

Research Article

Oscillation Criteria for Certain Even Order Neutral Delay Differential Equations

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We establish sufficient conditions for the oscillation of solutions of even order neutral type differential equations of the form $[r(t)[x(t) + p(t)x(\tau(t))]^{(n-1)'} + q(t)f(x(\sigma(t))) = 0$.

1. Introduction

This paper concerns the oscillatory behavior of solutions of higher order neutral type nonlinear differential equations of the following form:

$$\begin{aligned} [r(t)[x(t) + p(t)x(\tau(t))]^{(n-1)'} + q(t)f(x(\sigma(t))) = 0, \\ t \geq t_0, \end{aligned} \quad (1)$$

where $n \geq 2$ is even and the following conditions are assumed to hold:

- (H1) $p, q \in C([t_0, \infty))$, $q(t) > 0$, $0 \leq p(t) \leq p_0 < 1$, where p_0 is constant;
- (H2) $r \in C^1([t_0, \infty))$, $r(t) > 0$, $r'(t) \geq 0$;
- (H3) $\tau \in C([t_0, \infty), \mathbb{R})$, $\sigma \in C^1([t_0, \infty), \mathbb{R})$, $\tau(t) < t$, $\sigma(t) \leq \lambda t < t$, $\sigma'(t) > 0$, and $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \sigma(t) = \infty$;
- (H4) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(x)/x \geq K > 0$, for $x \neq 0$, and K is a constant.

Further, we will consider the two cases

$$\int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty, \quad (2)$$

$$\int_{t_0}^{\infty} \frac{1}{r(t)} dt < \infty. \quad (3)$$

By a solution of (1), we mean a real-valued function x which satisfies (1) and $\sup\{|x(t)| : t \geq t_x\} > 0$ for any $t_x \geq t_0$. Such a solution is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise.

Neutral differential equations arise in a number of important applications in natural science and technology. For instance, they are used in problems dealing with vibrating masses attached to an elastic bar and in the study of distributed networks containing lossless transmission lines which appears in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see Hale [1].

During the last 20 years, significant efforts have been devoted to investigate the oscillatory behaviour of neutral differential equations; see [1–16] and the references cited therein. In particular, (1) and related forms have been considered by several authors; see [10, 11, 13, 14, 16]. Several recent results are surveyed in Sun et al. [13]. In addition, we refer to [3–5, 8], where the oscillatory behaviour of solutions of (1) with $n = 2$ was studied.

In this paper, we establish oscillation theorems for solutions of (1). Our results generalize the results of Grammatikopoulos et al. [6] in some sense. Also, our results, in some sense, agree with the results of Sun et al. [13]. Here, we remove the restriction of [13] for τ and σ to be commute.

2. Auxiliary Lemmas

The following lemmas will be needed in the proof of our main results.

Lemma 1 (see [7, page 193]). *Let $y(t)$ be an n times differentiable function on $[0, \infty)$ of constant sign, let $y^{(n)}(t)$ be of constant sign and not identically equal to zero in any interval $[t_0, \infty)$, $t_0 \geq 0$, and let $y(t)y^{(n)}(t) \leq 0$. Then,*

- (i) *there exists a $t_1 \geq t_0$ such that $y^{(k)}(t)$, $k = 1, \dots, n - 1$, is of constant sign on $[t_1, \infty)$;*
- (ii) *there exists an integer l , $0 \leq l \leq n - 1$, with $n - l$ odd, such that*

$$y(t)y^{(k)}(t) > 0, \quad k = 0, 1, \dots, l, \quad t \geq t_1, \quad (4)$$

$$(-1)^{n+k-1}y(t)y^{(k)}(t) > 0, \quad k = l + 1, \dots, n - 1, \quad t \geq t_1. \quad (5)$$

Lemma 2 (see [12]). *Let $y(t)$ be a function as in Lemma 1. If*

$$y^{(n-1)}(t)y^{(n)}(t) \leq 0, \quad (6)$$

then, for every $\theta \in (0, 1)$, there exists a constant $M > 0$, such that

$$|y'(\theta t)| \geq Mt^{n-2}|y^{(n-1)}(t)|, \quad (7)$$

for sufficiently large t .

Lemma 3 (see [2, page 169]). *Let $y(t)$ be a function as in Lemma 1. If*

$$y^{(n-1)}(t)y^{(n)}(t) \leq 0, \quad (8)$$

and $\lim_{t \rightarrow \infty} y(t) \neq 0$, then for every $\theta \in (0, 1)$

$$y(t) \geq \frac{\theta}{(n-1)!}t^{n-1}y^{(n-1)}(t), \quad (9)$$

for sufficiently large t .

3. The Main Result

Theorem 4. *Assume that (2) holds. If*

$$\int_{t_0}^{\infty} q(t) dt = \infty, \quad (10)$$

then every solution $x(t)$ of (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and let $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.

Set

$$z(t) = x(t) + p(t)x(\tau(t)). \quad (11)$$

Since $p(t)$ is nonnegative, $z(t) > x(t) > 0$ for $t \geq t_1$.

From (1) and (11), we have

$$\begin{aligned} (r(t)z^{(n-1)}(t))' &= r(t)z^{(n)}(t) + r'(t)z^{(n-1)}(t) \\ &= -q(t)f(x(\sigma(t))) < 0. \end{aligned} \quad (12)$$

Thus, $r(t)z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t)$ is eventually of one sign. Hence, either

$$z^{(n-1)}(t) > 0, \quad \text{for } t \geq t_2 \geq t_1 \quad (13)$$

or

$$z^{(n-1)}(t) < 0, \quad \text{for } t \geq t_2 \geq t_1. \quad (14)$$

If (14) holds, then

$$r(t)z^{(n-1)}(t) \leq r(t_2)z^{(n-1)}(t_2) < 0, \quad \text{for } t \geq t_2. \quad (15)$$

Dividing this inequality by $r(t)$ and integrating from t_2 to t , then by using (2), we get

$$\lim_{t \rightarrow \infty} z^{(n-2)}(t) = -\infty. \quad (16)$$

This result along with (14) leads to $\lim_{t \rightarrow \infty} z(t) = -\infty$. But this contradicts the fact that $z(t) > 0$. Thus, (13) holds. Then, from (12) and the fact that $r(t)$ is a positive nondecreasing function, we conclude that $z^{(n)}(t) < 0$, for $t \geq t_2$. It follows that $z^{(i)}(t)$ ($i = 0, 1, \dots, n - 1$) is strictly monotonic and of constant sign eventually.

By applying Lemma 1, $z(t)$ satisfies (4) and (5). Since n is even, the integer l associated with $z(t)$ is odd; that is, $l \geq 1$. Hence, $z(t)$ is increasing for $t \geq t_3 \geq t_2$.

Then, from (11) and the fact that $z(t)$ is increasing, we have

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \\ &\geq (1 - p(t))z(t) \geq (1 - p_0)z(t), \end{aligned} \quad (17)$$

for $t \geq t_3$.

Let $t_4 \geq t_3$ be such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Combining (H4) and (17), we get

$$f(x(\sigma(t))) \geq K(1 - p_0)z(\sigma(t)), \quad \text{for } t \geq t_4. \quad (18)$$

It is clear that we can apply Lemma 2. Then, from (7) and the decreasing character of $z^{(n-1)}(t)$, we have

$$\begin{aligned} z'(\sigma(t)) &\geq \frac{M}{\lambda^{n-2}}\sigma^{n-2}(t)z^{(n-1)}\left(\frac{\sigma(t)}{\lambda}\right) \\ &\geq M_0\sigma^{n-2}(t)z^{(n-1)}(t), \end{aligned} \quad (19)$$

for $t \geq t_5 \geq t_3$,

where $M_0 = M/\lambda^{n-2}$.

Define

$$w(t) = \frac{r(t)z^{(n-1)}(t)}{z(\sigma(t))}, \quad (20)$$

and then $w(t) > 0$ for $t \geq t_6 = \max\{t_4, t_5\}$.

By differentiating w and using (12), (18), and (19), we obtain

$$w'(t) = \frac{(r(t)z^{(n-1)}(t))'}{z(\sigma(t))} - \frac{\sigma'(t)z'(\sigma(t))r(t)z^{(n-1)}(t)}{z^2(\sigma(t))} \leq -K(1-p_0)q(t) - \frac{M_0\sigma'(t)\sigma^{n-2}(t)}{r(t)}w^2(t). \tag{21}$$

Since $r(t) > 0$ and $\sigma'(t) > 0$, the term $(M_0\sigma'(t)\sigma^{n-2}(t)/r(t))w^2(t) > 0$. Hence, (21) reduces to

$$w'(t) \leq -K(1-p_0)q(t). \tag{22}$$

Integrating this inequality from t_7 to t , $t_7 > t_6$, and using assumption (10), we see that $w(t) \rightarrow -\infty$ as $t \rightarrow \infty$. But this contradicts the positivity of $w(t)$. Hence, the theorem is proved. \square

In the above proof, being $l \geq 1$ plays an important role. In fact, $l = 0$ is possible only for odd orders. In this case, the solutions are bounded. For unbounded solutions with n being odd, the integer l must be greater than or equal to 2. Thus, it is easy to show that if n is odd and the conditions of Theorem 4 are satisfied, then every unbounded solution of (1) is oscillatory.

Notice that if the solutions are assumed to be unbounded, then the restriction on $\sigma(t)$ in (H2) can be improved to be $\sigma(t) \leq t$. Indeed, under the assumption of unboundedness, $z'(t)$ is increasing. This modifies (19) as

$$z'(\sigma(t)) \geq z'(\theta\sigma(t)) \geq M\sigma^{n-2}(t)z^{(n-1)}(\sigma(t)) \geq M\sigma^{n-2}(t)z^{(n-1)}(t), \quad \theta \in (0, 1), \tag{23}$$

where the rest of the proof stays as above.

Remark 5. The condition (10) can be rewritten as

$$\int^{\infty} q(t)[1-p(\sigma(t))]dt = \infty. \tag{24}$$

Here, there is no need for bounded value p_0 for the function $p(t)$; that is, $0 \leq p(t) < 1$. When we take $n = 2$, $r(t) = 1$, $f(x) = x$, and $\sigma(t) = t - \sigma_0$, we recover the results of Grammatikopoulos et al. [6]. In this case, we consider unbounded solutions.

Remark 6. Theorem 4 remains true if the function f satisfies the condition that $xf(x) > 0$ and there exists a nondecreasing function $\phi \in C([t_0, \infty), (0, \infty))$ with

$$|f(x)| \geq \phi(|x|). \tag{25}$$

Theorem 7. Assume that (3) and (10) hold and $n \geq 4$ is even. Further, suppose that

$$\int^{\infty} \left[Cq(s)(\sigma(s))^{n-2}\delta(s) - \frac{1}{4r(s)\delta(s)} \right] ds = \infty, \tag{26}$$

where $C = \alpha K/(n-2)!$, $\alpha \in (0, 1)$ is a constant, and $\delta(t) = \int_t^{\infty} (1/r(s))ds$. Then, every solution $x(t)$ of (1) either is oscillatory or tends to zero as $t \rightarrow \infty$.

Proof. Assume that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$.

Proceeding as in the proof of Theorem 4, we conclude that $r(t)z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t)$ is eventually of one sign. Hence, either (13) or (14) holds.

If (13) holds, we obtain a contradiction by proceeding as in the proof of Theorem 4.

Suppose that (14) holds; that is, $z^{(n-1)}(t) < 0$, for $t \geq t_2 \geq t_1$. Now, we consider two assumptions: unbounded solutions and bounded solutions.

If the solution $x(t)$ is unbounded, it is obvious that $z(t)$ is also unbounded. Since $z(t)z^{(n-1)}(t) < 0$ and $n-1$ is odd, we have by Lemma 1 that $l \geq 2$ (if $l = 0$, then $z(t)$ is bounded). Hence, from (4), we have that $z'(t) > 0$, and $z''(t) > 0$. Therefore, $\lim_{t \rightarrow \infty} z(t) > 0$.

Since $z(t)$ is increasing, we obtain

$$x(t) \geq (1-p_0)z(t), \quad \text{for } t \geq t_3 \geq t_2. \tag{27}$$

By Lemma 3 and the fact that $z^{(n-2)}(t)$ is decreasing, we get

$$z(\sigma(t)) \geq \frac{\theta[\sigma(t)]^{n-2}}{(n-2)!}z^{(n-2)}(t), \quad t \geq t_4 \geq t_2. \tag{28}$$

Combining (H4), (27), and (28), we obtain

$$f(x(\sigma(t))) \geq C(\sigma(t))^{n-2}z^{(n-2)}(t), \tag{29}$$

$$\text{for } t \geq t_5 = \max\{t_3, t_4\},$$

where $C = \alpha K/(n-2)!$ with $\alpha = (1-p_0)\theta \in (0, 1)$.

Define

$$w(t) = \frac{r(t)z^{(n-1)}(t)}{z^{(n-2)}(t)}, \tag{30}$$

and then $w(t) < 0$ for $t \geq t_5$.

Differentiating w and using (12) and (29), we obtain

$$w'(t) = \frac{(r(t)z^{(n-1)}(t))'}{z^{(n-2)}(t)} - \frac{w^2(t)}{r(t)} \leq -Cq(t)(\sigma(t))^{n-2} - \frac{w^2(t)}{r(t)}. \tag{31}$$

Following [8, 13], we can show that $-1 \leq w(t)\delta(t) < 0$. Indeed, since $r(t)z^{(n-1)}(t)$ is decreasing,

$$r(s)z^{(n-1)}(s) \leq r(t)z^{(n-1)}(t), \quad s \geq t \geq t_5. \tag{32}$$

Dividing by $r(s)$, then integrating from t to l , and letting $l \rightarrow \infty$, we get

$$0 \leq z^{(n-2)}(t) + r(t)z^{(n-1)}(t)\delta(t). \tag{33}$$

Thus, we obtain

$$-1 \leq \frac{r(t)z^{(n-1)}(t)}{z^{(n-2)}(t)}\delta(t). \tag{34}$$

Hence,

$$-1 \leq w(t)\delta(t) < 0, \quad t \geq t_5. \tag{35}$$

Multiplying inequality (31) by $\delta(t)$ and integrating from t_5 to t , we get

$$\begin{aligned} w(t)\delta(t) - w(t_5)\delta(t_5) + \int_{t_5}^t \frac{w(s)}{r(s)} ds + \int_{t_5}^t \frac{w^2(s)\delta(s)}{r(s)} ds \\ \leq - \int_{t_5}^t Cq(s)(\sigma(s))^{n-2}\delta(s) ds, \end{aligned} \tag{36}$$

or

$$\begin{aligned} w(t)\delta(t) - w(t_5)\delta(t_5) \\ \leq - \int_{t_5}^t \left[Cq(s)(\sigma(s))^{n-2}\delta(s) - \frac{1}{4r(s)\delta(s)} \right] ds \\ - \int_{t_5}^t \frac{[w(s)\delta(s) + 1/2]^2}{r(s)\delta(s)} ds. \end{aligned} \tag{37}$$

Thus,

$$\begin{aligned} w(t)\delta(t) \leq - \int_{t_5}^t \left[Cq(s)(\sigma(s))^{n-2}\delta(s) - \frac{1}{4r(s)\delta(s)} \right] ds \\ + w(t_5)\delta(t_5). \end{aligned} \tag{38}$$

Using assumption (26), we see that $w(t)\delta(t) \rightarrow -\infty$ as $t \rightarrow \infty$. But this contradicts (35).

If the solution $x(t)$ is bounded, then $z(t)$ is also bounded. Since $z(t)z^{(n-1)}(t) < 0$ and $n - 1$ is odd, we have by Lemma 1 that $l = 0$ (otherwise, $z(t)$ is not bounded). Hence, from (4) and (5), we have

$$(-1)^i z^{(i)}(t) > 0, \quad i = 0, 1, \dots, n - 2 \text{ for } t \geq t_3 \geq t_2. \tag{39}$$

From (11) and the fact that $z(t) > x(t)$, we obtain

$$x(t) \geq z(t) - p_0 x(\tau(t)) \geq z(t) - p_0 z(\tau(t)), \tag{40}$$

or

$$x(t) \geq z(\tau(t)) \left[\frac{z(t)}{z(\tau(t))} - p_0 \right]. \tag{41}$$

From (39) $z(t) > 0$, $z'(t) < 0$, and $z''(t) > 0$, we have $\lim_{t \rightarrow \infty} z(t) = \lambda \geq 0$. Now, we consider two cases.

Case I. Consider that $\lambda > 0$. Since $z(t)$ is decreasing, there is an $\varepsilon > 0$ and $t_4 \geq t_3$ such that, for $t \geq t_4$,

$$\lambda \leq z(t) \leq z(\tau(t)) \leq \lambda + \varepsilon. \tag{42}$$

From this, we can conclude that

$$\frac{z(t)}{z(\tau(t))} \geq \frac{\lambda}{\lambda + \varepsilon}. \tag{43}$$

Choose $t_5 \geq t_4$ such that, for $t \geq t_5$, we have $p_0 + \varepsilon_1 \leq \lambda/(\lambda + \varepsilon)$, for some $\varepsilon_1 > 0$. Thus,

$$\frac{z(t)}{z(\tau(t))} \geq p_0 + \varepsilon_1, \quad t \geq t_5. \tag{44}$$

Using this inequality in (41) and the fact that $z(t)$ is decreasing, we obtain

$$x(t) \geq \varepsilon_1 z(t), \quad t \geq t_5. \tag{45}$$

By Lemma 3 and the fact that $z^{(n-2)}(t)$ is decreasing, we get

$$z(\sigma(t)) \geq \frac{\theta[\sigma(t)]^{n-2}}{(n-2)!} z^{(n-2)}(t), \quad t \geq t_6 \geq t_3. \tag{46}$$

Combining (H4), (45), and (46), we obtain

$$\begin{aligned} f(x(\sigma(t))) \geq C(\sigma(t))^{n-2} z^{(n-2)}(t), \\ \text{for } t \geq t_7 = \max\{t_5, t_6\}, \end{aligned} \tag{47}$$

where $C = \alpha K/(n-2)!$ with $\alpha = \varepsilon_1 \theta \in (0, 1)$.

By using the transformation (30) and proceeding as in the previous assumption ($x(t)$ unbounded), again we obtain a contradiction with (26).

Case II. Consider that $\lambda = 0$, since $x(t) \leq z(t)$, $x(t)$ tends to zero as $t \rightarrow \infty$, and this completes the proof. \square

4. Examples

In this section, we present some examples to illustrate the above results.

Example 1. Consider the following even order nonlinear neutral differential equation:

$$\begin{aligned} \left[\sqrt{t} \left[x(t) + \left(\frac{1}{2} - \frac{1}{2t+2} \right) x(t-3) \right]^{(n-1)} \right]' \\ + t^2 x \left(\frac{t-2}{2} \right) \left(1 + \frac{1}{x^2((t-2)/2) + 1} \right) = 0, \end{aligned} \tag{48}$$

$t > 0$.

Here, $r(t) = \sqrt{t}$, $p(t) = (1/2 - 1/(2t+2))$, $q(t) = t^2$, $\tau(t) = t-3$, $\sigma(t) = (t-2)/2$, and $f(x)/x \geq K = 1$. We can see that all conditions of Theorem 4 are satisfied. Thus, every solution of (48) is oscillatory.

The function $p(t) = (1/2 - 1/(2t+2))$ in (48) lies in the interval $(0, 1/2)$; that is, $p_0 = 1/2 < 1$. Now, for the same equation, if $p(t)$ is replaced by $p(t) = (1 - 1/(2t+2))$, then there is no such $p_0 < 1$. In this case, by using condition (24), we conclude again that every solution of (48) with $p(t) = (1 - 1/(2t+2))$ is oscillatory.

Example 2. Consider the following nonlinear neutral differential equation:

$$\left[e^{\alpha t} [x(t) + p_0 x(t - \tau_0)]^{(n-1)} \right]' + e^{\beta t} x(\lambda t) = 0, \quad t > 0, \tag{49}$$

where $n \geq 4$ is even, $\tau_0 > 0$, $0 < \lambda < 1$, $0 < p_0 < 1$, and $\beta \geq \alpha > 0$.

We can see that all conditions of Theorem 7 are satisfied. Thus, every solution of (49) either is oscillatory or tends to zero as $t \rightarrow \infty$.

Example 3. Consider the following even order nonlinear neutral differential equation:

$$\left[e^t \left[x(t) + \frac{1}{a} x \left(\frac{t}{a} - n \ln a \right) \right]^{(n-1)} \right]' + \left(1 - \frac{1}{a} \right) e^{(1+1/b-1/a)t} x \left(\frac{t}{b} \right) = 0, \quad (50)$$

$$t > 0,$$

where $n \geq 4$ and $a > b > 1$. It is easy to check that all conditions of Theorem 7 are satisfied. Thus, every solution of (50) either is oscillatory or tends to zero as $t \rightarrow \infty$. Indeed, $x(t) = e^{-t}$ is a solution that tends to zero as $t \rightarrow \infty$.

Note that the results of Sun et al. [13] cannot be applied in the above examples, since the τ and σ are not commute.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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