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Research Article

The Research of Periodic Solutions of Time-Varying Differential Models

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We have studied the periodicity of solutions of some nonlinear time-varying differential models by using the theory of reflecting functions. We have established a new relationship between the linear differential system and the Riccati equations and applied the obtained results to discuss the behavior of periodic solutions of the Riccati equations.

1. Introduction

From the book [1] we know that a lot of biological models can be expressed by the following differential system:

$$\begin{aligned} x' &= P(t,x), \\ y' &= q_0(t,x) + q_1(t,x)\,y + q_2(t,x)\,y^2 = Q(t,x,y) \end{aligned} \tag{1}$$

which has a continuous differentiable right-hand side, and have an unique solution for their initial value problem.

Because of this nonlinear and time-varying differential system, to discuss its qualitative behavior is very difficult. Now, we use the method of reflecting function [2, 3] to study the property of solutions of (1) and got some good results.

In the present section, we introduce the concept of the reflecting function, which will be used throughout the rest of this paper.

Consider differential system

$$x' = X(t,x), \quad t \in R,\ x \in R^n, \tag{2}$$

which has a continuous differentiable right-hand side, with a general solution $\varphi(t; t_0, x_0)$. For each such system, the *reflecting function* [2, 3] is defined as $F(t,x) := \varphi(-t,t,x)$. Therefore, for any solution $x(t)$ of (2), we have $F(t,x(t)) = x(-t)$.

If system (2) is 2ω-periodic with respect to t, then $T(x) := F(-\omega,x) = \varphi(\omega;-\omega,x)$ is the Poincaré mapping of (2) over the period $[-\omega,\omega]$. Thus, the solution $x = \varphi(t;-\omega,x_0)$ of (2) defined on $[-\omega,\omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$. The stability of this periodic solution is equivalent to the stability of the fixed point x_0.

A differentiable function $F(t,x)$ is a reflecting function of system (2) if and only if it is a solution of the Cauchy problem

$$F'_t + F'_x X(t,x) + X(-t,F) = 0, \qquad F(0,x) = x. \tag{3}$$

If the reflecting function of (2) has the form of $F(t,x) = F(t)x$, then the matrix $F(t)$ is called *reflecting matrix*. Thus, this matrix is a reflecting matrix of linear system $x' = A(t)x$, if and only if

$$F'(t) + F(t)A(t) + A(-t)F(t) = 0, \qquad F(0) = E. \tag{4}$$

There are many papers which are devoted to investigations of qualitative behavior of solutions of differential systems by help of reflecting functions. V. I. Mironenko and V. V. Mironenko [2–6] combined the theory of reflecting function with the integral manifold to discuss the symmetry and other geometric properties of solutions of (2), and obtained a lot of excellent new conclusions. Alisevich [7] has discussed when a linear system has triangular reflecting function. Musafirov [8] has studied when a linear system has reflecting function which can be expressed as a product of three-exponential matrix. Veresovich [9] has researched when the nonautonomous two-dimensional quadric systems are equivalent to a linear system. Maĭorovskaya [10] has established the sufficient conditions under which the quadratic systems have linear reflecting function. Zhou [11–13] has discussed the structure of reflecting function of quadratic systems and applied the obtained conclusions to study the qualitative behavior of solutions of such differential systems.

In this paper we will discuss when the system (1) has the reflecting function in the form of

$$F(t,x,y) = (F_1,F_2)^T = \left(x, \frac{\beta(t,x)+\varsigma(t,x)y}{\gamma(t,x)+\alpha(t,x)y}\right)^T, \quad (5)$$

where $\alpha(t,x)$, $\beta(t,x)$, $\gamma(t,x)$, and $\varsigma(t,x)$ are continuously differentiable functions in R^2. We will give the sufficient conditions for system (1) which has reflecting function of the form (5). We establish the relationship between the Riccati equation

$$\frac{dy}{dt} = a_0(t) + a_1(t)y + a_2(t)y^2 \quad (6)$$

and linear system

$$\frac{dz}{dt} = \begin{pmatrix} -\frac{1}{2}a_1(t) & -a_2(t) \\ a_0(t) & \frac{1}{2}a_1(t) \end{pmatrix} z. \quad (7)$$

We apply the obtained conclusions to study the behavior of solutions of system (1) and the above Riccati equations.

In the following, we will denote $a = a(t,x)$, $\overline{a} = a(-t,x)$, $F = F(t,x,y)$, and so forth.

2. Main Results

Lemma 1. *If the function* (5) *is the reflecting function of any system, then for small* $|t|$, *it can be written in the form* $F_2 = (\widehat{\beta} + \overline{\gamma}e^{\delta}y)/(\gamma e^{-\delta} + \widehat{\alpha}y)$, *in which*

$$\widehat{\alpha} + \overline{\widehat{\alpha}} = 0, \qquad \widehat{\beta} + \overline{\widehat{\beta}} = 0, \qquad \delta + \overline{\delta} = 0. \quad (8)$$

Proof. As the function (5) is a reflecting function, so $F_2(-t,F_1,F_2) = y$, $F_2(0,x,y) = y$; that is,

$$y\left(\gamma\overline{\gamma} + \overline{\alpha}\beta + \left(\alpha\overline{\gamma} + \overline{\alpha}\varsigma\right)y\right) = \gamma\overline{\beta} + \beta\overline{\varsigma} + \left(\alpha\overline{\beta} + \varsigma\overline{\varsigma}\right)y. \quad (9)$$

Equating the coefficients of the same power of y, we get

$$\overline{\beta}\gamma + \beta\overline{\varsigma} = 0; \quad (10)$$

$$\gamma\overline{\gamma} + \overline{\alpha}\beta = \alpha\overline{\beta} + \varsigma\overline{\varsigma}; \quad (11)$$

$$\alpha\overline{\gamma} + \overline{\alpha}\varsigma = 0, \quad (12)$$

$$\alpha(0,x) = \beta(0,x) = 0, \qquad \gamma(0,x) = \varsigma(0,x) = 1. \quad (13)$$

The relation (11) implies that

$$\gamma\overline{\gamma} = \varsigma\overline{\varsigma}, \qquad \alpha\overline{\beta} = \beta\overline{\alpha}. \quad (14)$$

Thus,

$$\gamma = \overline{\varsigma}e^{2\delta}, \qquad \delta + \overline{\delta} = 0. \quad (15)$$

Substituting it into (10) and (12), we get

$$\beta = \widehat{\beta}e^{\delta}, \qquad \alpha = \widehat{\alpha}e^{\delta}, \qquad \widehat{\beta} + \overline{\widehat{\beta}} = 0, \qquad \widehat{\alpha} + \overline{\widehat{\alpha}} = 0. \quad (16)$$

So

$$F_2 = \frac{\widehat{\beta}e^{\delta} + \overline{\gamma}e^{2\delta}y}{\gamma + \widehat{\alpha}e^{\delta}y} = \frac{\widehat{\beta} + \overline{\gamma}e^{\delta}y}{\gamma e^{-\delta} + \widehat{\alpha}y}. \quad (17)$$

The proof is finished. □

By Lemma 1, in the following, we all suppose $F_2 = (\beta + \overline{\gamma}y)/(\gamma + \alpha y)$, in which $\alpha + \overline{\alpha} = \beta + \overline{\beta} = 0$.

Theorem 2. *Suppose that* m, n, α, *and* β *are the solutions of the Cauchy problem*

$$\begin{aligned}
&m'_t + Pm'_x = q_{1e}n + q_{2e}\beta - q_{0e}\alpha, && m(0,x) = 1;\\
&n'_t + Pn'_x = -q_{0o}\alpha + q_{1e}m - q_{2o}\beta, && n(0,x) = 0;\\
&\alpha'_t + P\alpha'_x = -q_{1o}\alpha + 2q_{2e}m + 2q_{2o}n, && \alpha(0,x) = 0;\\
&\beta'_t + P\alpha'_x = 2q_{0o}n + \beta q_{1o} - 2q_{0e}m, && \beta(0,x) = 0,
\end{aligned} \quad (18)$$

where $P + \overline{P} = 0$ *and* $q_{ie} = (q_i + \overline{q}_i)/2$, $q_{io} = (q_i - \overline{q}_i)/2$, $i = 0,1,2$.

Then

$$m = \overline{m}, \qquad n + \overline{n} = \alpha + \overline{\alpha} = \beta + \overline{\beta} = 0, \qquad m^2 - n^2 - \alpha\beta = 1 \quad (19)$$

$$F = \left(x, \frac{\beta + \overline{\gamma}y}{\gamma + \alpha y}\right)^T \quad (20)$$

is the reflecting function of system (1). *Therefore, nearby* $t = 0$, $\gamma + \alpha y > 0$, *where* $\gamma = m + n$.

Proof. Putting

$$y_1 = m - \overline{m}; \qquad y_2 = n + \overline{n}, \qquad y_3 = \alpha + \overline{\alpha}, \qquad y_4 = \beta + \overline{\beta}, \quad (21)$$

then by (18), we have

$$\begin{aligned}
&y'_{1t} + y'_{1x}P = q_{1e}y_2 + q_{2e}y_4 - q_{0e}y_3, && y_1(0,x) = 0;\\
&y'_{2t} + y'_{2x}P = -q_{0o}y_3 + q_{1e}y_1 - q_{2o}y_4, && y_2(0,x) = 0;\\
&y'_{3t} + y'_{3x}P = -q_{1o}y_3 + 2q_{2e}y_1 + 2q_{2o}y_2, && y_3(0,x) = 0;\\
&y'_{4t} + y'_{4x}P = 2q_{0o}y_2 + q_{1o}y_4 - 2q_{0e}y_1, && y_4(0,x) = 0.
\end{aligned} \quad (22)$$

By the uniqueness of the solution of linear partial differential equations, we get

$$y_1 = y_2 = y_3 = y_4 \equiv 0; \tag{23}$$

that is,

$$m = \overline{m}, \qquad n + \overline{n} = \alpha + \overline{\alpha} = \beta + \overline{\beta} = 0. \tag{24}$$

Obviously, $\gamma\overline{\gamma} - \alpha\beta = C$ is the first integral of system (18); by the initial conditions, we have $\gamma\overline{\gamma} - \alpha\beta \equiv 1$. Thus, the functions γ and α have not the common zeros. As $\gamma(0, x) + \alpha(0, x)y = 1 > 0$, so in the nearby area of $t = 0$, $\gamma + \alpha y > 0$.

Using the conditions of the present theorem, it is not difficult to check that function (20) satisfies the relation (3) with respect to $X = (P, Q)^T$. So, the function (20) is the reflecting function of (1). The proof is completed. □

Corollary 3. *If all the conditions of Theorem 2 are satisfied, and the system* (1) *is 2ω-periodic with respect to t, then its Poincaré mapping can be expressed by*

$$T(x, y) = \left(x, \frac{-\beta(\omega, x) + \gamma(\omega, x)y}{\gamma(-\omega, x) - \alpha(\omega, x)y}\right)^T. \tag{25}$$

So, the solution $(x(t), y(t))$ of (1) *defined on $[-\omega, \omega]$ is 2ω-periodic, if and only if $(x(-\omega), y(-\omega))$ is the solution of equations*

$$\alpha(\omega, x)y^2 + 2\gamma_o(\omega, x)y - \beta(\omega, x) = 0, \qquad x(-\omega) = x(\omega). \tag{26}$$

Corollary 4. *Suppose that*

$$q_0 + \overline{q}_0 = 0, \qquad P + \overline{P} = 0, \qquad \lim_{t\to 0}\frac{q_{1e}}{q_0} = 0, \tag{27}$$

and $\alpha = q_{1e}/q_0$ is continuously differentiable and satisfies

$$\alpha'_t + \alpha'_x P = -q_{1o}\alpha + 2q_{2e}. \tag{28}$$

Then $F = (x, y/(1 + \alpha y))^T$ is the reflecting function of system (1)*. In addition, if the system* (1) *is a 2ω-periodic system with respect to t, then all the solutions of* (1) *defined on $[-\omega, \omega]$ are 2ω-periodic.*

Proof. It is not difficult to check that, under the conditions of above corollary, the function $F = (x, y/(1 + \alpha y))^T$ is the solution of the Cauchy problem (3), so it is a reflecting function of (1). In view of $\alpha = q_{1e}/q_0$, so $\alpha(-\omega, x) \equiv 0$ and $F(-\omega, x, y) \equiv (x, y)^T$. Thus, the conclusion of above corollary is true. □

Example 5. Differential system

$$\begin{aligned} x' &= \phi(t, x)\sin t\cos x, \\ y' &= \sin t\left(1 + \cos^2 x\right) \\ &\quad + y\left(\phi(t, x)\sin t\sin x + \sin^2 t\left(1 + \cos^2 x\right)\cos x\right) \\ &\quad + y^2\left(\frac{1}{2}\cos t\cos x + \psi(t, x)\right) \end{aligned} \tag{29}$$

has reflecting function $F = (x, y/(1 + y\sin t\cos x))^T$. Where $\phi(t, x) = \phi(-t, x)$, $\psi(t, x) + \psi(-t, x) = 0$. When $\phi(t + 2\pi, x) = \phi(t, x)$, $\psi(t + 2\pi, x) = \psi(t, x)$, all the solutions of above system defined on $[-\pi, \pi]$ are 2π-periodic.

Example 6. Differential system

$$\begin{aligned} x' &= \lambda(t, x)\cos x, \\ y' &= \mu(t, x)\sin t\cos x + y\left(\mu\sin t\cos^2 x + \lambda(t, x)\sin x\right) \\ &\quad + y^2\left(\frac{1}{2}\cos t\cos x + \kappa(t, x)\right), \end{aligned} \tag{30}$$

has reflecting function $F = (x, y/(1 + y\sin t\cos x))^T$. Where $\lambda(t, x) + \lambda(-t, x) = 0$, $\mu(t, x) = \mu(-t, x)$, $\kappa(t, x) + \kappa(-t, x) = 0$.

If the functions $\lambda(t, x)$, $\mu(t, x)$, $\kappa(t, x)$ are 2π-periodic with respect to t, then all the solutions of above system defined on $[-\pi, \pi]$ are 2π-periodic.

Similar to Corollary 4, we get the following.

Corollary 7. *If*

$$P + \overline{P} = 0, \qquad q_2 + \overline{q}_2 = 0, \qquad \lim_{t\to 0}\frac{q_{1e}}{q_{2o}} = 0, \tag{31}$$

$\beta = q_{1e}/q_{2o}$ is continuously differentiable and satisfies

$$\beta'_t + \beta'_x P = \beta q_{1o} - 2q_{0e}. \tag{32}$$

Then $F = (x, \beta + y)^T$ is the reflecting function of system (1)*. In addition, if the system* (1) *is a 2ω-periodic system, then all the solutions of* (1) *defined on $[-\omega, \omega]$ are 2ω-periodic.*

Example 8. Differential system

$$\begin{aligned} x' &= \rho(t, x)\cos x, \\ y' &= \xi(t, x) - \frac{1}{2}\cos t\cos x \\ &\quad + \left(-\rho(t, x)\sin x + \eta(t, x)\sin^2 t\cos x\right)y \\ &\quad + \eta(t, x)\sin t y^2 \end{aligned} \tag{33}$$

has reflecting function $F = (x, \sin t\cos x + y)^T$.

Where $\rho(t, x) + \rho(-t, x) = 0$, $\xi(t, x) + \xi(-t, x) = 0$, $\eta(t, x) = \eta(-t, x)$. Besides, if functions $\rho(t, x)$, $\xi(t, x)$, $\eta(t, x)$ are 2π-periodic with respect to t, then all the solutions of above system defined on $[-\pi, \pi]$ are 2π-periodic.

Corollary 9. *Suppose that $P + \overline{P} = 0$,*

$$\begin{aligned} &\delta'_t + \delta'_x P = q_{2e} + q_{0e}, \qquad \delta(0, x) = 0, \\ &\left(q_{0e} - q_{2e}\right)\cos\delta + q_{1o}\sin\delta = 0; \\ &q_{1e}\cos\delta + \left(q_{2o} - q_{0o}\right)\sin\delta = 0. \end{aligned} \tag{34}$$

and then $F = (x, (-\sin\delta + y\cos\delta)/(\cos\delta + y\sin\delta))^T$ is the reflecting function of (1)*. In addition, if the system* (1) *is a 2ω-periodic system, then the solution $(x(t), y(t))$ of* (1) *defined on*

$[-\omega,\omega]$ is 2ω-periodic, and if and only if $\delta(\omega, x(-\omega)) = k\pi$, k is a natural number.

Proof. In Theorem 2 taking $n = 0$, $\beta = -\alpha = -\sin\delta$, $m = \cos\delta$, under the above hypothesis, the function $F = (x, (-\sin\delta + y\cos\delta)/(\cos\delta + y\sin\delta))^T$ is the reflecting function of (1). Thus, when the system (1) is 2ω-periodic, its Poincaré mapping can be expressed by $T(x,y) = F(-\omega, x, y)$, its solution $(x(t), y(t))$ defined on $[-\omega,\omega]$ is 2ω-periodic, and if and only if $(y(-\omega)^2 + 1)\sin\delta(\omega, x(-\omega)) = 0$, it implies the conclusion of the present corollary. □

Example 10. Differential system

$$\begin{gathered} x' = P(t,x), \\ y' = q(t,x) + \frac{1+\sin^2 t}{2} + \left(\sin^2 t - \sin t\cos t\right) y \\ + \left(q(t,x) - \sin t\cos t + \frac{1-\sin^2 t}{2}\right) y^2 \end{gathered} \tag{35}$$

has reflecting function

$$F = \left(x, \frac{-\sin t + y\cos t}{\cos t + y\sin t}\right)^T \tag{36}$$

and all the solutions of above system defined on $[-\pi,\pi]$ are 2π-periodic, where functions $P(t,x)$ and $q(t,x)$ are arbitrary continuously differentiable odd and 2π-periodic with respect to t.

Theorem 11. *Let the reflecting function of linear system*

$$\frac{dz}{dt} = A(t)z, \quad A(t) = \begin{pmatrix} -\frac{1}{2}a_1(t) & -a_2(t) \\ a_0(t) & \frac{1}{2}a_1(t) \end{pmatrix}, \tag{37}$$

have the form

$$G(t,z) = \begin{pmatrix} g_1(t) & g_2(t) \\ g_3(t) & g_1(-t) \end{pmatrix} z. \tag{38}$$

Then the function

$$F = \frac{g_3(t) + g_1(-t)y}{g_1(t) + g_2(t)y} \tag{39}$$

is the reflecting function of the Riccati equation

$$\frac{dy}{dt} = a_0(t) + a_1(t)y + a_2(t)y^2. \tag{40}$$

Proof. As $G(t)$ is the reflecting matrix of the linear system (37), thus,

$$\begin{gathered} G'(t) + G(t)A(t) + A(-t)G(t) = 0, \\ G(0) = E, \quad |G(t)| \equiv 1, \end{gathered} \tag{41}$$

and it implies

$$\begin{aligned} g_{1e}' &= a_{1e}g_{1o} + a_{2e}g_3 - a_{0e}g_2, & g_{1e}(0) &= 1; \\ g_{1o}' &= -a_{0o}g_2 + a_{1e}g_{1e} - a_{2o}g_3, & g_{1o}(0) &= 0; \\ g_2' &= -a_{1o}g_2 + 2a_{2e}g_{1e} + 2a_{2o}g_{1o}, & g_2(0) &= 0; \\ g_3' &= 2a_{0o}g_{1o} + g_3a_{1o} - 2a_{0e}g_{1e}, & g_3(0) &= 0, \end{aligned} \tag{42}$$

where $a_{ie} = (a(t)+a(-t))/2$, $a_{io} = (a(t)-a(-t))/2$ $(i = 0,1,2)$, $g_{1e} = (g_1(t)+g_1(-t))/2$, and $g_{1o} = (g_1(t)-g_1(-t))/2$. Similar to Theorem 2, we get function (39) is the reflecting function of the Riccati equation (40). □

Corollary 12. *If $a_i(t+2\omega) = a_i(t)$ $(i = 0,1,2)$ and the conditions of Theorem 11 are satisfied, then*

(1) *if $g_2(\omega) \neq 0$ and $|g_{1e}(\omega)| > 1$, then the Riccati (40) has two 2ω-periodic solutions. If $g_2(\omega) \neq 0$ and $|g_{1e}(\omega)| < 1$, (40) has no one 2ω-periodic solution. If $g_2(\omega) \neq 0$ and $|g_{1e}(\omega)| = 1$, (40) has a unique 2ω-periodic solution;*

(2) *if $g_2(\omega) = 0$ and $g_{1o}(\omega) \neq 0$, then (40) has a unique 2ω-periodic solution. If $g_2(\omega) = 0$ and $g_{1o}(\omega) = 0$, $g_3(\omega) = 0$, all the solutions of (40) defined on $[-\omega,\omega]$ are 2ω-periodic; If $g_2(\omega) = 0$ and $g_{1o}(\omega) = 0$, $g_3(\omega) \neq 0$, (40) has no one 2ω-periodic solution.*

Proof. As (39) is the reflecting function of (40) and $g_1(t)g_1(-t) - g_2(t)g_3(t) \equiv 1$. Thus, the Poincaré mapping can be expressed by $T(y) = F(-\omega, y)$, and the solutions $y(t)$ of (40) defined on $[-\omega,\omega]$ are 2ω-periodic, if and only if $y(-\omega)$ is a solution of equation: $F(-\omega, y) = y$; that is,

$$g_2(\omega)y^2 + 2g_{1o}(\omega)y - g_3(\omega) = 0, \tag{43}$$

and from this relation and $g_{1e}^2(t) - 1 = g_{1o}^2(t) + g_2(t)g_3(t)$, it is easy to deduce the present conclusion. □

Theorem 13. *Suppose that*

(1) *$\lambda_1 = (a_{0e}+a_{2e})/a_{1o}$, $\lambda_0 = (a_{2o}-a_{0o})/a_{1o}$ are continuous and $\lim_{t\to 0}\lambda_1 = 0$;*

(2) *μ_1/μ_0 is continuously differentiable and $\lim_{t\to 0}(\mu_1/\mu_0) = 0$ and satisfies*

$$\begin{aligned} \left(\frac{\mu_1}{\mu_0}\right)' &= \left(a_{1e} + (a_{2e} - a_{0e})\lambda_0\right)\left(\frac{\mu_1}{\mu_0}\right)^2 \\ &\quad + \left((a_{0e} - a_{2e})\lambda_1 - (a_{0o} + a_{2o})\lambda_0\right)\frac{\mu_1}{\mu_0} \\ &\quad + (a_{0o} + a_{2o})\lambda_1 - a_{1e}. \end{aligned} \tag{44}$$

Then the function (39) is the reflecting function of the Riccati equation (40). In addition, if $a_i(t+2\omega) = a_i(t)$ $(i = 0,1,2)$, *then all the solutions of* (40) *defined on* $[-\omega,\omega]$ *are* 2ω-*periodic, where*

$$g_{1e}=\exp\int_0^t\left(\left(a_{2e}-a_{0e}\right)\lambda_1-\frac{\mu_1}{\mu_0}\left(a_{1e}+\lambda_0a_{2e}-\lambda_0a_{0e}\right)\right)dt,$$
$$g_{1o}=-\frac{\mu_1}{\mu_0}g_{1e},\qquad g_2=g_3=\lambda_1g_{1e}+\lambda_0g_{1o};$$
$$\mu_0=\lambda_0'+a_{1e}\lambda_1+a_{1o}\lambda_0-2a_{2o}+\left(a_{2e}-a_{0e}\right)\lambda_1\lambda_0-\left(a_{0o}+a_{2o}\right)\lambda_0^2;$$
$$\mu_1=\lambda_1'+\lambda_1^2\left(a_{2e}-a_{0e}\right)-\lambda_1\lambda_0\left(a_{0o}+a_{2o}\right)-\lambda_0a_{1e}+\lambda_1a_{1o}-2a_{2e}. \tag{45}$$

Proof. It is not difficult to check that under the hypothesis of Theorem 13 the functions $g_{1e}, g_{1o}, g_2=g_3=\lambda_1g_{1e}+\lambda_0g_{1o}$ are the solution of the Cauchy problem (42), so, the function (39) is 2ω-periodic reflecting function of (40); by [1] it implies the result of the present theorem. □

From a similar discussion we can get the following results.

Theorem 14. *Suppose that*

(1) $k_1=a_{1e}/a_{0o}$, $k_3=-a_{2o}/a_{0o}$ *are continuous and* $\lim_{t\to0}k_1=0$;

(2) ρ_1/ρ_3 *is continuously differentiable and* $\lim_{t\to0}(\rho_1/\rho_3)=0$ *and satisfies*

$$\left(\frac{\rho_1}{\rho_3}\right)'=\left(a_{2e}-a_{0e}k_3\right)\left(\frac{\rho_1}{\rho_3}\right)^2+\left(a_{1o}+a_{0e}k_1\right)\frac{\rho_1}{\rho_3}+2a_{0e}. \tag{46}$$

Then the function (39) *is the reflecting function of the Riccati equation* (40). *In addition, if* $a_i(t+2\omega)=a_i(t)$ $(i=0,1,2)$, *then all the solutions of* (40) *defined on* $[-\omega,\omega]$ *are* 2ω-*periodic, where*

$$g_1=\exp\int_0^t\left(\left(a_{0e}k_3-a_{2e}\right)\frac{\rho_1}{\rho_3}-a_{0e}k_1\right)dt;$$
$$g_3=-\frac{\rho_1}{\rho_3}g_1,\qquad g_2=k_1g_1+k_3g_3; \tag{47}$$
$$\rho_1=k_1'-a_{0e}k_1^2-2a_{0e}k_3+a_{1o}k_1-a_{2e};$$
$$\rho_3=k_3'+a_{2e}k_1-a_{0e}k_1k_3+2a_{1o}k_3.$$

Theorem 15. *If there is a continuously differentiable even function* $h(t)$ *and*

(1) $\eta_1=(a_{0e}h(t)+a_{2e})/(h(t)a_{0o}-a_{2o})$ *and* $\eta_3=(h'(t)+2a_{1o}h(t))/(2a_{2o}-2h(t)a_{0o})$ *are continuous and* $\lim_{t\to0}\eta_1=0$;

(2) ζ_1/ζ_3 *is continuously differentiable and* $\lim_{t\to0}(\zeta_1/\zeta_3)=0$ *and satisfies*

$$\left(\frac{\zeta_1}{\zeta_3}\right)'=\left(a_{2e}-a_{0e}h(t)+a_{1e}\eta_3\right)\left(\frac{\zeta_1}{\zeta_3}\right)^2+\left(a_{1o}++2a_{0o}\eta_3-a_{1e}\eta_1\right)\frac{\zeta_1}{\zeta_3}+2a_{0e}-2a_{0o}\eta_1. \tag{48}$$

Then the function (39) *is the reflecting function of the Riccati equation* (40). *In addition, if* $a_i(t+2\omega)=a_i(t)$ $(i=0,1,2)$, $h(t+2\omega)=h(t)$, *then all the solutions of* (40) *defined on* $[-\omega,\omega]$ *are* 2ω-*periodic, where*

$$g_{1e}=\exp\int_0^t\left(a_{1e}\eta_1-\frac{\zeta_1}{\zeta_3}\left(\eta_3a_{1e}+a_{2e}-a_{0e}h(t)\right)\right)dt;$$
$$g_{1o}=\left(\eta_1-\frac{\zeta_1}{\zeta_3}\eta_3\right)g_{1e},\qquad g_3=-\frac{\zeta_1}{\zeta_3}g_{1e},$$
$$g_2=h(t)\,g_3; \tag{49}$$
$$\zeta_1=\eta_1'+2a_{0o}\eta_1^2-2a_{0e}\eta_1+a_{1e}\eta_1\eta_3-a_{1e};$$
$$\zeta_3=\eta_3'+\left(a_{1o}+2a_{0o}\eta_3\right)\eta_1+a_{1e}\eta_3^2+\left(a_{2e}-a_{0e}h(t)\right)\eta_3+a_{2o}+a_{0o}h(t).$$

Example 16. The Riccati equation

$$y'=\frac{1}{2}\cos t\left(\sin^4t+\sin^2t+2\right)-2\sin t+\left(2\sin^2t-\sin^3t\cos t\right)y+\left(\frac{1}{2}\cos^3t+2\sin t\right)y^2 \tag{50}$$

has reflecting function

$$F=\frac{\sin t\left(2+\sin^2t\right)+\left(1+\sin^2t\right)y}{1+\sin^2t+y\sin t}. \tag{51}$$

So, all the solutions of the equation above defined on $[-\pi,\pi]$ are 2π-periodic.

Example 17. The Riccati equation

$$y'=-\frac{1}{2}Ce^S\left(2-2S+S^2-3S^3+S^4-S^5\right)+C\left(1+2S-S^3+S^4\right)y+\frac{1}{2}Ce^{-S}\left(1+S^2-3S^3+S^4-S^5\right)y^2 \tag{52}$$

has reflecting function

$$F=\frac{S\left(2+S^2\right)+\left(1+S^2\right)e^{-S}y}{\left(1+S^2\right)e^S+yS}, \tag{53}$$

where $S:=\sin t$ and $C=\cos t$. And all the solutions of the equation above defined on $[-\pi,\pi]$ are 2π-periodic.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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