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Research Article

Infinitely Many Homoclinic Solutions for Nonperiodic Fourth Order Differential Equations with General Potentials

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We investigate a class of nonperiodic fourth order differential equations with general potentials. By using variational methods and genus properties in critical point theory, we obtain that such equations possess infinitely homoclinic solutions.

1. Introduction

In this paper, we consider the following a class of fourth order differential equations:

$$u^{(4)} + wu'' + a(x)u = f(x,u), \quad x \in \mathbb{R}, \tag{1}$$

where w is a constant, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $a \in C(\mathbb{R}, \mathbb{R})$.

Recently, a lot of attention has been focused on the study of homoclinic and heteroclinic solutions for this problem; see [1–8]. This may be due to its concrete applications, such as physics and mechanics; see [9, 10]. More precisely, Tersian and Chaparova [5] studied periodic case. They obtained nontrivial homoclinic solutions by using mountain pass theorem. For nonperiodic case, Li [7] studied the existence of nontrivial homoclinic solutions for this class of equations. Sun and Wu [8] studied multiple homoclinic solutions for the following nonperiodic fourth order equations with a perturbation:

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R}, \quad 1 \le p < 2. \tag{2}$$

By using the mountain pass theorem and Ekeland variational principle, two homoclinic solutions for these equations are obtained under the assumption that A_0 and f are superlinear or asymptotically linear as $|u| \to +\infty$, where (A_0) is the following condition:

(A_0) there exists a positive constant a_1 such that $0 < a_1 \le a(x) \to +\infty$ as $|x| \to +\infty$ and $w \le 2\sqrt{a_1}$.

The assumption (A_0) is too strict to be satisfied by many general functions $a(x)$; for example, $a(x) = 1$. In addition, although there is perturbation, the right of (2) is superlinear or asymptotically linear as $|u| \to +\infty$. In the present paper we study the infinitely many homoclinic solutions for (1) under more general assumption than A_0 and sublinear condition on f.

Before stating our results we introduce some notations. Throughout this paper, we denote by $\|\|_r$ the L^r-norm, $2 \le r \le \infty$. $L^\infty(\mathbb{R})$ is the Banach space of essentially bounded functions equipped with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}\{|u(x)| : x \in \mathbb{R}\}. \tag{3}$$

If we take a subsequence of a sequence $\{u_n\}$ we will denote it again by $\{u_n\}$.

Now we state our main result.

Theorem 1. *Assume that the following conditions hold:*

(A) *there exists a positive constant a_1 such that $a(x) \ge a_1 > 0$ and $2\sqrt{a_1} \ge w$;*

(F_1) *there exists a constant $1 < \gamma < 2$ and positive function $b(x) \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R}^+)$ such that*

$$|f(x,u)| \le \gamma b(x)|u|^{\gamma-1}, \quad \forall (x,u) \in \mathbb{R} \times \mathbb{R}; \tag{4}$$

(F_2) *there exist* $x_0 \in \mathbb{R}$ *and* $\nu \in (1,2)$ *such that*

$$\liminf_{(x,u)\to(x_0,0)} \frac{F(x,u)}{|u|^{\nu}} > 0, \tag{5}$$

where F *is the primitive* $F(x,u) = \int_0^u f(x,t)dt$.

Then, problem (1) *possesses at least one nontrivial homoclinic solution.*

In addition, if f *is odd symmetry in* u, *that is,*

(F_3) $f(x,u) = -f(x,-u)$, $\forall (x,u) \in \mathbb{R}\times\mathbb{R}$,

then one gets the existence of infinitely many nontrivial homoclinic solutions.

Theorem 2. *Under the assumptions of* (A), (F_1)–(F_3), *problem* (1) *possesses infinitely many nontrivial homoclinic solutions.*

Example 3. If $f(x,u) = (4\cos x/3e^{|x|})|u|^{-2/3}u$, clearly,

$$\left|f(x,u)\right| \le \frac{4}{3e^{|x|}}|u|^{1/3}u, \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R},$$

$$F(x,u) = \frac{\cos x}{e^{|x|}}|u|^{4/3} \ge \frac{\cos 1}{e}|u|^{4/3}, \quad \forall (x,u) \in [0,1]\times\mathbb{R}. \tag{6}$$

Thus (F_1), (F_2), and (F_3) are satisfied with $x_0 = 0$, $\nu = 4/3$, $\gamma = 4/3$.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of our main results.

2. Preliminaries

In order to prove our main results, we first give some properties of space X on which the variational setting for problem (1) is defined.

Lemma 4 (see [5]). *Assume that* (A) *hold. Then there exists a constant* $c_0 > 0$ *such that*

$$\int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2\right]dx \ge c_0\|u\|_{H^2}^2, \quad \forall u \in H^2(\mathbb{R}), \tag{7}$$

where $\|u\|_{H^2}^2 = \left(\int_{\mathbb{R}}[u''(x)^2 + u'(x)^2 + u(x)^2]dx\right)^{1/2}$ *is the norm of Sobolev space* $H^2(\mathbb{R})$.

Letting

$$X = \left\{u \in H^2(\mathbb{R}) \mid \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2\right]dx < +\infty\right\}, \tag{8}$$

then X *is a Hilbert space with the inner product*

$$(u,v) = \int_{\mathbb{R}} \left[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)\right]dx \tag{9}$$

and the corresponding norm $\|u\|^2 = (u,u)$. *Note that*

$$X \subset H^2(\mathbb{R}) \subset L^r(\mathbb{R}), \tag{10}$$

for all $r \in [2,+\infty]$, *with the embedding being continuous. Hence, for any* $r \in [2,+\infty]$, *there is* $C_r > 0$ *such that*

$$\|u\|_r \le C_r\|u\|, \quad \forall u \in X. \tag{11}$$

Now we begin describing the variational formulation of problem (1). Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)\,dx. \tag{12}$$

Lemma 5. *Under the conditions of* (A), (F_1)–(F_2), $J \in C^1(X,\mathbb{R})$ *and its derivative is given by the following;*

$$J'(u)v = \int_{\mathbb{R}} \left[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)\right]dx - \int_{\mathbb{R}} f(x,u(x))v(x)\,dx, \tag{13}$$

for all $u, v \in X$. *In addition, any critical point of* J *on* X *is a classical solution of problem* (1).

Proof. We firstly show that $J : X \to \mathbb{R}$. From (F_1), one has

$$|F(x,u)| \le b(x)|u|^{\gamma}, \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R}. \tag{14}$$

By the Hölder inequality and (14), we have

$$\begin{aligned}\int_{\mathbb{R}} |F(x,u(x))|\,dx &\le \int_{\mathbb{R}} b(x)|u(x)|^{\gamma}dx \\ &\le \|b\|_{2/(2-\gamma)}\left(\int_{\mathbb{R}}|u(x)|^2dx\right)^{\gamma/2} \\ &\le C_2^{\gamma}\|b\|_{2/(2-\gamma)}\|u\|^{\gamma},\end{aligned} \tag{15}$$

where C_2 is constant in (11). Hence, J defined by (12) is well defined on X.

Next we prove that $J \in C^1(X,\mathbb{R})$. To this end, we rewrite J as follows:

$$A(u) = \frac{1}{2}\|u\|^2, \qquad B(u) = \int_{\mathbb{R}} F(x,u)\,dx. \tag{16}$$

It is easy to check that $A \in C^1(X,\mathbb{R})$, and we have $A'(u)v = \int_{\mathbb{R}}[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx$, for all $u, v \in X$. On the other hand, we will show that $B \in C^1(X,\mathbb{R})$ and

$$B'(u)v = \int_{\mathbb{R}} f(x,u(x))v(x)\,dx, \tag{17}$$

for any given $u, v \in X$. For any given $u \in X$, let us define

$$K(u)v = \int_{\mathbb{R}} f(x,u(x))v(x)\,dx, \quad \forall v \in X. \tag{18}$$

It is obvious that $K(u)$ is linear. Now we show that $K(u)$ is bounded. In fact, for any $u \in X$, by the Hölder inequality and (F_1), we can obtain that

$$\begin{aligned}
&|K(u)v| \\
&\quad \leq \int_{\mathbb{R}} |f(x,u(x))v(x)|\,dx \\
&\quad \leq \gamma \int_{\mathbb{R}} b(x)|u(x)|^{\gamma-1}|v(x)|\,dx \\
&\quad \leq \gamma \|b\|_{2/(2-\gamma)} \left(\int_{\mathbb{R}} |u(x)|^2 dx\right)^{(\gamma-1)/2} \left(\int_{\mathbb{R}} |v(x)|^2 dx\right)^{1/2} \\
&\quad \leq \gamma C_2^{\gamma} \|b\|_{2/(2-\gamma)} \|u\|^{\gamma-1} \|v\|.
\end{aligned} \tag{19}$$

Moreover, for $u, v \in X$, using the Mean Value Theorem, we get

$$\begin{aligned}
&\int_{\mathbb{R}} F(x,u(x)+v(x))\,dx - \int_{\mathbb{R}} F(x,u)\,dx \\
&\quad = \int_{\mathbb{R}} f(x,u(x)+\theta(x)v(x))v(x)\,dx,
\end{aligned} \tag{20}$$

for some $0 < \theta(x) < 1$. On the other hand, $b(x) \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R}^+)$, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\left(\int_{|x|>T} b^{2/(2-\gamma)}(x)\,dx\right)^{(2-\gamma)/2} < \varepsilon. \tag{21}$$

Therefore, on account of the Sobolev compact theorem ($X|_{[-T,T]}$ is compactly embedded in $L^{\infty}([-T,T],\mathbb{R})$) and Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{\|v\|}\left[\int_{\mathbb{R}} |f(x,u(x)+\theta(x)v(x))v(x) - f(x,u(x))v(x)|\,dx\right] \\
&\quad \leq \frac{1}{\|v\|}\left[\int_{|x|\leq T} |f(x,u(x)+\theta(x)v(x))v(x) - f(x,u(x))v(x)|\,dx\right] \\
&\qquad + \frac{1}{\|v\|}\left[\int_{|x|>T} |f(x,u(x)+\theta(x)v(x))v(x) - f(x,u(x))v(x)|\,dx\right] \\
&\quad \leq \frac{1}{\|v\|}\left[\int_{|x|\leq T} |f(x,u(x)+\theta(x)v(x)) - f(x,u(x))|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\leq T} |v(x)|^2 dx\right]^{1/2} \\
&\qquad + \frac{2}{\|v\|}\left[\int_{|x|>T} \gamma b(x)\left[|u(x)|^{\gamma-1} + |v(x)|^{\gamma-1}\right]|v(x)|\,dx\right] \\
&\quad \leq \frac{1}{\|v\|}\left[\int_{|x|\leq T} |f(x,u(x)+\theta(x)v(x)) - f(x,u(x))|^2 dx\right]^{1/2} C_2\|v\| \\
&\qquad + 2\gamma\left(\int_{|x|>T} |b(x)|^{2/(2-\gamma)}dx\right)^{(2-\gamma)/2} \\
&\qquad \times \left(\int_{|x|>T} |u(x)|^2 dx\right)^{(\gamma-1)/2} \left(\int_{|x|>T} |v(x)|^2 dx\right)^{1/2} \\
&\qquad + \gamma\left(\int_{|x|>T} |b(x)|^{2/(2-\gamma)}dx\right)^{(2-\gamma)/2} \\
&\qquad \times \left(\int_{|x|>T} |v(x)|^2 dx\right)^{\gamma/2} \\
&\quad \leq C_2\left[\int_{|x|\leq T} |f(x,u(x)+\theta(x)v(x)) - f(x,u(x))|^2 dx\right]^{1/2} \\
&\qquad + \varepsilon\gamma C_2^{\gamma}\left(\|u\|^{\gamma-1} + \|v\|^{\gamma-1}\right) \longrightarrow 0, \quad \text{as } v \longrightarrow 0,
\end{aligned} \tag{22}$$

which, together with (19), implies that (17) holds. It remains to show that B' is continuous. Suppose that $u \to u_0$ in X, then we have

$$\begin{aligned}
&\sup_{\|v\|=1} \left|B'(u)v - B'(u_0)v\right| \\
&\quad = \sup_{\|v\|=1}\left[\int_{\mathbb{R}} |f(x,u(x))v(x) - f(x,u_0(x))v(x)|\,dx\right] \\
&\quad \leq \sup_{\|v\|=1}\left[\int_{|x|\leq T} |f(x,u(x))v(x) - f(x,u_0(x))v(x)|\,dx\right] \\
&\qquad + \sup_{\|v\|=1}\left[\int_{|x|>T} |f(x,u(x))v(x) - f(x,u_0(x))v(x)|\,dx\right] \\
&\quad \leq \sup_{\|v\|=1}\left[\int_{|x|\leq T} |f(x,u(x)) - f(x,u_0(x))|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\leq T} |v(x)|^2 dx\right]^{1/2} \\
&\qquad + \sup_{\|v\|=1}\left[\int_{|x|>T} \gamma b(x)\left[|u(x)|^{\gamma-1} + |u_0(x)|^{\gamma-1}\right]|v(x)|\,dx\right] \\
&\quad \leq \sup_{\|v\|=1}\left[\int_{|x|\leq T} |f(x,u(x)) - f(x,u_0(x))|^2 dx\right]^{1/2} C_2\|v\| \\
&\qquad + \gamma \sup_{\|v\|=1}\left(\int_{|x|>T} |b(x)|^{2/(2-\gamma)}dx\right)^{(2-\gamma)/2}
\end{aligned}$$

$$\times \left(\int_{|x|>T} |u(x)|^2 dx \right)^{(\gamma-1)/2} \left(\int_{|x|>T} |v(x)|^2 dx \right)^{1/2}$$
$$+ \gamma \sup_{\|v\|=1} \left(\int_{|x|>T} |b(x)|^{2/(2-\gamma)} dx \right)^{(2-\gamma)/2}$$
$$\times \left(\int_{|x|>T} |u_0(x)|^2 dx \right)^{(\gamma-1)/2}$$
$$\times \left(\int_{|x|>T} |v(x)|^2 dx \right)^{1/2}$$
$$\le C_2 \left[\int_{|x|\le T} \left| f(x,u(x)) - f(x,u_0(x)) \right|^2 dx \right]^{1/2}$$
$$+ \varepsilon\gamma C_2^{\gamma} \left(\|u\|^{\gamma-1} + \|u_0\|^{\gamma-1} \right) \longrightarrow 0, \quad \text{as } u \longrightarrow u_0, \tag{23}$$

uniformly with respect to v, which implies that B' is continuous. Therefore, we obtain $J \in C^1(X,\mathbb{R})$ and its derivative is given by

$$J'(u)v = \int_{\mathbb{R}} \left[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x) \right] dx - \int_{\mathbb{R}} f(x,u(x))v(x)\,dx, \tag{24}$$

for all $u, v \in X$. In addition, from [8], we can know that any critical point of J on X is a classical solution of problem (1). □

Next, we give some useful Lemmas which can be seen in [11].

Definition 6. $I \in C^1(E,\mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_n\} \subset E, n \in \mathbb{N}$, for which $\{I(u_n)\}$ is bounded and $I'(u_n)) \to 0$ as $n \to +\infty$ and possesses a convergent subsequence in E.

Lemma 7. *Let E be a real Banach space and let $I \in C^1(E,\mathbb{R})$ satisfy the (PS) condition. If I is bounded from below, then $c = \inf_E I(u)$ is a critical value of I.*

To obtain the existence of infinitely many homoclinic solutions for problem (1) under the assumptions of Theorem 2, we will employ the "genus" properties in critical point theory; see [11].

Let E be a Banach space, $I \in C^1(E,\mathbb{R})$, and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$
$$K_c = \left\{ u \in E : I(u) = c, I'(u) = 0 \right\},$$
$$I^c = \{u \in E : I(u) \le c\}. \tag{25}$$

Definition 8. For $A \in \Sigma$, we say the genus of A is J (denoted by $\gamma(A) = j$) if there is an odd map $\psi \in C((A,\mathbb{R})^j \setminus \{0\})$ and j is the smallest integer with this property.

Lemma 9. *Let I be an even C^1 functional on E and satisfy the (PS) condition. For any $j \in \mathbb{N}$, set*

$$\Sigma_j = \{A \in \Sigma : \gamma(A) \ge j\}, \qquad c_j = \inf_{A\in\Sigma_j} \sup_{u\in A} I(u). \tag{26}$$

(i) *If $\Sigma_j \neq \emptyset$ and $c_j \in \mathbb{R}$, then c_j is a critical value of I;*
(ii) *if there exists $r \in \mathbb{N}$ such that*

$$c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R}, \tag{27}$$

and $c \neq I(0)$, then $\gamma(K_c) \ge r+1$.

Remark 10. From Remark 7.3 in [11], we know that if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then I has infinitely many distinct critical points in E.

3. Proof of the Main Results

To prove our main results, we first give the following Lemma.

Lemma 11. *If (A) and (F_2) hold, then J defined by (12) satisfies (PS) condition.*

Proof. By (12) and (14) and Hölder inequality, one has

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} b(x)|u(x)|^{\gamma} dx \\ &\ge \frac{1}{2}\|u\|^2 - \|b\|_{2/(2-\gamma)} \left(\int_{\mathbb{R}} |u(x)|^2 dx \right)^{\gamma/2} \\ &\ge \frac{1}{2}\|u\|^2 - C_2^{\gamma}\|b\|_{2/(2-\gamma)}\|u\|^{\gamma}. \end{aligned} \tag{28}$$

Since $1 < \gamma < 2$, (28) implies that $J(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, J is bounded from below.

Now, we show that J satisfies the (PS) condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset X$ is a sequence such that $\{J(u_k)\}_{k\in\mathbb{N}}$ is bounded and $J'(u_k) \to 0$ as $k \to +\infty$. Then by (28), there exists a constant $C > 0$ such that

$$\|u_k\|_2 \le C_2\|u_k\| \le C, \quad k \in \mathbb{N}. \tag{29}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in X. Since $b(x) \in L^{2/(2-\gamma)}(\mathbb{R},\mathbb{R}^+)$, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\left(\int_{|x|>T} b^{2/(2-\gamma)}(x)\,dx \right)^{(2-\gamma)/2} < \varepsilon. \tag{30}$$

Since the embedding of $X \hookrightarrow L^2_{\text{loc}}(\mathbb{R})$ is compact, $u_k \rightharpoonup u_0$ in X implies

$$\lim_{k\to\infty} \int_{|x|\le T} \left| u_k(x) - u_0(x) \right|^2 dx = 0. \tag{31}$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|f(x,u_k(x)) - f(x,u_0(x))\right| \left|u_k(x) - u_0(x)\right| dx \\
&\quad \le \left[\int_{|x|\le T} \left|f(x,u_k(x)) - f(x,u_0(x))\right| \left|u_k(x) - u_0(x)\right| dx\right] \\
&\qquad + \left[\int_{|x|>T} \left|f(x,u_k(x)) - f(x,u_0(x))\right| \right. \\
&\qquad\qquad \left. \times \left|u_k(x) - u_0(x)\right| dx\right] \\
&\quad \le \left[\int_{|x|\le T} \left|f(x,u_k(x)) - f(x,u_0(x))\right|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\le T} \left|u_k(x) - u_0(x)\right|^2 dx\right]^{1/2} \\
&\qquad + \int_{|x|>T} \gamma b(x) \left[\left|u_k(x)\right|^{\gamma-1} + \left|u_0(x)\right|^{\gamma-1}\right] \\
&\qquad\qquad \times \left[\left|u_k(x)\right| + \left|u_0(x)\right|\right] dx \\
&\quad \le \left[\int_{|x|\le T} \left|f(x,u_k(x)) - f(x,u_0(x))\right|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\le T} \left|u_k(x) - u_0(x)\right|^2 dx\right]^{1/2} \\
&\qquad + 2\gamma \int_{|x|>T} b(x) \left[\left|u_k(x)\right|^{\gamma} + \left|u_0(x)\right|^{\gamma}\right] dx \\
&\quad \le \left[\int_{|x|\le T} \left|f(x,u_k(x)) - f(x,u_0(x))\right|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\le T} \left|u_k(x) - u_0(x)\right|^2 dx\right]^{1/2} \\
&\qquad + 2\gamma \left[\int_{|x|>T} |b(x)|^{2/(2-\gamma)}\right]^{(2-\gamma)/2} \left[\|u_k\|_2^{\gamma} + \|u_0\|_2^{\gamma}\right] \\
&\quad \le \left[\int_{|x|\le T} \left|f(x,u_k(x)) - f(x,u_0(x))\right|^2 dx\right]^{1/2} \\
&\qquad \times \left[\int_{|x|\le T} \left|u_k(x) - u_0(x)\right|^2 dx\right]^{1/2} \\
&\qquad + 2\gamma \left[\int_{|x|>T} |b(x)|^{2/(2-\gamma)}\right]^{(2-\gamma)/2} \left[C^{\gamma} + \|u_0\|_2^{\gamma}\right].
\end{aligned} \tag{32}
$$

Hence, by (30), (31), and the fact that ε is arbitrary, one can get

$$
\int_{\mathbb{R}} \left|f(x,u_k(x)) - f(x,u_0(x))\right| \left|u_k(x) - u_0(x)\right| dx \longrightarrow 0 \tag{33}
$$

as $k \to \infty$. It follows from (13) that

$$
\begin{aligned}
&\left\langle J'(u_k) - J'(u_0), u_k - u_0 \right\rangle \\
&\quad = \|u_k - u_0\|^2 \\
&\qquad - \int_{\mathbb{R}} \left|f(x,u_k(x)) - f(x,u_0(x))\right| \left|u_k(x) - u_0(x)\right| dx.
\end{aligned} \tag{34}
$$

In view of the definition of weak convergence, we have

$$
\left\langle J'(u_k) - J'(u_0), u_k - u_0 \right\rangle \longrightarrow 0. \tag{35}
$$

Therefore, we can obtain that $u_k \to u_0$ in X. Hence, J satisfies (PS) condition. □

Now we are in the position to complete the proof of Theorems 1 and 2.

Proof of Theorem 1. It is obvious that $J(0) = 0$, and by Lemmas 5 and 11 we know that J is a C^1 functional on X satisfying the (PS) condition. In view of (28), we have J is bounded below on X. Hence, by Lemma 7, $c = \inf_E J(u)$ is a critical value of J; that is, there exists a critical point $u^* \in X$ such that $J(u^*) = c$.

In addition, by (F_2), there exists an open set $D \subset \mathbb{R}$ with $x_0 \in D$, $\sigma > 0$, $\eta > 0$ such that

$$
F(x,u) \ge \eta |u|^{\nu}, \quad \forall (x,u) \in D \times \mathbb{R}, |u| \le \sigma. \tag{36}
$$

Let $u_0 \in W_0^{2,2}(D) \cap X \setminus \{0\}$ and $\|u_0\|_{\infty} \le \sigma$; then we have

$$
\begin{aligned}
J(su_0) &= \frac{1}{2}s^2 \|u_0\|^2 - \int_{\mathbb{R}} F(x, su_0)\, dx \\
&= \frac{1}{2}s^2 \|u_0\|^2 - \int_{D} F(x, su_0)\, dx \\
&\le \frac{1}{2}s^2 \|u_0\|^2 - \eta s^{\nu} \int_{D} |u_0|^{\nu} dx,
\end{aligned} \tag{37}
$$

where $0 < s < 1$. Since $1 < \nu < 2$, one has $J(su_0) < 0$, for $s > 0$ small enough. Hence, $u^* \neq 0$, $J(u^*) < 0$; therefore u^* is a nontrivial homoclinic solution for problem (1). □

Proof of Theorem 2. Now, by (F_3), we have J is even and $J(0) = 0$. In order to apply Lemma 9, we prove that there exists $\varepsilon > 0$ such that

$$
\gamma\left(J^{-\varepsilon}\right) \ge j, \tag{38}
$$

for any $j \in \mathbb{N}$. For any $j \in \mathbb{N}$, we take j disjoint open sets D_i such that $\cup_{i=1}^{j} D_i \subset D$. For $i = 1, 2, \ldots, j$, let $u_i \in W_0^{2,2}(D_i) \cap X \setminus \{0\}$ with $\|u_i\| = 1$, and

$$
\begin{aligned}
E_j &= \operatorname{span}\left\{u_1, u_2, \ldots, u_j\right\}, \\
S_j &= \left\{u \in E_j : \|u\| = 1\right\}.
\end{aligned} \tag{39}
$$

Then, for any $u \in E_j$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, j$ such that

$$u(x) = \sum_{i=1}^{j} \lambda_i u_i(x). \tag{40}$$

Then,

$$\|u\|_\nu = \left(\int_{\mathbb{R}} |u(x)|^\nu dx \right)^{1/\nu} = \left(\sum_{i=1}^{j} \lambda_i^\nu \int_{D_i} |u(x)|^\nu dx \right)^{1/\nu}, \tag{41}$$

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x) u(x)^2 \right] dx \\ &= \sum_{i=1}^{j} \lambda_i^2 \int_{D_i} \left[u_i''(x)^2 - wu_i'(x)^2 + a(x) u_i(x)^2 \right] dx \\ &= \sum_{i=1}^{j} \lambda_i^2 \int_{\mathbb{R}} \left[u_i''(x)^2 - wu_i'(x)^2 + a(x) u_i(x)^2 \right] dx \\ &= \sum_{i=1}^{j} \lambda_i^2 \|u_i\|^2 = \sum_{i=1}^{j} \lambda_i^2. \end{aligned} \tag{42}$$

Since all norms of a finite dimensional norm space are equivalent, so there exists a constant $d > 0$ such that

$$d\|u\| \le \|u\|_\nu, \quad \forall u \in E_j. \tag{43}$$

For all $u \in S_j$ and sufficient small $s > 0$, we have

$$\begin{aligned} J(su) &= \frac{1}{2} s^2 \|u\|^2 - \int_{\mathbb{R}} F(x, su)\, dx \\ &= \frac{1}{2} s^2 \|u\|^2 - \sum_{i=1}^{j} \int_{D_i} F(x, s\lambda_i u_i(x))\, dx \\ &\le \frac{1}{2} s^2 \|u\|^2 - \eta s^\nu \sum_{i=1}^{j} |\lambda_i|^\nu \int_{D_i} |u_i|^\nu dx \\ &\le \frac{1}{2} s^2 \|u\|^2 - \eta s^\nu \|u\|_\nu^\nu \\ &\le \frac{1}{2} s^2 \|u\|^2 - \eta (ds)^\nu \|u\|^\nu \le \frac{1}{2} s^2 - \eta (ds)^\nu. \end{aligned} \tag{44}$$

In this case (36) is applicable, since u is continuous on $\overline{D}$ and so $|s\lambda_i u_i(x)| \le \sigma$, $\forall x \in D$, $i = 1, 2, \ldots, j$ can be true for sufficiently small s. Therefore, it follows from (44) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\delta u) < -\varepsilon \quad \text{for } u \in S_j. \tag{45}$$

Let

$$S_j^\delta = \left\{ \delta u : u \in S_j \right\}, \qquad \Omega = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_j) \in \mathbb{R}^j : \sum_{i=1}^{j} \lambda_i^2 < \delta^2 \right\}. \tag{46}$$

Then, it follows from (45) that

$$J(u) < -\varepsilon, \quad \forall u \in S_j^\delta, \tag{47}$$

which together with the fact that J is even C^1 functional on X, yields that

$$S_j^\delta \subset J^{-\varepsilon} \in \Sigma, \tag{48}$$

where $J^{-\varepsilon}$ and Σ have been previously introduced in Section 2. On the other hand, it follows from (40) and (42) that there exists an odd homeomorphism $\psi \in C(S_j^\delta, \partial\Omega)$. By some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [11]), we infer

$$\gamma(J^{-\varepsilon}) \ge \gamma\left(S_j^\delta\right) = j, \tag{49}$$

so the proof of (38) follows. Set

$$c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} J(u), \tag{50}$$

where Σ_j is defined in Lemma 9. It follows from (50) and the fact that J is bounded from below in X that we have $-\infty < c_j \le -\varepsilon < 0$, which implies that, for any $j \in \mathbb{N}$, c_j is a real negative number. By Lemma 9 and Remark 10, J has infinitely many nontrivial critical points, and consequently, problem (1) possesses infinitely many nontrivial homoclinic solutions. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. Chen and P. J. McKenna, "Traveling waves in a nonlinearly suspended beam: theoretical results and numerical observations," *Journal of Differential Equations*, vol. 136, no. 2, pp. 325–355, 1997.

[2] B. Buffoni, "Periodic and homoclinic orbits for Lorentz-Lagrangian systems via variational methods," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 3, pp. 443–462, 1996.

[3] D. Smets and J. B. van den Berg, "Homoclinic solutions for Swift-Hohenberg and suspension bridge type equations," *Journal of Differential Equations*, vol. 184, no. 1, pp. 78–96, 2002.

[4] S. Santra and J. Wei, "Homoclinic solutions for fourth order traveling wave equations," *SIAM Journal on Mathematical Analysis*, vol. 41, no. 5, pp. 2038–2056, 2009.

[5] S. Tersian and J. Chaparova, "Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations," *Journal of Mathematical Analysis and Applications*, vol. 260, no. 2, pp. 490–506, 2001.

[6] Y. L. Yeun, "Heteroclinic solutions for the extended Fisher-Kolmogorov equation," *Journal of Mathematical Analysis and Applications*, vol. 407, no. 1, pp. 119–129, 2013.

[7] C.-y. Li, "Remarks on homoclinic solutions for semilinear fourth-order ordinary differential equations without periodicity," *Applied Mathematics*, vol. 24, no. 1, pp. 49–55, 2009.

[8] J. Sun and T.-f. Wu, "Two homoclinic solutions for a nonperiodic fourth order differential equation with a perturbation," *Journal of Mathematical Analysis and Applications*, vol. 413, no. 2, pp. 622–632, 2014.

[9] P. Coullet, C. Elphick, and D. Repaux, "Nature of spatial chaos," *Physical Review Letters*, vol. 58, no. 5, pp. 431–434, 1987.

[10] J. Lega, J. V. Moloney, and A. C. Newell, "Swift-Hohenberg equation for lasers," *Physical Review Letters*, vol. 73, no. 22, pp. 2978–2981, 1994.

[11] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.