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Research Article

On Certain Class of Non-Bazilevič Functions of Order $\alpha + i\beta$ Defined by a Differential Subordination

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We introduce a new subclass $N_n(\lambda, \alpha, \beta, A, B)$ of Non-Bazilevič functions of order $\alpha + i\beta$. Some subordination relations and inequality properties are discussed. The results obtained generalize the related work of some authors. In addition, some other new results are also obtained.

1. Introduction

Let A_n denote the class of the functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $f(z)$ and $F(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \le 1$ and $f(z) = F(w(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Assume that $0 < \alpha < 1$, a function $f(z) \in A_n$, is in $N(\alpha)$ if and only if

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}\right\} > 0, \quad (z \in \mathbb{U}). \tag{2}$$

The class $N(\alpha)$ was introduced by Obradović [1] recently. This class of functions was said to be of Non-Bazilevič type. To this date, this class was studied in a direction of finding necessary conditions over α that embeds this class into the class of univalent functions or its subclasses which is still an open problem.

Assume that $0 < \alpha < 1$, $\lambda \in \mathbb{C}$, $-1 \le B \le 1$, $A \ne B$, and $A \in \mathbb{R}$, we consider the following subclass of A_n:

$$N(\lambda, \alpha, A, B) = \left\{f(z) \in A_n : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)} \times\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}\right\}, \quad (z \in \mathbb{U}), \tag{3}$$

where all the powers are principal values, and we apply this agreement to get the following definition.

Definition 1. Let $N(\lambda, \alpha, \mu)$ denote the class of functions in A_n satisfying the inequality

$$\Re\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > \mu, \tag{4}$$

where $0 < \alpha < 1$, $\lambda \in \mathbb{C}$, $0 \le \mu < 1$, and $z \in \mathbb{U}$.

The classes $N(\lambda, \alpha, A, B)$ and $N(\lambda, \alpha, \mu)$ were studied by Wang et al. [2].

In the present paper, similarly we define the following class of analytic functions.

Definition 2. Let $N_n(\lambda, \alpha, \beta, A, B)$ denote the class of functions in A_n satisfying the inequality

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}), \tag{5}$$

where $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $-1 \leq B \leq 1$, $A \neq B$, and $A \in \mathbb{R}$. All the powers in (5) are principal values.

We say that the function $f(z)$ in this class is Non-Bazilevič functions of type $\alpha + i\beta$.

Definition 3. Let $f(z) \in N_n(\lambda, \alpha, \beta, \mu)$ if and only if $f(z) \in A_n$ and it satisfies

$$\Re\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} > \mu, \tag{6}$$

where $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $0 \leq \mu < 1$, and $z \in \mathbb{U}$.

In particular, if $\beta = 0$, it reduces to the class $N(\lambda, \alpha, A, B)$ studied in [2].

If $\beta = 0$, $\lambda = -1$, $n = 1$, $A = 1$, and $B = -1$, then the class $N_n(\lambda, \alpha, \beta, A, B)$ reduces to the class of non-Bazilevič functions. If $\beta = 0$, $\lambda = -1$, $n = 1$, $A = 1 - 2\mu$, and $B = -1$, then the class $N_n(\lambda, \alpha, \beta, A, B)$ reduces to the class of non-Bazilevič functions of order μ $(0 \leq \mu < 1)$. Tuneski and Darus studied the Fekete-Szegö problem of the class $N(-1, \alpha, 0, 1 - 2\mu, -1)$ [3]. Other works related to Bazilevič and non-Bazilevič can be found in ([4–9]).

In the present paper, we will discuss the subordination relations and inequality properties of the class $N_n(\lambda, \alpha, \beta, A, B)$. The results presented here generalize and improve some known results, and some other new results are obtained.

2. Some Lemmas

Lemma 4 (see [10]). *Let $F(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex in $\mathbb{U}$, $h(0) = 1$. If*

$$F(z) + \frac{1}{c} zF'(z) \prec h(z), \tag{7}$$

where $c \neq 0$ and $\operatorname{Re} c \geq 0$, then

$$F(z) \prec \frac{c}{n} z^{-c/n} \int_0^z t^{(c/n)-1} h(t)\, dt \prec h(z), \tag{8}$$

and $(c/n) z^{-c/n} \int_0^z t^{(c/n)-1} h(t) dt$ is the best dominant for the differential subordination (7).

Lemma 5 (see [11]). *Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$; then*

$$\frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z}. \tag{9}$$

Lemma 6 (see [12]). *Let $F(z)$ be analytic and convex in $\mathbb{U}$, $f(z) \in A_n$, $g(z) \in A_n$. If*

$$f(z) \prec F(z), \quad g(z) \prec F(z), \quad 0 \leq \lambda \leq 1, \tag{10}$$

then

$$\lambda f(z) + (1 - \lambda) g(z) \prec F(z). \tag{11}$$

Lemma 7 (see [13]). *Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be analytic in $\mathbb{U}$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ analytic and convex in $\mathbb{U}$. If $f(z) \prec g(z)$, then $|a_k| \leq |b_k|$, for $k = 1, 2, \ldots$.*

Lemma 8. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, $-1 \leq B \leq 1$, $A \neq B$, and $A \in \mathbb{R}$. Then $f(z) \in N_n(\lambda, \alpha, \beta, A, B)$ if and only if*

$$F(z) + \frac{\lambda}{\alpha + i\beta} zF'(z) \prec \frac{1 + Az}{1 + Bz}, \tag{12}$$

where

$$F(z) = \left(\frac{z}{f(z)}\right)^{\alpha+i\beta}. \tag{13}$$

Proof. Let

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} = F(z). \tag{14}$$

Then, by taking the derivatives of both sides of (14) and through simple calculation, we have

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} = F(z) + \frac{\lambda}{\alpha + i\beta} zF'(z); \tag{15}$$

since $f(z) \in N_n(\lambda, \alpha, \beta, A, B)$, we have

$$F(z) + \frac{\lambda}{\alpha + i\beta} zF'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{16}$$

□

3. Main Results

Theorem 9. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, $-1 \leq B \leq 1$, $A \neq B$, and $A \in \mathbb{R}$. If $f(z) \in N_n(\lambda, \alpha, \beta, A, B)$, then*

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{\alpha + i\beta}{\lambda n} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{((\alpha+i\beta)/\lambda n)-1} du \prec \frac{1 + Az}{1 + Bz}. \tag{17}$$

Proof. First let $F(z) = (z/f(z))^{\alpha+i\beta}$; then $F(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$. Now, suppose that $f(z) \in N_n(\lambda, \alpha, \beta, A, B)$; by Lemma 8, we know that

$$F(z) + \frac{\lambda}{\alpha + i\beta} zF'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{18}$$

It is obvious that $h(z) = (1 + Az)/(1 + Bz)$ is analytic and convex in $\mathbb{U}$, $h(0) = 1$. Since $\alpha + i\beta \neq 0$, $\alpha \geq 0$, $\lambda \neq 0$, and $\Re\{(\alpha + i\beta)/\lambda\} \geq 0$; therefore, it follows from Lemma 4 that

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} = F(z) \prec \frac{\alpha+i\beta}{\lambda n} z^{-(\alpha+i\beta)/\lambda n} \int_0^z t^{((\alpha+i\beta)/\lambda n)-1} h(t)\,dt = \frac{\alpha+i\beta}{\lambda n} \int_0^1 \frac{1+Azu}{1+Bzu} u^{((\alpha+i\beta)/\lambda n)-1} du \prec \frac{1+Az}{1+Bz}. \tag{19}$$

□

Corollary 10. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, and $\mu \neq 1$. If $f(z) \in A_n$ satisfies*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+(1-2\mu)z}{1-z} \quad (z \in \mathbb{U}), \tag{20}$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{\alpha+i\beta}{\lambda n} \times \int_0^1 \frac{1+(1-2\mu)zu}{1-zu} u^{((\alpha+i\beta)/\lambda n)-1} dt \quad (z \in \mathbb{U}), \tag{21}$$

or equivalent to

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \mu + \frac{(1-\mu)(\alpha+i\beta)}{\lambda n} \times \int_0^1 \frac{1+zu}{1-zu}\, u^{((\alpha+i\beta)/\lambda n)-1} dt \quad (z \in \mathbb{U}). \tag{22}$$

Corollary 11. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, and $\Re\{\lambda\} \geq 0$; then*

$$N_n(\lambda, \alpha, \beta, A, B) \subset N_n(0, \alpha, \beta, A, B). \tag{23}$$

Theorem 12. *Let $0 \leq \lambda_1 \leq \lambda_2$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$; then*

$$N_n(\lambda_2, \alpha, \beta, A_2, B_2) \subset N_n(\lambda_1, \alpha, \beta, A_1, B_1). \tag{24}$$

Proof. Suppose that $f(z) \in N_n(\lambda_2, \alpha, \beta, A_2, B_2)$ we have $f(z) \in A_n$, and

$$(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda_2 \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+A_2 z}{1+B_2 z} \quad (z \in \mathbb{U}). \tag{25}$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore it follows from Lemma 5 that

$$(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda_2 \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+A_1 z}{1+B_1 z} \quad (z \in \mathbb{U}); \tag{26}$$

that is $f(z) \in N_n(\lambda_2, \alpha, \beta, A_1, B_1)$. So Theorem 12 is proved when $\lambda_1 = \lambda_2 \geq 0$. □

When $\lambda_2 > \lambda_1 \geq 0$, then we can see from Corollary 11 that $f(z) \in N_n(0, \alpha, \beta, A_1, B_1)$; then

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+A_1 z}{1+B_1 z}. \tag{27}$$

But

$$(1+\lambda_1)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda_1 \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} = \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} + \frac{\lambda_1}{\lambda_2} \times \left[(1+\lambda_2)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda_2 \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right]. \tag{28}$$

It is obvious that $h_1(z) = (1 + A_1 z)/(1 + B_1 z)$ is analytic and convex in $\mathbb{U}$. So we obtain from Lemma 6 and differential subordinations (26) and (27) that

$$(1+\lambda_1)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda_1 \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{1+A_1 z}{1+B_1 z}; \tag{29}$$

that is, $f(z) \in N_n(\lambda_1, \alpha, \beta, A_1, B_1)$. Thus we have

$$N_n(\lambda_2, \alpha, \beta, A_2, B_2) \subset N_n(\lambda_1, \alpha, \beta, A_1, B_1). \tag{30}$$

Corollary 13. *Let $0 \leq \lambda_1 \leq \lambda_2$, $0 \leq \mu_1 \leq \mu_2 < 1$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, and $\alpha + i\beta \neq 0$; then*

$$N_n(\lambda_2, \alpha, \beta, \mu_2) \subset N_n(\lambda_1, \alpha, \beta, \mu_1). \tag{31}$$

Theorem 14. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha + i\beta \neq 0$, $-1 \leq B \leq 1$, $A \neq B$, and $A \in \mathbb{R}$. If $f(z) \in N_n(\lambda, \alpha, \beta, A, B)$, then*

$$\inf_{z\in\mathbb{U}} \Re\left\{\frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu} u^{((\alpha+i\beta)/\lambda n)-1} du\right\} < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} < \sup_{z\in\mathbb{U}} \Re\left\{\frac{\alpha+i\beta}{\lambda n} \times \int_0^1 \frac{1+Azu}{1+Bzu} u^{((\alpha+i\beta)/\lambda n)-1} du\right\}. \tag{32}$$

Proof. Suppose that $f(z) \in N_{\text{n}}(\lambda, \alpha, \beta, A, B)$; then from Theorem 9 we know that

$$\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} \prec \frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{((\alpha+i\beta)/\lambda n)-1}du. \quad (33)$$

Therefore, from the definition of the subordination, we have

$$\begin{aligned} &\Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} \\ &\quad > \inf_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\}, \\ &\Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} \\ &\quad < \sup_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\}. \end{aligned} \quad (34)$$

□

Corollary 15. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha+i\beta \neq 0$, and $\mu < 1$. If $f(z) \in N_n(\lambda, \alpha, \beta, 1-2\mu, -1)$, then*

$$\begin{aligned} &\mu + (1-\mu)\inf_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+zu}{1-zu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\} \\ &\quad < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} \\ &\quad < \mu + (1-\mu)\sup_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\right. \\ &\qquad\qquad \left.\times\int_0^1 \frac{1+zu}{1-zu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\}. \end{aligned} \quad (35)$$

Corollary 16. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha+i\beta \neq 0$, and $\mu > 1$. If $f(z) \in A_n$; then*

$$\Re\left((1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right) < \mu \quad (z \in \mathbb{U}), \quad (36)$$

then

$$\begin{aligned} &\mu + (1-\mu)\sup_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\int_0^1 \frac{1+zu}{1-zu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\} \\ &\quad < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right\} \\ &\quad < \mu + (1-\mu)\inf_{z\in\mathbb{U}}\Re\left\{\frac{\alpha+i\beta}{\lambda n}\right. \\ &\qquad\qquad \left.\times\int_0^1 \frac{1+zu}{1-zu}u^{((\alpha+i\beta)/\lambda n)-1}du\right\}. \end{aligned} \quad (37)$$

Corollary 17. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, and $-1 \leq B < A \leq 1$. If $f(z) \in N_n(\lambda, \alpha, 0, A, B)$, then*

$$\begin{aligned} &\frac{\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}u^{(\alpha/\lambda n)-1}du \\ &\quad < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \\ &\quad < \frac{\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}u^{(\alpha/\lambda n)-1}du \quad (z \in \mathbb{U}), \end{aligned} \quad (38)$$

and inequality (38) *is sharp, with the extremal function defined by*

$$f_{\lambda,\alpha,B,A}(z) = z\left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Az^n u}{1+Bz^n u}u^{(\alpha/\lambda n)-1}du\right)^{-(1/\alpha)}. \quad (39)$$

Proof. Suppose that $f(z) \in N_n(\lambda, \alpha, 0, A, B)$; from Theorem 9 we know

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{\alpha}{\lambda n}\int_0^1 \frac{1+Auz}{1+Buz}u^{(\alpha/\lambda n)-1}du. \quad (40)$$

Therefore, from the definition of the subordination and $A > B$, we have that

$$\begin{aligned} \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} &< \sup_{z\in\mathbb{U}}\Re\left\{\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Auz}{1+Buz}u^{(\alpha/\lambda n)-1}du\right\} \\ &\leq \frac{\alpha}{\lambda n}\int_0^1 \sup_{z\in\mathbb{U}}\left\{\frac{1+Auz}{1+Buz}\right\}u^{(\alpha/\lambda n)-1}du \\ &< \frac{\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}u^{(\alpha/\lambda n)-1}du, \\ \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} &> \inf_{z\in\mathbb{U}}\Re\left\{\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Auz}{1+Buz}u^{(\alpha/\lambda n)-1}du\right\} \\ &\geq \frac{\alpha}{\lambda n}\int_0^1 \inf_{z\in\mathbb{U}}\left\{\frac{1+Auz}{1+Buz}\right\}u^{(\alpha/\lambda n)-1}du \\ &> \frac{\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}u^{(\alpha/\lambda n)-1}du. \end{aligned} \quad (41)$$

It is obvious that inequality (38) is sharp, with the extremal function given by (39). □

Corollary 18. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, and $\mu < 1$. If $f(z) \in N_n(\lambda, \alpha, 0, 1-2\mu, -1)$, then*

$$\begin{aligned} &\frac{\alpha}{\lambda n}\int_0^1 \frac{1-(1-2\mu)u}{1+u}u^{(\alpha/\lambda n)-1}du \\ &\quad < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \\ &\quad < \frac{\alpha}{\lambda n}\int_0^1 \frac{1+(1-2\mu)u}{1-u}u^{(\alpha/\lambda n)-1}du \quad (z \in \mathbb{U}), \end{aligned} \quad (42)$$

and inequality (42) *is equivalent to*

$$\mu + \frac{(1-\mu)\alpha}{\lambda n}\int_0^1 \frac{1-u}{1+u}u^{(\alpha/\lambda n)-1}du < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \mu + \frac{(1-\mu)\alpha}{\lambda n}\int_0^1 \frac{1+u}{1-u}u^{(\alpha/\lambda n)-1}du \quad (z \in \mathbb{U}). \tag{43}$$

The inequality (42) *is sharp, with the extremal function defined by*

$$f_{\lambda,\alpha,\beta}(z) = z\left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1+(1-2\beta)z^n u}{1-z^n u}u^{(\alpha/\lambda n)-1}du\right)^{-1/\alpha} dz. \tag{44}$$

Corollary 19. *Let $\lambda \in \mathbb{C}$, $\alpha \ge 0$, and $-1 \le A < B \le 1$. If $f(z) \in N_n(\lambda, \alpha, 0, A, B)$, then*

$$\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}u^{(\alpha/\lambda n)-1}du < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \frac{\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}u^{(\alpha/\lambda n)-1}du \quad (z \in \mathbb{U}), \tag{45}$$

and inequality (45) *is sharp, with the extremal function given by* (39).

Proof. Applying similar method as in Corollary 17, we get the result. □

Corollary 20. *Let $\lambda \in \mathbb{C}$, $\alpha \ge 0$, and $\mu > 1$. If $f(z) \in A_n$ satisfies*

$$\Re\left((1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\right) < \mu \quad (z \in \mathbb{U}), \tag{46}$$

then

$$\frac{\alpha}{\lambda n}\int_0^1 \frac{1+(1-2\mu)u}{1-u}u^{(\alpha/\lambda n)-1}du < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \frac{\alpha}{\lambda n}\int_0^1 \frac{1-(1-2\mu)u}{1+u}u^{(\alpha/\lambda n)-1}du, \tag{47}$$

and inequality (47) *is equivalent to*

$$\mu + \frac{(1-\mu)\alpha}{\lambda n}\int_0^1 \frac{1+u}{1-u}u^{(\alpha/\lambda n)-1}du < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \mu + \frac{(1-\mu)\alpha}{\lambda n}\int_0^1 \frac{1-u}{1+u}u^{(\alpha/\lambda n)-1}du \quad (z \in \mathbb{U}), \tag{48}$$

and inequality (47) *is sharp, with the extremal function defined by equality* (44).

If $\Re w \ge 0$, then $(\Re w)^{1/2} \le \Re w^{1/2} \le \Re z^{1/2}$ (see [2, 12]). So we have the following.

Corollary 21. *Let $\lambda \in \mathbb{C}$, $\alpha \ge 0$, and $-1 \le B < A \le 1$. If $f(z) \in N_n(\lambda, \alpha, 0, A, B)$, then*

$$\left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}u^{(\alpha/\lambda n)-1}du\right)^{1/2} < \Re\left\{\left[\left(\frac{z}{f(z)}\right)^{\alpha}\right]^{1/2}\right\} < \left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}u^{(\alpha/\lambda n)-1}du\right)^{1/2} \quad (z \in \mathbb{U}), \tag{49}$$

and inequality (49) *is sharp, with the extremal function defined by equality* (39).

Proof. From Theorem 9 we have

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}. \tag{50}$$

Since $-1 \le A < B \le 1$, we have

$$0 \le \frac{1-A}{1-B} < \Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} < \frac{1+A}{1+B}. \tag{51}$$

Thus, from inequality (38), we can get inequality (49). It is obvious that inequality (49) is sharp, with the extremal function defined by equality (39). □

Corollary 22. *Let $\lambda \in \mathbb{C}$, $\alpha \ge 0$, and $-1 \le A < B \le 1$. If $f(z) \in N_n(\lambda, \alpha, 0, A, B)$, then*

$$\left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}u^{(\alpha/\lambda n)-1}du\right)^{1/2} < \Re\left\{\left[\left(\frac{z}{f(z)}\right)^{\alpha}\right]^{1/2}\right\} < \left(\frac{\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}u^{(\alpha/\lambda n)-1}du\right)^{1/2} \quad (z \in \mathbb{U}), \tag{52}$$

and inequality (52) *is sharp, with the extremal function defined by equality* (39).

Proof. Applying similar method as in Corollary 21, we get the required result. □

Remark 23. From Corollaries 21 and 22, we can generalize the corresponding results and some other special classes of analytic functions.

Corollary 24. *Let $\lambda \in \mathbb{C}$, $\alpha \geq 0$, $-1 \leq B < A \leq 1$, and $A \in \mathbb{R}$; if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N_n(\lambda, \alpha, 0, A, B)$, then one has*

$$\left|a_{n+1}\right| \leq \frac{|A-B|}{|n\lambda + \alpha|} \tag{53}$$

and inequality (53) *is sharp, with the extremal function defined by equality* (39).

Proof. Suppose that $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N_n(\lambda, \alpha, 0, A, B)$; then we have

$$\begin{aligned}(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda \frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \\ = 1 + (-n\lambda - \alpha) a_{n+1} z^n + \cdots \prec \frac{1+Az}{1+Bz}.\end{aligned} \tag{54}$$

It follows from Lemma 7 that

$$\left|a_{n+1}\right| \leq \frac{|A-B|}{|n\lambda + \alpha|}. \tag{55}$$

Thus, we can get (53). Notice that

$$f(z) = z + \frac{A-B}{n\lambda + \alpha} z^{n+1} + \cdots \in N_n(\lambda, \alpha, 0, A, B); \tag{56}$$

we obtain that the inequality (53) is sharp. □

Remark 25. Setting $\lambda = n = A = 1$, and $B = -1$ in Corollary 24 we get the results obtained by [14].

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

Both authors read and approved the final paper.

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References

[1] M. Obradović, "A class of univalent functions," *Hokkaido Mathematical Journal*, vol. 27, no. 2, pp. 329–335, 1998.

[2] Z. Wang, C. Gao, and M. Liao, "On certain generalized class of non-Bazilevič functions," *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, vol. 21, no. 2, pp. 147–154, 2005.

[3] N. Tuneski and M. Darus, "Fekete-Szegö functional for non-Bazilevic functions," *Acta Mathematica Academiae Paedagogicae Nyí regyháziensis*, vol. 18, pp. 63–65, 2002.

[4] A. A. Amer and M. Darus, "Distortion theorem for certain class of Bazilevic functions," *International Journal of Mathematical Analysis*, vol. 6, no. 9–12, pp. 591–597, 2012.

[5] R. W. Ibrahim and M. Darus, "Mixed class of functions of non-Bazilevic type and bounded turning," *Far East Journal of Mathematical Sciences*, vol. 67, no. 1, pp. 141–152, 2012.

[6] R. W. Ibrahim and M. Darus, "Argument estimate for non-Bazilevic type and bounded turning functions," *Far East Journal of Mathematical Sciences*, vol. 68, no. 2, pp. 175–183, 2012.

[7] R. W. Ibrahim, M. Darus, and N. Tuneski, "On subordination for classes of non-Bazilevic type," *Annales Universitatis Mariae Curie-Sklodowska Lublin-Polonia A*, vol. 64, no. 2, pp. 49–60, 2010.

[8] M. Darus and R. W. Ibrahim, "On subclasses of uniformly Bazilevic type functions involving generalised differential and integral operators," *Far East Journal of Mathematical Sciences (FJMS)*, vol. 33, no. 3, pp. 401–411, 2009.

[9] T. N. Shanmugam, S. Sivasubramanian, M. Darus, and S. Kavitha, "On sandwich theorems for certain subclasses of non-Bazilevic functions involving Cho-Kim transformation," *Complex Variables and Elliptic Equations*, vol. 52, no. 10-11, pp. 1017–1028, 2007.

[10] S. S. Miller and P. T. Mocanu, "Differential subordinations and univalent functions," *The Michigan Mathematical Journal*, vol. 28, no. 2, pp. 157–172, 1981.

[11] M. S. Liu, "On a subclass of p-valent close-to-convex functions of order β and type α," *Journal of Mathematical Study*, vol. 30, no. 1, pp. 102–104, 1997.

[12] M.-S. Liu, "On certain class of analytic functions defined by differential subordination," *Acta Mathematica Scientia B*, vol. 22, no. 3, pp. 388–392, 2002.

[13] W. Rogosinski, "On the coefficients of subordination functions," *Proceedings of the London Mathematical Society 2*, vol. 48, pp. 48–82, 1943.

[14] R. Singh, " On Bazilevic functions," *Proceedings of the American Mathematical Society*, vol. 38, pp. 261–271, 1973.