Research Article

On Certain Subclass of Harmonic Starlike Functions

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Coefficient conditions, distortion bounds, extreme points, convolution, convex combinations, and neighborhoods for a new class of harmonic univalent functions in the open unit disc are investigated. Further, a class preserving integral operator and connections with various previously known results are briefly discussed.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. There is a close interrelation between analytic functions and harmonic functions. For example, for real harmonic functions u and v, there exist analytic functions U and V so that $u = \text{Re}(U)$ and $v = \text{Im}(V)$. Then $f(z) = h(z) + \overline{g(z)}$, where h and g are, respectively, the analytic functions $(U + V)/2$ and $(U - V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by $J_f = |h'(z)|^2 - |g'(z)|^2$. The mapping $z \to f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f > 0$ in D. The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \to f(z)$ is orientation preserving, harmonic, and one-to-one in D. We call h the analytic part and g the coanalytic part of f (see Clunie and Sheil-Small [1]).

Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

Note that $\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the coanalytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

A function $f = h + \overline{g}$ with h and g given by (1) is said to be harmonic starlike of order β ($0 \le \beta < 1$) for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta} \left(\arg f\left(re^{i\theta}\right) \right) = \text{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) \ge \beta. \tag{3}$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}_H^*(\beta)$ and extensively studied by Jahangiri [2]. The cases $\beta = 0$ and $b_1 = 0$ were studied by Silverman and Silvia [3] and Silverman [4]. Other related works of the class $\mathcal{H}$ also appeared in [5–16].

Definition 1. Let $f = h + \overline{g}$ where h and g are given by (1). Let $0 \le \beta < 1$ and $\alpha \ge 0$. Then $f \in \mathcal{S}_H^*(\alpha, \beta)$ if and only if

$$\text{Re}\left(\frac{\alpha z^2 h''(z) + zh'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) zg'(z)}}{h(z) + \overline{g(z)}} \right)$$
$$\ge \beta. \tag{4}$$

We note that for $\alpha = 0$, the class $\mathcal{S}_H^*(\alpha, \beta)$ reduces to the class $\mathcal{S}_H^*(\beta)$. Further, if the coanalytic part $g(z)$ is zero, the

class $\mathcal{S}_H^*(\alpha, \beta)$ reduces to the class $\mathcal{P}(\alpha, \beta)$ of functions $f \in \mathcal{S}$ which satisfy the condition

$$\text{Re}\left(\frac{\alpha z^2 f''(z) + z f'(z)}{f(z)}\right) \geq \beta \tag{5}$$

for some β $(0 \leq \beta < 1)$, $\alpha \geq 0$, $f(z)/z \neq 0$ and $z \in \mathbb{U}$. Observe that the classes $\mathcal{P}(\alpha, \beta)$ and $\mathcal{P}(\alpha, 0)$ were introduced and studied by many authors and these include, for example, by Obradovic and Joshi [17], Padmanabhan [18], Li and Owa [19], Xu and Yang [20], Singh and Gupta [21], and Lashin [22]. We also note that for $\alpha = 0$, the class $\mathcal{P}(0, \beta)$ was studied by Silverman [23].

2. Main Results

The first theorem of this section determines the sufficient coefficient condition for functions $f = h + \overline{g}$ to belong to the class $\mathcal{S}_H^*(\alpha, \beta)$. The following lemma obtained by Jahangiri is needed.

Lemma 2 (see [2, Theorem 1]). *Let* $f = h + \overline{g}$ *with* h *and* g *of the form* (1) *and let*

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| + \sum_{n=1}^{\infty} \frac{n + \beta}{1 - \beta} |b_n| \leq 1, \tag{6}$$

where $0 \leq \beta < 1$. *Then* f *is harmonic, orientation preserving, and univalent in* $\mathbb{U}$, *and* $f \in \mathcal{S}_H^*(\beta)$.

Theorem 3. *Let* $f = h + \overline{g}$ *where* h *and* g *are of the form* (1). *If*

$$\sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_n|$$
$$+ \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_n| \leq 1 \tag{7}$$

for some β, $(0 \leq \beta < 1)$ *and* $\alpha \geq 0$, *then* f *is harmonic, orientation preserving, and univalent in* $\mathbb{U}$ *and* $f \in \mathcal{S}_H^*(\alpha, \beta)$.

Proof. Since $(n - \beta) \leq \alpha n(n - 1) + n - \beta$ and $n + \beta \leq \alpha n(n + 1) + n + \beta$, $(n \geq 1)$, it follows from Lemma 2 that $f \in \mathcal{S}_H^*(\beta)$ and hence f is harmonic, orientation preserving, and univalent in $\mathbb{U}$. Now, we only need to show that if (7) holds then

$$\text{Re}\left(\frac{\alpha z^2 h''(z) + z h'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) z g'(z)}}{h(z) + \overline{g(z)}}\right)$$
$$= \text{Re}\,\frac{A(z)}{B(z)} \geq \beta. \tag{8}$$

Using the fact that $\text{Re}(w) \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$, it suffices to show that

$$|A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)| \geq 0, \tag{9}$$

where

$$A(z) = \alpha z^2 h''(z) + z h'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) z g'(z)},$$
$$B(z) = h(z) + \overline{g(z)}. \tag{10}$$

Substituting for $A(z)$ and $B(z)$ in (9), we obtain

$$|A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)|$$

$$= \left| (2 - \beta) z + \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) + 2 - \beta) a_n z^n \right.$$

$$\left. + \sum_{n=1}^{\infty} (\alpha n(n+1) - (n - 1 + \beta)) \overline{b_n z^n} \right|$$

$$- \left| -\beta z + \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) - \beta) a_n z^n \right.$$

$$\left. + \sum_{n=1}^{\infty} (\alpha n(n+1) - (n + 1 + \beta)) \overline{b_n z^n} \right|$$

$$\geq (2 - \beta) |z|$$

$$- \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) + 2 - \beta) |a_n| |z|^n$$

$$- \sum_{n=1}^{\infty} |\alpha n(n+1) - (n - 1 + \beta)| |b_n| |z|^n - \beta |z|$$

$$- \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) - \beta) |a_n| |z|^n$$

$$- \sum_{n=1}^{\infty} |\alpha n(n+1) - (n + 1 + \beta)| |b_n| |z|^n$$

$$\geq 2(1 - \beta) |z|$$

$$- 2 \sum_{n=2}^{\infty} (\alpha n(n - 1) + n - \beta) |a_n| |z|^n$$

$$- 2 \sum_{n=1}^{\infty} (\alpha n(n+1) + (n + \beta)) |b_n| |z|^n$$

$$\geq 2(1 - \beta) |z|$$

$$\times \left(1 - \sum_{n=2}^{\infty} \frac{\alpha n(n - 1) + n - \beta}{1 - \beta} |a_n| \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + (n + \beta)}{1 - \beta} |b_n| \right) \geq 0, \tag{11}$$

by the given condition (7).

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} x_n z^n$$

$$+ \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n+1) + (n+\beta)} \overline{y}_n \overline{z}^n, \tag{12}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$ shows that the coefficient bound given in (7) is sharp. The functions of the form (12) are in $\mathscr{S}_H^*(\alpha, \beta)$ since

$$\sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \tag{13}$$

$$= \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1.$$

$\square$

Remark 4. Setting $g(z) = 0$ in Theorem 3 yields the result obtained by Lashin [22, Theorem 2.1].

We denote by $\overline{\mathscr{S}}_H^*(\alpha, \beta)$ the class of functions $f \in \mathscr{S}_H^*(\alpha, \beta)$ whose coefficients satisfy the condition (7).

Theorem 5. *Let* $0 \leq \alpha_1 < \alpha_2$ *and* $0 \leq \beta < 1$. *Then* $\overline{\mathscr{S}}_H^*(\alpha_2, \beta) \subset \overline{\mathscr{S}}_H^*(\alpha_1, \beta)$.

Proof. For $f \in \overline{\mathscr{S}}_H^*(\alpha_2, \beta)$, it follows from (7) that

$$\sum_{n=2}^{\infty} \frac{\alpha_1 n(n-1) + n - \beta}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{\alpha_1 n(n+1) + n + \beta}{1-\beta} |b_n|$$

$$< \sum_{n=2}^{\infty} \frac{\alpha_2 n(n-1) + n - \beta}{1-\beta} |a_n|$$

$$+ \sum_{n=1}^{\infty} \frac{\alpha_2 n(n+1) + n + \beta}{1-\beta} |b_n| \leq 1. \tag{14}$$

Hence $f \in \overline{\mathscr{S}}_H^*(\alpha_1, \beta)$.

$\square$

As a consequence of Theorem 5, the functions in $\overline{\mathscr{S}}_H^*(\alpha, \beta)$ are starlike harmonic in $\mathbb{U}$.

Corollary 6. *For* $\alpha \geq 0$ *and* $0 \leq \beta < 1$, $\overline{\mathscr{S}}_H^*(\alpha, \beta) \subset \mathscr{S}_H^*(\beta)$.

3. Distortion Bounds and Extreme Points

In this section, we obtain the distortion bounds and extreme points for functions in the class $\overline{\mathscr{S}}_H^*(\alpha, \beta)$.

Theorem 7. *Let* $f = h + \overline{g}$ *where* h *and* g *are of the form* (1) *and* $f \in \overline{\mathscr{S}}_H^*(\alpha, \beta)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \leq (1 + |b_1|) r$$

$$+ \left(\frac{1-\beta}{2\alpha + 2 - \beta} - \frac{2\alpha + 1 + \beta}{2\alpha + 2 - \beta} |b_1| \right) r^2, \tag{15}$$

$$|f(z)| \geq (1 - |b_1|) r$$

$$- \left(\frac{1-\beta}{2\alpha + 2 - \beta} - \frac{2\alpha + 1 + \beta}{2\alpha + 2 - \beta} |b_1| \right) r^2, \tag{16}$$

where

$$|b_1| \leq \frac{1-\beta}{2\alpha + 1 + \beta}. \tag{17}$$

The result is sharp.

Proof. We shall prove the first inequality. Let $f \in \overline{\mathscr{S}}_H^*(\alpha, \beta)$. Then we have

$$|f(z)| \leq (1 + |b_1|) r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n$$

$$\leq (1 + |b_1|) r + r^2 \sum_{n=2}^{\infty} (|a_n| + |b_n|)$$

$$= (1 + |b_1|) r + \frac{1-\beta}{2\alpha + 2 - \beta} \tag{18}$$

$$\times \sum_{n=2}^{\infty} \frac{2\alpha + 2 - \beta}{1-\beta} (|a_n| + |b_n|) r^2$$

and so

$$|f(z)| \leq (1 + |b_1|) r$$

$$+ \frac{1-\beta}{2\alpha + 2 - \beta}$$

$$\times \sum_{n=2}^{\infty} \left(\frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| \right.$$

$$\left. + \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \right) r^2 \tag{19}$$

$$\leq (1 + |b_1|) r$$

$$+ \frac{1-\beta}{2\alpha + 2 - \beta} \left(1 - \frac{2\alpha + 1 + \beta}{1-\beta} |b_1| \right) r^2$$

$$= (1 + |b_1|) r$$

$$+ \left(\frac{1-\beta}{2\alpha + 2 - \beta} - \frac{2\alpha + 1 + \beta}{2\alpha + 2 - \beta} |b_1| \right) r^2.$$

The proof of the inequality (16) is similar to the proof of the inequality (15), thus we omit it.

The upper bound given for $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1| \overline{z}$$
$$+ \left(\frac{1-\beta}{2\alpha + 2 - \beta} - \frac{2\alpha + 1 + \beta}{2\alpha + 2 - \beta} |b_1| \right) \overline{z}^2 \quad (z = r), \tag{20}$$

where $|b_1| \leq (1-\beta)/(2\alpha + 1 + \beta)$. This completes the proof of Theorem 7. □

Now, we determine the extreme points of the closed convex hull of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ denoted by clco $\overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Theorem 8. *Let $f = h + \overline{g}$ where h and g are given by (1). Then $f \in clco\overline{\mathcal{S}}_H^*(\alpha, \beta)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n), \tag{21}$$

where

$$h_1(z) = z,$$

$$h_n(z) = z + \frac{1-\beta}{\alpha n(n-1) + n - \beta} z^n \quad (n = 2, 3, \ldots);$$

$$g_n(z) = z + \frac{1-\beta}{\alpha n(n+1) + n + \beta} \overline{z}^n \quad (n = 1, 2, 3, \ldots),$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1, \quad X_n \geq 0, \ Y_n \geq 0. \tag{22}$$

In particular, the extreme points of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ are $\{h_n\}$ and $\{g_n\}$, respectively.

Proof. For a function f of the form (21), we have

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$$
$$= \sum_{n=1}^{\infty} (X_n + Y_n) z$$
$$+ \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n z^n$$
$$+ \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \overline{z}^n \tag{23}$$
$$= z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n z^n$$
$$+ \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \overline{z}^n.$$

But

$$\sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n \right)$$
$$+ \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \right)$$
$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1. \tag{24}$$

Thus $f \in$ clco $\overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Conversely, suppose that $f \in$ clco $\overline{\mathcal{S}}_H^*(\alpha, \beta)$. Set

$$X_n = \frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| \quad (n = 2, 3, \ldots),$$

$$Y_n = \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \quad (n = 1, 2, 3, \ldots). \tag{25}$$

Then by the inequality (7), we have $0 \leq X_n \leq 1 \ (n = 2, 3, \ldots)$ and $0 \leq Y_n \leq 1 \ (n = 1, 2, \ldots)$. Define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that $X_1 \geq 0$. Thus we obtain $f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$. This completes the proof of the theorem. □

4. Convolution and Convex Combinations

For two harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n,$$

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n \tag{26}$$

we define their convolution

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n. \tag{27}$$

Using this definition, we show that the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convolution.

Theorem 9. *For $0 \leq \beta < 1$ and $\alpha \geq 0$, let $f, F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then $f * F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$.*

Proof. We note that $|A_n| \leq 1$ and $|B_n| \leq 1$. For the convolution $(f * F)$, we have

$$
\sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} |A_n a_n|
$$

$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} |B_n b_n|
$$

$$
\leq \sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} |a_n|
$$

$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} |b_n| \leq 1.
\tag{28}
$$

Therefore $f * F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. $\square$

We show that the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination of its members.

Theorem 10. *The class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$ where $f_i(z)$ is given by

$$
f_i(z) = z + \sum_{n=2}^{\infty} a_{ni} z^n + \sum_{n=1}^{\infty} \overline{b}_{ni} \overline{z}^n.
\tag{29}
$$

Then by (7), we have

$$
\sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} |a_{ni}|
$$

$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} |b_{ni}| \leq 1.
\tag{30}
$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$
\sum_{i=1}^{\infty} t_i f_i = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{ni} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b}_{ni} \right) \overline{z}^n.
\tag{31}
$$

Then by (7), we have

$$
\sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} \left| \sum_{i=1}^{\infty} t_i a_{ni} \right|
$$

$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} \left| \sum_{i=1}^{\infty} t_i \overline{b}_{ni} \right|
$$

$$
\leq \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} |a_{ni}| \right.
$$

$$
\left. + \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} |b_{ni}| \right)
\tag{32}
$$

$$
\leq \sum_{i=1}^{\infty} t_i = 1.
$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. $\square$

5. Neighborhood Results

Following the earlier investigations by Goodman [24], Ruscheweyh [25], Altintas et al. [26], and Porwal and Aouf [27], we define the δ-neighborhood of function $f(z) \in \mathcal{H}$ by

$$
N_\delta(f) = \left\{ F \in \mathcal{H} : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n, \right.
$$

$$
\left. \sum_{n=2}^{\infty} n |a_n - A_n| + \sum_{n=1}^{\infty} n |b_n - B_n| \leq \delta \right\}.
\tag{33}
$$

In particular, for the identity function $e(z) = z$, we immediately have

$$
N_\delta(e) = \left\{ F \in \mathcal{H} : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n, \right.
$$

$$
\left. \sum_{n=2}^{\infty} n |A_n| + \sum_{n=1}^{\infty} n |B_n| \leq \delta \right\}.
\tag{34}
$$

Theorem 11. $\overline{\mathcal{S}}_H^*(\alpha, \beta) \subseteq N_\delta(e)$, *where*

$$
\delta = \frac{(1 - \beta)(2\alpha + 1)}{(\alpha + 1)(2\alpha + 1 - \beta)}.
\tag{35}
$$

Proof. Let $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then, in view of (7), since $\alpha n(n-1) + n - \beta$ and $\alpha n(n+1) + n + \beta$ are increasing functions of n ($n \geq 1$), we have

$$
(2\alpha + 1 - \beta) \left(\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right)
$$

$$
\leq (2\alpha + 2 - \beta) \sum_{n=2}^{\infty} |a_n| + (2\alpha + 1 + \beta) \sum_{n=1}^{\infty} |b_n|
$$

$$
\leq \sum_{n=2}^{\infty} (\alpha n(n-1) + n - \beta) |a_n|
\tag{36}
$$

$$
+ \sum_{n=1}^{\infty} (\alpha n(n+1) + n + \beta) |b_n| \leq (1 - \beta),
$$

which yields

$$
\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \leq \frac{1 - \beta}{(2\alpha + 1 - \beta)}.
\tag{37}
$$

On the other hand, we also find from (7)

$$
\left((\alpha + 1) \sum_{n=2}^{\infty} n \, |a_n| - \beta \sum_{n=2}^{\infty} |a_n| \right)
$$
$$
+ \left((\alpha + 1) \sum_{n=1}^{\infty} n \, |b_n| - \beta \sum_{n=1}^{\infty} |b_n| \right)
$$
$$
\leq \sum_{n=2}^{\infty} \left((\alpha (n-1) + 1) n - \beta \right) |a_n| \tag{38}
$$
$$
+ \sum_{n=1}^{\infty} \left((\alpha (n+1) + 1) n + \beta \right) |b_n|
$$
$$
\leq (1 - \beta).
$$

From (37) and (38), we obtain

$$
(\alpha + 1) \left(\sum_{n=2}^{\infty} n \, |a_n| + \sum_{n=1}^{\infty} n \, |b_n| \right)
$$
$$
\leq (1 - \beta) + \beta \left(\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right) \tag{39}
$$
$$
= (1 - \beta) + \beta \frac{1 - \beta}{(2\alpha + 1 - \beta)},
$$

which is equivalent to

$$
\sum_{n=2}^{\infty} n \, |a_n| + \sum_{n=1}^{\infty} n \, |b_n| \leq \frac{(1 - \beta)(2\alpha + 1)}{(\alpha + 1)(2\alpha + 1 - \beta)} = \delta. \tag{40}
$$

$\square$

6. A Family of Class Preserving Integral Operator

In this section, we consider the closure property of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ under the Bernardi integral operator $F(z)$, which is defined by

$$
F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) \, dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t) \, dt} \tag{41}
$$
$$
(c > -1).
$$

Theorem 12. *Let $f = h + \overline{g}$ be in the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$, where h and g are given by (1). Then $F(z)$ defined by (41) also belongs to the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$.*

Proof. From the representation of F, it follows that

$$
F(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n + \sum_{n=1}^{\infty} \frac{c+1}{c+n} \overline{b_n} \overline{z}^n. \tag{42}
$$

Now

$$
\sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} \left(\frac{c+1}{c+n} |a_n| \right)
$$
$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} \left(\frac{c+1}{c+n} |b_n| \right) \tag{43}
$$
$$
\leq \sum_{n=2}^{\infty} \frac{\alpha n (n-1) + n - \beta}{1 - \beta} |a_n|
$$
$$
+ \sum_{n=1}^{\infty} \frac{\alpha n (n+1) + n + \beta}{1 - \beta} |b_n| \leq 1
$$

by (7). Thus $F(z) \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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