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Research Article

Algorithmic Approach to the Equilibrium Points and Fixed Points

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The equilibrium and fixed point problems are considered. An iterative algorithm is presented. Convergence analysis of the algorithm is provided.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $\Phi : C \to H$ be a nonlinear operator and let $\Theta : C \times C \to R$ be a bifunction. The equilibrium problem is formulated as finding $x^\dagger \in C$ such that

$$\Theta\left(x^\dagger, x\right) + \left\langle \Phi\left(x^\dagger\right), x - x^\dagger \right\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

The solution set of (1) is denoted by EP. The problem (1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others. For related work, please see, for example, [1–17]. Next, we recall several interesting results where Θ verifies the following usual conditions $(C1)$–$(C4)$ which will be used in the sequel:

$(C1)$ $\Theta(u, u) = 0$ for all $u \in C$;

$(C2)$ Θ is monotone; that is, $\Theta(u, v) + \Theta(v, u) \leq 0$ for all $u, v \in C$;

$(C3)$ for each $u, v, w \in C$, $\lim_{t\downarrow 0}\Theta(tw+(1-t)u, v) \leq \Theta(u, v)$;

$(C4)$ for each $u \in C$, $v \mapsto \Theta(u, v)$ is convex and lower semicontinuous.

Theorem 1. *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to R$ be a bifunction which satisfies conditions (C1)–(C4). Let $S : C \to C$ be a nonexpansive mapping. Let $\phi : H \to H$ be a contraction. For $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated by*

$$\begin{gathered} \Theta\left(y_n, x^\dagger\right) + \frac{1}{\lambda_n}\langle x^\dagger - y_n, y_n - x_n \rangle \geq 0, \quad \forall x^\dagger \in C, \\ x_{n+1} = \alpha_n \phi\left(x_n\right) + \left(1 - \alpha_n\right) S y_n, \quad n \geq 0, \end{gathered} \tag{2}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty}\lambda_n > 0$, and $\sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$. Then, the sequence $\{x_n\}$ converges strongly to $x^\ddagger = P_{F(S)\cap EP}\phi(x^\ddagger)$ provided $F(S) \cap EP \neq \emptyset$.

Chuang et al. [18] considered an iteration process of Halpern's type for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points for a nonexpansive mapping with perturbation in a Hilbert space and they proved a strong convergence theorem for such iterations.

Theorem 2. *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to R$ be a bifunction which satisfies conditions (C1)–(C4). Let $S : C \to H$ be a nonexpansive mapping. Let*

$\{u_n\} \subset H$ be a sequence. For $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\Theta\left(y_n, x^{\dagger}\right) + \frac{1}{\lambda_n}\langle x^{\dagger} - y_n, y_n - x_n\rangle \geq 0, \quad \forall x^{\dagger} \in C,$$
$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)\left(\beta_n y_n + (1-\beta_n) S y_n\right), \quad n \geq 0, \tag{3}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [a, \infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\beta_n(1-\beta_n) > 0$, and $\lim_{n\to\infty}u_n = u \in H$. Then, the sequence $\{x_n\}$ converges strongly to $x^{\ddagger} = P_{F(S)\cap EP}(u)$ provided $F(S) \cap EP \neq \emptyset$.

S. Takahashi and W. Takahashi [17] introduced the following iterative algorithm for finding an element of $F(S)\cap \mathrm{EP}$:

$$\Theta\left(y_n, x^{\dagger}\right) + \langle \Phi(x_n), x^{\dagger} - y_n\rangle + \frac{1}{\lambda_n}\langle x^{\dagger} - y_n, y_n - x_n\rangle \geq 0, \quad \forall x^{\dagger} \in C, \tag{4}$$
$$x_{n+1} = \beta_n x_n + S\left[\alpha_n u + (1-\beta_n) y_n\right], \quad n \geq 0.$$

And they proved that the sequence $\{x_n\}$ converges strongly to $x^{\ddagger} = P_{F(S)\cap \mathrm{EP}}(u)$.

Remark 3. Algorithm (3) is involved in a variant anchor $\{u_n\}$ and the parameters are also relaxed. In [17], the authors considered a general equilibrium problem.

In this paper, our main purpose is to introduce a new iteration process for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points for a nonexpansive mapping in a Hilbert space and then we prove a strong convergence theorem for such iterations. Our iterations are very different from (2)–(4). As a special case, we can find the minimum norm solution of $F(S) \cap \mathrm{EP}$.

2. Preliminaries

In the sequel, we assume H is a real Hilbert space. Let $C \subset H$ be a nonempty closed convex set. Recall that a mapping $\Phi : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle \Phi(u) - \Phi(v), u - v\rangle \geq \alpha\|\Phi(u) - \Phi(v)\|^2, \quad \forall u, v \in C. \tag{5}$$

A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C. \tag{6}$$

We use $F(T)$ to denote the set of fixed points of S.

The following lemmas are useful for the next section.

Lemma 4 (see [11]). *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to R$ be a bifunction which satisfies conditions (C1)–(C4). Let $\lambda > 0$ and $x \in H$. Then, there exists $y \in C$ such that*

$$\Theta\left(y, x^{\dagger}\right) + \frac{1}{\lambda}\langle x^{\dagger} - y, y - x\rangle \geq 0, \quad \forall x^{\dagger} \in C. \tag{7}$$

Set $T_\lambda(x) = \{y \in C : \Theta(y, x^{\dagger}) + (1/\lambda)\langle x^{\dagger} - y, y - x\rangle \geq 0$, for all $x^{\dagger} \in C\}$. Then the following hold:

(i) *T_λ is single-valued and T_λ is firmly nonexpansive; that is for any $x, y \in H$, $\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle$;*

(ii) *EP is closed and convex and $EP = F(T_\lambda)$.*

Lemma 5. *Let C, H, F, and $T_\lambda x$ be as in Lemma 4. Then the following holds:*

$$\left\|T_\lambda x - T_\gamma x\right\|^2 \leq \frac{\lambda - \gamma}{\lambda}\langle T_\lambda x - T_\gamma x, T_\lambda x - x\rangle \tag{8}$$

for all $\lambda, \gamma > 0$ and $x \in H$.

Lemma 6 (see [19]). *Let C be a nonempty closed convex subset of H. Let the mapping $\Phi : C \to H$ be α-inverse-strongly monotone and let $\lambda > 0$ be a constant. Then, we have*

$$\|(I - \lambda\Phi) u - (I - \lambda\Phi) v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|\Phi(u) - \Phi(v)\|^2, \quad \forall u, v \in C. \tag{9}$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda\Phi$ is nonexpansive.

Lemma 7 (see [20]). *Let C be a closed convex subset of H and let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 8 (see [21]). *Let $\{R_n\}$ and $\{W_n\}$ be two bounded sequences in H. Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $R_{n+1} = (1-\beta_n)W_n + \beta_n R_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|W_{n+1} - W_n\| - \|R_{n+1} - R_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|W_n - R_n\| = 0$.*

Lemma 9 (see [22]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n) a_n + \delta_n \gamma_n, \tag{10}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) *$\sum_{n=1}^{\infty}\gamma_n = \infty$;*

(2) *$\limsup_{n\to\infty}\delta_n \leq 0$ or $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.*

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

In this section, we will prove our main results.

Theorem 10. *Let C be a nonempty closed convex subset of H and let $\Theta : C \times C \to R$ be a bifunction satisfying conditions (C1)–(C4). Let $\Phi : C \to H$ be an ς-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping.*

Suppose that $F(S) \cap EP \neq \emptyset$. Let $\{u_n\}$ be a sequence in H. For $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \varpi_n x_n + (1-\varpi_n) ST_{\rho_n}\left[(1-\varsigma_n) x_n + \varsigma_n u_n - \rho_n \Phi(x_n)\right], \quad n \geq 0, \tag{11}$$

where T_{ρ_n} is defined as that in Lemma 4 and $\{\rho_n\} \subset (0, 2\varsigma)$, $\{\varsigma_n\} \subset (0,1)$, and $\{\varpi_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty}\varsigma_n = 0$ *and* $\sum_{n=1}^{\infty}\varsigma_n = \infty$;

(ii) $0 < c \leq \varpi_n \leq d < 1$;

(iii) $a(1-\varsigma_n) \leq \rho_n \leq b(1-\varsigma_n)$, *where* $[a,b] \subset (0,2\varsigma)$ *and* $\lim_{n\to\infty}(\rho_{n+1}-\rho_n) = 0$;

(iv) $\lim_{n\to\infty}u_n = u$.

Then $\{x_n\}$ generated by (11) converges strongly to $P_{F(S)\cap EP}(u)$.

We divide our proofs into several conclusions.

Conclusion 1. The sequence $\{x_n\}$ is bounded.

Proof. Let $z \in F(S)\cap \mathrm{EP}$. We have $z = Sz = T_{\rho_n}(z-\rho_n\Phi(z)) = T_{\rho_n}[\varsigma_n z + (1-\varsigma_n)(z-\rho_n\Phi(z)/(1-\varsigma_n))]$ for all $n \geq 0$. Set $z_n = T_{\rho_n}[(1-\varsigma_n)x_n + \varsigma_n u_n - \rho_n\Phi(x_n)]$ for all $n \geq 0$. By Lemma 4, we know that T_{ρ_n} is nonexpansive. By the convexity of $\|\cdot\|$, we derive

$$\begin{aligned}
\|z_n - z\|^2 &= \left\|T_{\rho_n}\left[(1-\varsigma_n)x_n + \varsigma_n u_n - \rho_n\Phi(x_n)\right] - T_{\rho_n}(z-\rho_n\Phi(z))\right\|^2 \\
&= \left\|T_{\rho_n}\left[\varsigma_n u_n + (1-\varsigma_n)\left(x_n - \frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\right)\right] - T_{\rho_n}\left[\varsigma_n z + (1-\varsigma_n)\left(z - \frac{\rho_n}{1-\varsigma_n}\Phi(z)\right)\right]\right\|^2 \\
&\leq \left\|\left[\varsigma_n u_n + (1-\varsigma_n)\left(x_n - \frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\right)\right] - \left[\varsigma_n z + (1-\varsigma_n)\left(z - \frac{\rho_n}{1-\varsigma_n}\Phi(z)\right)\right]\right\|^2 \\
&= \left\|(1-\varsigma_n)\left[\left(x_n - \frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\right) - \left(z - \frac{\rho_n}{1-\varsigma_n}\Phi(z)\right)\right] + \varsigma_n(u_n - z)\right\|^2 \\
&\leq (1-\varsigma_n)\left\|\left(x_n - \frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\right) - \left(z - \frac{\rho_n}{1-\varsigma_n}\Phi(z)\right)\right\|^2 + \varsigma_n\|u_n - z\|^2.
\end{aligned} \tag{12}$$

Since Φ is ς-inverse-strongly monotone, we know from Lemma 6 that

$$\begin{aligned}
&\left\|\left(x_n - \frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\right) - \left(z - \frac{\rho_n}{1-\varsigma_n}\Phi(z)\right)\right\|^2 \\
&\quad\leq \|x_n - z\|^2 + \frac{\rho_n(\rho_n - 2(1-\varsigma_n)\varsigma)}{(1-\varsigma_n)^2}\|\Phi(x_n) - \Phi(z)\|^2.
\end{aligned} \tag{13}$$

It follows that

$$\begin{aligned}
\|z_n - z\|^2 &\leq (1-\varsigma_n)\left(\|x_n - z\|^2 + \frac{\rho_n(\rho_n - 2(1-\varsigma_n)\varsigma)}{(1-\varsigma_n)^2} \times \|\Phi(x_n) - \Phi(z)\|^2\right) + \varsigma_n\|u_n - z\|^2, \\
&\leq (1-\varsigma_n)\|x_n - z\|^2 + \varsigma_n\|u_n - z\|^2.
\end{aligned} \tag{14}$$

So, we have that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\varpi_n(x_n - z) + (1-\varpi_n)(Sz_n - z)\|^2 \\
&\leq \varpi_n\|x_n - z\|^2 + (1-\varpi_n)\|z_n - z\|^2 \\
&\leq \varpi_n\|x_n - z\|^2 + (1-\varpi_n)\left((1-\varsigma_n)\|x_n - z\|^2 + \varsigma_n\|u_n - z\|^2\right) \\
&= [1-(1-\varpi_n)\varsigma_n]\|x_n - z\|^2 + (1-\varpi_n)\varsigma_n\|u_n - z\|^2 \\
&\leq \max\left\{\|u_n - z\|^2, \|u_n - z\|^2\right\}.
\end{aligned} \tag{15}$$

Note that $\lim_{n\to\infty}u_n = u \in H$. Without loss of generality, we can assume that $\sup_n\|u_n - u\| \leq M$ for some $M > 0$.

By induction, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \max\left\{\|x_0 - z\|^2, (\|u_n - u\| + \|u - z\|)^2\right\} \\
&\leq \max\left\{\|x_0 - z\|^2, (M + \|u - z\|)^2\right\}.
\end{aligned} \tag{16}$$

Therefore, $\{x_n\}$ is bounded. □

Conclusion 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_{\rho_n}[(1-\varsigma_n)x_n + \varsigma_n u_n - \rho_n\Phi(x_n)]\| = 0$.

Proof. Putting $w_n = (1-\varsigma_n)x_n + \varsigma_n u_n - \rho_n\Phi(x_n)$ for all $n \geq 0$, we have

$$z_{n+1} - z_n = T_{\rho_{n+1}}w_{n+1} - T_{\rho_{n+1}}w_n + T_{\rho_{n+1}}w_n - T_{\rho_n}w_n. \tag{17}$$

It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \left\|T_{\rho_{n+1}}w_{n+1} - T_{\rho_{n+1}}w_n\right\| + \left\|T_{\rho_{n+1}}w_n - T_{\rho_n}w_n\right\| \\
&\leq \|w_{n+1} - w_n\| + \left\|T_{\rho_{n+1}}w_n - T_{\rho_n}w_n\right\|.
\end{aligned} \tag{18}$$

From Lemma 6, we know that $I-\rho\Phi$ is nonexpansive for all $\rho\in(0,2\varsigma)$. Thus, we have that $I-((\rho_{n+1})/(1-\varsigma_{n+1}))\Phi$ is nonexpansive for all n due to the fact that $(\rho_{n+1})/(1-\varsigma_{n+1})\in(0,2\varsigma)$. Then, we get

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|(1-\varsigma_{n+1})x_{n+1}+\varsigma_{n+1}u_{n+1}-\rho_{n+1}\Phi(x_{n+1}) \\ &\quad -((1-\varsigma_n)x_n+\varsigma_n u_n-\rho_n\Phi(x_n))\| \\ &= \Big\|(1-\varsigma_{n+1})\Big(x_{n+1}-\frac{\rho_{n+1}}{1-\varsigma_{n+1}}\Phi(x_{n+1})\Big) \\ &\quad -(1-\varsigma_n)\Big(x_n-\frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\Big) \\ &\quad +(\varsigma_{n+1}-\varsigma_n)u_{n+1}+\varsigma_n(u_{n+1}-u_n)\Big\| \\ &\le (1-\varsigma_{n+1})\Big\|\Big(I-\frac{\rho_{n+1}}{1-\varsigma_{n+1}}\Phi\Big)x_{n+1} \\ &\quad -\Big(I-\frac{\rho_{n+1}}{1-\varsigma_{n+1}}\Phi\Big)x_n\Big\| \\ &\quad +\|(1-\varsigma_{n+1})x_n-\rho_{n+1}\Phi(x_n) \\ &\quad -(1-\varsigma_n)x_n+\rho_n\Phi(x_n)\| \\ &\quad +|\varsigma_{n+1}-\varsigma_n|\,\|u_{n+1}\|+\varsigma_n\|u_{n+1}-u_n\| \\ &\le \|x_{n+1}-x_n\|+|\varsigma_{n+1}-\varsigma_n|(\|x_n\|+\|u_{n+1}\|) \\ &\quad +\varsigma_n\|u_{n+1}-u_n\|+|\rho_{n+1}-\rho_n|\,\|\Phi(x_n)\|.\end{aligned} \tag{19}$$

By Lemma 5, we have

$$\|T_{\rho_{n+1}}w_n-T_{\rho_n}w_n\|\le\frac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|T_{\rho_{n+1}}w_n-w_n\|. \tag{20}$$

From (18)–(20), we obtain

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|x_{n+1}-x_n\|+|\varsigma_{n+1}-\varsigma_n|(\|x_n\|+\|u_{n+1}\|) \\ &\quad +\varsigma_n\|u_{n+1}-u_n\|+|\rho_{n+1}-\rho_n|\,\|\Phi(x_n)\| \\ &\quad +\frac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|T_{\rho_{n+1}}w_n-w_n\|.\end{aligned} \tag{21}$$

Then,

$$\begin{aligned}\|Sz_{n+1}-Sz_n\| &\le \|z_{n+1}-z_n\| \\ &\le \|x_{n+1}-x_n\|+|\varsigma_{n+1}-\varsigma_n| \\ &\quad \times(\|x_n\|+\|u_{n+1}\|)+\varsigma_n\|u_{n+1}-u_n\| \\ &\quad +|\rho_{n+1}-\rho_n|\,\|\Phi(x_n)\| \\ &\quad +\frac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|T_{\rho_{n+1}}w_n-w_n\|.\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}\|Sz_{n+1}-Sz_n\|-\|x_{n+1}-x_n\| &\le |\varsigma_{n+1}-\varsigma_n|(\|x_n\|+\|u_{n+1}\|) \\ &\quad +\varsigma_n\|u_{n+1}-u_n\| \\ &\quad +|\rho_{n+1}-\rho_n|\,\|\Phi(x_n)\| \\ &\quad +\frac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|T_{\rho_{n+1}}w_n-w_n\|.\end{aligned} \tag{23}$$

Since $\varsigma_n\to 0$, $\rho_{n+1}-\rho_n\to 0$ and $\liminf_{n\to\infty}\rho_n>0$, we obtain

$$\limsup_{n\to\infty}(\|Sz_{n+1}-Sz_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{24}$$

By Lemma 8, we get

$$\lim_{n\to\infty}\|Sz_n-x_n\|=0. \tag{25}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\varpi_n)\|Sz_n-x_n\|=0. \tag{26}$$

From (11) and (14), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \varpi_n\|x_n-z\|^2+(1-\varpi_n) \\ &\quad \times\|ST_{\rho_n}[(1-\varsigma_n)x_n+\varsigma_n u_n-\rho_n\Phi(x_n)]-z\|^2 \\ &\le (1-\varpi_n)\Big\{(1-\varsigma_n)\Big(\|x_n-z\|^2+\frac{\rho_n}{(1-\varsigma_n)^2} \\ &\quad \times(\rho_n-2(1-\varsigma_n)\varsigma) \\ &\quad \times\|\Phi(x_n)-\Phi(z)\|^2\Big) \\ &\quad +\varsigma_n\|u_n-z\|^2\Big\}+\varpi_n\|x_n-z\|^2 \\ &= [1-(1-\varpi_n)\varsigma_n]\|x_n-z\|^2+\frac{(1-\varpi_n)\rho_n}{1-\varsigma_n} \\ &\quad \times(\rho_n-2(1-\varsigma_n)\varsigma)\|\Phi(x_n)-\Phi(z)\|^2 \\ &\quad +(1-\varpi_n)\varsigma_n\|u_n-z\|^2 \\ &\le \|x_n-z\|^2+\frac{(1-\varpi_n)\rho_n}{1-\varsigma_n}(\rho_n-2(1-\varsigma_n)\varsigma) \\ &\quad \times\|\Phi(x_n)-\Phi(z)\|^2+(1-\varpi_n)\varsigma_n\|u_n-z\|^2.\end{aligned} \tag{27}$$

Then, we obtain

$$\begin{aligned} &\frac{(1-\varpi_n)\rho_n}{1-\varsigma_n}\left(2\left(1-\varsigma_n\right)\varsigma-\rho_n\right)\|\Phi(x_n)-\Phi(z)\|^2 \\ &\quad \le \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\varpi_n)\varsigma_n\|u_n-z\|^2 \\ &\quad \le \left(\|x_n-z\|-\|x_{n+1}-z\|\right)\|x_{n+1}-x_n\| \\ &\qquad +(1-\varpi_n)\varsigma_n\|u_n-z\|^2. \end{aligned} \tag{28}$$

Since $\lim_{n\to\infty}\varsigma_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\liminf_{n\to\infty}((1-\varpi_n)\rho_n/(1-\varsigma_n))(2(1-\varsigma_n)\varsigma-\rho_n) > 0$, we have

$$\lim_{n\to\infty}\|\Phi(x_n)-\Phi(z)\| = 0. \tag{29}$$

Next, we show $\|x_n - T_{\rho_n}w_n\| \to 0$. By using the firm nonexpansivity of T_{ρ_n}, we have

$$\begin{aligned} \left\|T_{\rho_n}w_n-z\right\|^2 &= \left\|T_{\rho_n}w_n-T_{\rho_n}(z-\rho_n\Phi(z))\right\|^2 \\ &\le \left\langle w_n-(z-\rho_n\Phi(z)), T_{\rho_n}w_n-z\right\rangle \\ &= \frac{1}{2}\Big(\|w_n-(z-\rho_n\Phi(z))\|^2+\left\|T_{\rho_n}w_n-z\right\|^2 \\ &\qquad -\big\|(1-\varsigma_n)x_n+\varsigma_n u_n \\ &\qquad\quad -\rho_n\left(\Phi(x_n)-\Phi(z)\right)-T_{\rho_n}w_n\big\|^2\Big). \end{aligned} \tag{30}$$

We note that

$$\|w_n-(z-\rho_n\Phi(z))\|^2 \le (1-\varsigma_n)\|x_n-z\|^2+\varsigma_n\|u_n-z\|^2. \tag{31}$$

Thus,

$$\begin{aligned} \left\|T_{\rho_n}w_n-z\right\|^2 \le \frac{1}{2}\Big(&(1-\varsigma_n)\|x_n-z\|^2 \\ &+\varsigma_n\|u_n-z\|^2+\left\|T_{\rho_n}w_n-z\right\|^2 \\ &-\big\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n \\ &\quad -\rho_n\left(\Phi(x_n)-\Phi(z)\right)\big\|^2\Big); \end{aligned} \tag{32}$$

that is,

$$\begin{aligned} \left\|T_{\rho_n}w_n-z\right\|^2 &\le (1-\varsigma_n)\|x_n-z\|^2+\varsigma_n\|u_n-z\|^2 \\ &\quad -\big\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n \\ &\qquad -\rho_n\left(\Phi(x_n)-\Phi(z)\right)\big\|^2 \\ &= (1-\varsigma_n)\|x_n-z\|^2+\varsigma_n\|u_n-z\|^2 \\ &\quad -\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|^2 \\ &\quad +2\rho_n\big\langle(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n, \Phi(x_n) \\ &\qquad -\Phi(z)\big\rangle-\rho_n^2\|\Phi(x_n)-\Phi(z)\|^2 \\ &\le (1-\varsigma_n)\|x_n-z\|^2+\varsigma_n\|u_n-z\|^2 \\ &\quad -\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|^2 \\ &\quad +2\rho_n\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\| \\ &\quad \times\|\Phi(x_n)-\Phi(z)\|. \end{aligned} \tag{33}$$

It follows that

$$\begin{aligned} \|x_{n+1}-z\|^2 &\le \varpi_n\|x_n-z\|^2+(1-\varpi_n)(1-\varsigma_n) \\ &\quad \times\|x_n-z\|^2+(1-\varpi_n)\varsigma_n\|u_n-z\|^2 \\ &\quad -(1-\varpi_n)\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|^2 \\ &\quad +2\rho_n(1-\varpi_n)\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\| \\ &\quad \times\|\Phi(x_n)-\Phi(z)\| \\ &= \left[1-(1-\varpi_n)\varsigma_n\right]\|x_n-z\|^2 \\ &\quad +(1-\varpi_n)\varsigma_n\|u_n-z\|^2-(1-\varpi_n) \\ &\quad \times\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|^2 \\ &\quad +2\rho_n(1-\varpi_n)\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\| \\ &\quad \times\|\Phi(x_n)-\Phi(z)\|. \end{aligned} \tag{34}$$

Hence,

$$\begin{aligned} &(1-\varpi_n)\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|^2 \\ &\quad \le \|x_n-z\|^2-\|x_{n+1}-z\|^2 \\ &\qquad -(1-\varpi_n)\varsigma_n\|x_n-z\|^2 \\ &\qquad +(1-\varpi_n)\varsigma_n\|u_n-z\|^2+2\rho_n(1-\varpi_n) \\ &\qquad \times\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|\|\Phi(x_n)-\Phi(z)\| \\ &\quad \le \left(\|x_n-z\|+\|x_{n+1}-z\|\right)\|x_{n+1}-x_n\| \\ &\qquad +(1-\varpi_n)\varsigma_n\|u_n-z\|^2+2\rho_n(1-\varpi_n) \\ &\qquad \times\left\|(1-\varsigma_n)x_n+\varsigma_n u_n-T_{\rho_n}w_n\right\|\|\Phi(x_n)-\Phi(z)\|. \end{aligned} \tag{35}$$

Since $\limsup_{n\to\infty}\varpi_n < 1$, $\|x_{n+1} - x_n\| \to 0$, $\varsigma_n \to 0$, and $\|\Phi(x_n) - \Phi(z)\| \to 0$, we deduce

$$\lim_{n\to\infty}\left\|(1-\varsigma_n)x_n + \varsigma_n u_n - T_{\rho_n}w_n\right\| = 0. \tag{36}$$

This implies that

$$\lim_{n\to\infty}\left\|x_n - T_{\rho_n}w_n\right\| = \lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{37}$$

□

Conclusion 3. Consider

$$\limsup_{n\to\infty}\langle\tilde{x} - u, x_n - \tilde{x}\rangle \geq 0, \tag{38}$$

where $\tilde{x} = P_{F(S)\cap EP}(u)$.

Proof. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to w$ weakly and

$$\limsup_{n\to\infty}\langle\tilde{x} - u, x_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle\tilde{x} - u, x_{n_i} - \tilde{x}\rangle. \tag{39}$$

By (25) and (37), we deduce

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{40}$$

Hence,

$$\lim_{i\to\infty}\left\|x_{n_i} - Sx_{n_i}\right\| = 0. \tag{41}$$

This together with Lemma 7 implies that $w \in F(S)$.

Next we show that $\tilde{x} \in \mathrm{EP}$. Since $z_n = T_{\rho_n}[(1-\varsigma_n)x_n + \varsigma_n u_n - \rho_n\Phi(x_n)]$ for any $y \in C$, we have

$$\begin{aligned}&\Theta(z_n, y) + \langle\Phi(x_n), y - z_n\rangle \\ &\quad + \frac{1}{\rho_n}\langle y - z_n, z_n - (1-\varsigma_n)x_n - \varsigma_n u_n\rangle \geq 0.\end{aligned} \tag{42}$$

From $(C2)$, we have

$$\begin{aligned}&\langle\Phi(x_n), y - z_n\rangle + \frac{1}{\rho_n}\langle y - z_n, z_n - (1-\varsigma_n)x_n - \varsigma_n u_n\rangle \\ &\quad \geq \Theta(y, z_n).\end{aligned} \tag{43}$$

Put $x_t = ty + (1-t)\tilde{x}$ for all $t \in (0,1)$ and $y \in C$. Then, we have $x_t \in C$. So, from (43), we have

$$\begin{aligned}\langle x_t - z_n, \Phi(x_t)\rangle &\geq \langle x_t - z_n, \Phi(x_t)\rangle - \langle\Phi(x_n), x_t - z_n\rangle \\ &\quad - \frac{1}{\rho_n}\langle x_t - z_n, z_n - (1-\varsigma_n)x_n - \varsigma_n u_n\rangle \\ &\quad + \Theta(x_t, z_n) \\ &= \langle x_t - z_n, \Phi(x_t) - \Phi(z_n)\rangle \\ &\quad + \langle x_t - z_n, \Phi(z_n) - \Phi(x_n)\rangle \\ &\quad - \frac{1}{\rho_n}\langle x_t - z_n, z_n - (1-\varsigma_n)x_n - \varsigma_n u_n\rangle \\ &\quad + \Theta(x_t, z_n).\end{aligned} \tag{44}$$

Since Φ is ς-inverse-strongly monotone, Φ is $(1/\varsigma)$-Lipschitzian. By (37), we derive that $\|\Phi(z_n) - \Phi(x_n)\| \to 0$. Further, from monotonicity of Φ, we have $\langle x_t - z_n, \Phi(x_t) - \Phi(z_n)\rangle \geq 0$. Letting $n \to \infty$ in (45) and noting $(C4)$, we have

$$\langle x_t - \tilde{x}, \Phi(x_t)\rangle \geq \Theta(x_t, \tilde{x}). \tag{45}$$

By $(C1)$, $(C4)$, and (45), we deduce

$$\begin{aligned}0 &= \Theta(x_t, x_t) \\ &\leq t\Theta(x_t, y) + (1-t)\Theta(x_t, \tilde{x}) \\ &\leq t\Theta(x_t, y) + (1-t)\langle x_t - \tilde{x}, \Phi(x_t)\rangle \\ &= t\Theta(x_t, y) + (1-t)t\langle y - \tilde{x}, \Phi(x_t)\rangle,\end{aligned} \tag{46}$$

and hence

$$0 \leq \Theta(x_t, y) + (1-t)\langle y - \tilde{x}, \Phi(x_t)\rangle. \tag{47}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta(\tilde{x}, y) + \langle y - \tilde{x}, \Phi(\tilde{x})\rangle. \tag{48}$$

This implies $\tilde{x} \in \mathrm{EP}$. Therefore, we have $w \in F(S) \cap \mathrm{EP}$. So,

$$\begin{aligned}\limsup_{n\to\infty}\langle\tilde{x} - u, x_n - \tilde{x}\rangle &= \lim_{i\to\infty}\left\langle\tilde{x} - u, x_{n_i} - \tilde{x}\right\rangle \\ &= \langle\tilde{x} - u, w - \tilde{x}\rangle \geq 0.\end{aligned} \tag{49}$$

Setting $v_n = x_n - (\rho_n/(1-\varsigma_n))(\Phi(x_n) - \Phi(\tilde{x}))$ for all n and taking $z = \tilde{x}$ in (29) to get $\|\Phi(x_n) - \Phi(\tilde{x})\| \to 0$, so, $v_n - x_n \to 0$. Since $u_n \to u$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u_n - \tilde{x}, v_n - \tilde{x}\rangle &= \limsup_{n\to\infty}\langle u - \tilde{x}, v_n - \tilde{x}\rangle \\ &= \limsup_{n\to\infty}\langle u - \tilde{x}, x_n - \tilde{x}\rangle \leq 0.\end{aligned} \tag{50}$$

□

Conclusion 4 ($x_n \to \tilde{x}$).

Proof. From (11), we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\leq \varpi_n\|x_n - \tilde{x}\|^2 + (1-\varpi_n)\left\|ST_{\rho_n}w_n - \tilde{x}\right\|^2 \\ &\leq \varpi_n\|x_n - \tilde{x}\|^2 + (1-\varpi_n)\left\|T_{\rho_n}w_n - \tilde{x}\right\|^2 \\ &= \varpi_n\|x_n - \tilde{x}\|^2 + (1-\varpi_n) \\ &\quad \times \left\|T_{\rho_n}w_n - T_{\rho_n}(\tilde{x} - \rho_n\Phi(\tilde{x}))\right\|^2\end{aligned}$$

$$\begin{aligned}
&\le \varpi_n\|x_n-\tilde{x}\|^2+(1-\varpi_n)\|w_n-(\tilde{x}-\rho_n\Phi(\tilde{x}))\|^2\\
&=\varpi_n\|x_n-\tilde{x}\|^2+(1-\varpi_n)\\
&\quad\times\|(1-\varsigma_n)x_n+\varsigma_n u_n-\rho_n\Phi(x_n)-(\tilde{x}-\rho_n\Phi(\tilde{x}))\|^2\\
&=(1-\varpi_n)\Big\|(1-\varsigma_n)\Big(\Big(x_n-\frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\Big)\\
&\qquad-\Big(\tilde{x}-\frac{\rho_n}{1-\varsigma_n}\Phi(\tilde{x})\Big)\Big)\\
&\qquad+\varsigma_n(u_n-\tilde{x})\Big\|^2\\
&\quad+\varpi_n\|x_n-\tilde{x}\|^2\\
&=(1-\varpi_n)\Big((1-\varsigma_n)^2\Big\|\Big(x_n-\frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\Big)\\
&\qquad-\Big(\tilde{x}-\frac{\rho_n}{1-\varsigma_n}\Phi(\tilde{x})\Big)\Big\|^2\\
&\qquad+2\varsigma_n(1-\varsigma_n)\\
&\qquad\times\Big\langle u_n-\tilde{x},\Big(x_n-\frac{\rho_n}{1-\varsigma_n}\Phi(x_n)\Big)\\
&\qquad-\Big(\tilde{x}-\frac{\rho_n}{1-\varsigma_n}\Phi(\tilde{x})\Big)\Big\rangle\\
&\qquad+\varsigma_n^2\|u_n-\tilde{x}\|^2\Big)\\
&\quad+\varpi_n\|x_n-\tilde{x}\|^2\\
&\le\varpi_n\|x_n-\tilde{x}\|^2+(1-\varpi_n)\\
&\quad\times\Big((1-\varsigma_n)^2\|x_n-\tilde{x}\|^2+2\varsigma_n(1-\varsigma_n)\\
&\qquad\times\Big\langle u_n-\tilde{x},x_n-\frac{\rho_n}{1-\varsigma_n}\\
&\qquad\times(\Phi(x_n)-\Phi(\tilde{x}))\\
&\qquad-\tilde{x}\Big\rangle\\
&\qquad+\varsigma_n^2\|u_n-\tilde{x}\|^2\Big)\\
&\le[1-(1-\varpi_n)\varsigma_n]\|x_n-\tilde{x}\|^2+(1-\varpi_n)\varsigma_n\\
&\quad\times\left\{2(1-\varsigma_n)\langle u_n-\tilde{x},v_n-\tilde{x}\rangle+\varsigma_n\|u_n-\tilde{x}\|^2\right\}.
\end{aligned}\tag{51}$$

It is clear that $\sum_{n=1}^{\infty}(1-\varpi_n)\varsigma_n=\infty$ and $\limsup_{n\to\infty}(2(1-\varsigma_n)\langle u_n-\tilde{x},v_n-\tilde{x}\rangle+\varsigma_n\|u_n-\tilde{x}\|^2)\le 0$. We can therefore apply Lemma 9 to conclude that $x_n\to\tilde{x}$. This completes the proof. □

Corollary 11. *Let C be a nonempty closed convex subset of H and let $\Theta: C\times C\to R$ be a bifunction satisfying conditions (C1)–(C4). Let $\Phi: C\to H$ be an ς-inverse-strongly monotone mapping and let $S: C\to C$ be a nonexpansive mapping. Suppose that $F(S)\cap EP\ne\emptyset$. For $x_0\in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1}=\varpi_n x_n+(1-\varpi_n)ST_{\rho_n}\left[(1-\varsigma_n)x_n-\rho_n\Phi(x_n)\right],\quad n\ge 0,\tag{52}$$

where T_{ρ_n} is defined as that in Lemma 4 and $\{\rho_n\}\subset(0,2\varsigma)$, $\{\varsigma_n\}\subset(0,1)$, and $\{\varpi_n\}\subset(0,1)$ satisfy

(i) $\lim_{n\to\infty}\varsigma_n=0$ *and* $\sum_{n=1}^{\infty}\varsigma_n=\infty$;

(ii) $0<c\le\varpi_n\le d<1$;

(iii) $a(1-\varsigma_n)\le\rho_n\le b(1-\varsigma_n)$, *where* $[a,b]\subset(0,2\varsigma)$ *and* $\lim_{n\to\infty}(\rho_{n+1}-\rho_n)=0$.

Then $\{x_n\}$ generated by (52) converges strongly to $P_{F(S)\cap EP}(0)$ which is the minimum norm element in $F(S)\cap EP$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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