansive)} \\ & \leq (1 - \alpha_n)^2 \|x_n - p_0\|^2 + \alpha_n^2 O(1) \\ & \quad + 2\alpha_n(1 - \alpha_n) \langle U_n x_n - x_n, u - p_0 \rangle \\ & \quad + 2\alpha_n(1 - \alpha_n) \langle x_n - p_0, u - p_0 \rangle \\ & \leq (1 - \alpha_n) \|x_n - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1) \|U_n x_n - x_n\| \\ & \quad + 2\alpha_n(1 - \alpha_n) \langle x_n - p_0, u - p_0 \rangle. \end{aligned} \tag{93}$$

If we set $\sigma_n = \alpha_n O(1) + O(1) \|U_n x_n - x_n\| + 2(1 - \alpha_n) \langle x_n - p_0, u - p_0 \rangle$, we have

$$\|x_{n+1} - p_0\|^2 \leq (1 - \alpha_n) \|x_n - p_0\|^2 + \alpha_n \sigma_n. \tag{94}$$

From $\sum_{n=1}^\infty \alpha_n = \infty$, (91), and $\limsup_{n \rightarrow \infty} \langle u - p_0, x_n - p_0 \rangle \leq 0$ we conclude that $(x_n)_{n \in \mathbb{N}}$ strongly converges to p_0 .

Case B. $(\|x_n - p_0\|)_{n \in \mathbb{N}}$ is not definitively nonincreasing. This means that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_{k+1}} - p_0\| \quad \forall k \in \mathbb{N}. \tag{95}$$

Then by Maingé Lemma 12 there exists a sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ that satisfies some properties defined previously.

Consequently,

$$\begin{aligned} 0 & \leq \liminf_{n \rightarrow \infty} (\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|) \\ & \leq \limsup_{n \rightarrow \infty} (\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|) \\ & \leq \limsup_{n \rightarrow \infty} (\|x_{n+1} - p_0\| - \|x_n - p_0\|) \\ & \text{(by (79))} \\ & = \limsup_{n \rightarrow \infty} (\|\alpha_n(u - p_0) + (1 - \alpha_n)(U_n x_n - p_0)\| \\ & \quad - \|x_n - p_0\|) \end{aligned} \tag{96}$$

$(U_n$ quasi-nonexpansive)

$$\leq \limsup_{n \rightarrow \infty} (\alpha_n O(1) + \|x_n - p_0\| - \|x_n - p_0\|) = 0,$$

and hence

$$\lim_{n \rightarrow \infty} (\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|) = 0. \tag{97}$$

By (83) we get

$$\begin{aligned} & (1 - \beta_{\tau(n)})(1 - \delta) \|x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \\ & \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)} O(1), \end{aligned} \tag{98}$$

and by (84) we have

$$\begin{aligned} & \beta_{\tau(n)}(1 - \beta_{\tau(n)}) \|T_\delta x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \\ & \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)} O(1). \end{aligned} \tag{99}$$

As in Case A, we get

- (a) $\lim_{n \rightarrow \infty} \|x_{\tau(n)} - Sx_{\tau(n)}\| = 0$,
- (b) $\lim_{n \rightarrow \infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0$,
- (c) $\lim_{n \rightarrow \infty} \|x_{\tau(n)} - U_{\tau(n)} x_{\tau(n)}\| = 0$.

By Lemma 10, (a) and (b) we have

$$\limsup_{n \rightarrow \infty} \langle u - p_0, x_{\tau(n)} - p_0 \rangle \leq 0. \tag{100}$$

Finally, we prove that $(x_n)_{n \in \mathbb{N}}$ converges strongly to p_0 .

As in Case A, using (c), the assumption $\sum_{n=1}^\infty \alpha_n = \infty$, and (100) we can apply Xu's Lemma 11 and conclude that

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - \hat{p}\| = 0, \tag{101}$$

and then from property (d) of Maingé Lemma and (97) we can derive that

$$\lim_{n \rightarrow \infty} \|x_n - p_0\| = 0. \tag{102}$$

□

Remark 15. We remark that in order to prove that

$$\begin{aligned} \|x_n - Tx_n\| & \longrightarrow 0, & \|x_n - Sx_n\| & \longrightarrow 0, \\ \|x_n - U_n x_n\| & \longrightarrow 0, \end{aligned} \tag{103}$$

we do not follow the line of the proof of Song and Chai in [4] because their techniques seem questionable.

Remark 16. The main result of this paper gives a positive answer to the question of Kurokawa and Takahashi; see Remark page 1567 in [19].

Conflict of Interests

The authors declare that they have no competing interests.

References

- [1] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
- [2] C. E. Chidume and C. O. Chidume, "Iterative approximation of fixed points of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 288–295, 2006.
- [3] T. Suzuki, "A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 135, no. 1, pp. 99–106, 2007.
- [4] Y. Song and X. Chai, "Halpern iteration for firmly type nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4500–4506, 2009.
- [5] S. Saejung, "Halpern's iteration in Banach spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 10, pp. 3431–3439, 2010.
- [6] F. Kohsaka and W. Takahashi, "Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces," *Archiv der Mathematik*, vol. 91, no. 2, pp. 166–177, 2008.
- [7] S. Iemoto and W. Takahashi, "Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e2082–e2089, 2009.
- [8] S. Itoh and W. Takahashi, "The common fixed point theory of singlevalued mappings and multivalued mappings," *Pacific Journal of Mathematics*, vol. 79, no. 2, pp. 493–508, 1978.
- [9] M. O. Osilike and F. O. Isiogugu, "Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 5, pp. 1814–1822, 2011.
- [10] A. Moudafi, "Krasnoselski-Mann iteration for hierarchical fixed-point problems," *Inverse Problems*, vol. 23, no. 4, pp. 1635–1640, 2007.
- [11] P. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.
- [12] P. E. Maingé, "The viscosity approximation process for quasicontractive mappings in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 74–79, 2010.
- [13] K. Wongchan and S. Saejung, "On the strong convergence of viscosity approximation process for quasicontractive mappings in Hilbert spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 385843, 9 pages, 2011.
- [14] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.
- [15] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
- [16] F. E. Browder and W. V. Petryshyn, "Semicontractive and semiaccretive nonlinear mappings in Banach spaces," *Bulletin of the American Mathematical Society*, vol. 74, pp. 660–665, 1968.
- [17] A. Moudafi, "Split monotone variational inclusions," *Journal of Optimization Theory and Applications*, vol. 150, no. 2, pp. 275–283, 2011.
- [18] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
- [19] Y. Kurokawa and W. Takahashi, "Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 6, pp. 1562–1568, 2010.