Research Article

Oscillation Behavior for a Class of Differential Equation with Fractional-Order Derivatives

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By using a generalized Riccati transformation technique and an inequality, we establish some oscillation theorems for the fractional differential equation $[a(t)(p(t) + q(t)(D_-^\alpha x)(t))^\gamma]' - b(t)f\left(\int_t^\infty (s-t)^{-\alpha} x(s)ds\right) = 0$, for $t \geq t_0 > 0$, where $D_-^\alpha x$ is the Liouville right-sided fractional derivative of order $\alpha \in (0, 1)$ of x and γ is a quotient of odd positive integers. The results in this paper extend and improve the results given in the literatures (Chen, 2012).

1. Introduction

Differential equations with fractional-order derivatives have gained importance due to their various applications in science and engineering such as rheology, dynamical processes in self-similar and porous structures, heat conduction, control theory, electroanalytical chemistry, chemical physics, and economics; for example, see [1–7]. It is well recognized that fractional calculus leads to better results than classical calculus.

Many articles have investigated some aspects of differential equation with fractional-order derivatives, such as the existence and uniqueness for p-type fractional neutral differential equations, smoothness and stability of the solutions, and the methods for explicit and numerical solutions; for example, see [8–16]. However, to the best of the author's knowledge very little is known regarding the oscillatory behavior of differential equation with fractional-order derivatives up to now except for [17–27].

Grace initiated the study of oscillatory theory of FDE, and he considered the equations of the form

$$D_a^q x + f_1(t, x) = v(t) + f_2(t, x), \qquad \lim_{t \to a^+} J_a^{1-q} x(t) = b_1, \tag{1}$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0 < q < 1$. In fact, the IVP is equivalent to the Volteria fractional integral equation:

$$x(t) = \frac{b_1(t-a)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left[v(s) + f_2(s, x(s)) - f_1(s, x(s)) \right] ds. \tag{2}$$

He made use of the conditions:

$$x f_i(t, x) > 0 \quad (i = 1, 2), \; x \neq 0, \; t \geq a,$$
$$\left| f_1(t, x) \right| \geq p_1(t) |x|^\beta, \qquad \left| f_2(t, x) \right| \leq p_2(t) |x|^\gamma, \tag{3}$$
$$x \neq 0, \quad t \geq a,$$

where $p_1, p_2 \in C([a, \infty), \mathbb{R}^+)$ and $\beta, \gamma > 0$ are real numbers. He talked over the four cases of $f_2 = 0$; $\beta > 1$ and $\gamma = 1$; $\beta = 1$ and $\gamma < 1$; $\beta > 1$ and $\gamma < 1$. Besides that, he replaced $0 < q < 1$ with $m - 1 < q < m$ and got some results on the same cases by using an inequality; refer to [17].

Chen studied the oscillation of the differential equation with fractional-order derivatives:

$$\left[r(t) \left(D_-^\alpha y \right)^\eta (t) \right]' - q(t) f \left(\int_t^\infty (v-t)^{-\alpha} y(v) \, dv \right) = 0,$$

for $t > 0$,

(4)

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$\left(D_-^\alpha y \right)(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (v-t)^{-\alpha} y(v) \, dv$$

(5)

for $t \in \mathbb{R}_+ := (0, \infty)$,

and he obtained four main results under the condition of

$$\int_{t_0}^\infty r^{-1/\eta}(t) \, dt = \infty,$$

$$\int_{t_0}^\infty r^{-1/\eta}(t) \, dt < \infty$$

(6)

by using a generalized Riccati transformation technique and an inequality; see [18].

Using the same method, in 2013, Chen [23] studied oscillatory behavior of the fractional differential equation with the form

$$\left(D_-^{1+\alpha} y \right)(t) - p(t) \left(D_-^\alpha y \right)(t)$$

$$+ q(t) f \left(\int_t^\infty (v-t)^{-\alpha} y(v) \, dv \right) = 0 \quad \text{for } t > 0,$$

(7)

where $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0, 1)$ of y.

Zheng [24] considered the oscillation of the nonlinear fractional differential equation with damping term:

$$\left[a(t) (D_-^\alpha x(t))^\gamma \right]' + p(t) \left(D_-^\alpha x(t) \right)^\gamma$$

$$- q(t) f \left(\int_t^\infty (\xi-t)^{-\alpha} x(\xi) \, d\xi \right) = 0, \quad t \in [t_0, \infty),$$

(8)

where $D_-^\alpha x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. Using a generalized Riccati function and inequality technique, he established some new oscillation criteria.

Han et al. [19] considered the oscillation for a class of fractional differential equation:

$$\left[r(t) g \left((D_-^\alpha y)(t) \right) \right]' - p(t) f \left(\int_t^\infty (s-t)^{-\alpha} y(s) \, ds \right) = 0,$$

for $t > 0$,

(9)

where $0 < \alpha < 1$ is a real number and $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order α of y.

By generalized Riccati transformation technique, oscillation criteria for the nonlinear fractional differential equation are obtained.

Qi and Cheng [20] studied the oscillation behavior of the equation with the form

$$\left(a(t) \left[r(t) D_-^\alpha x(t) \right]' \right)' + p(t) \left[r(t) D_-^\alpha x(t) \right]'$$

$$- q(t) \int_t^\infty (\xi-t)^{-\alpha} x(\xi) \, d\xi = 0, \quad t \in [t_0, \infty),$$

(10)

where $D_-^\alpha x(t)$ also denotes the Liouville right-sided fractional derivative and some sufficient conditions for the oscillation of the equation were given.

The above works on the oscillation are all on fractional equations with Liouville right-sided fractional derivative by Riccati transformation technique.

We notice that very little attention is paid to oscillation of fractional differential equations with Riemann-Liouville derivative. For the relative works of study for oscillatory behavior of fractional differential equations Riemann-Liouville derivative we refer to [17, 21, 25, 26].

Marian et al. [25] presented the oscillatory behavior of forced nonlinear fractional difference equation of the form

$$\Delta^\alpha x(t) + f_1(t, x(t+\alpha)) = v(t) + f_2(t, x(t+\alpha)),$$

$$t \in N_0, \quad 0 < \alpha \le 1, \quad \Delta^{\alpha-1} x(t) \big|_{t=0} = x_0,$$

(11)

where Δ^α is a Riemann-Liouville like discrete fractional difference operator of order α, and some oscillation criteria are established by the same method with [17].

In 2013, Chen et al. [21] improved and extended some work in [17] by considering the forced oscillation of fractional differential equation:

$$D_a^q x + f_1(t, x) = v(t) + f_2(t, x),$$

$$\lim_{t \to a^+} J_a^{1-q} x(t) = b_1,$$

(12)

with the conditions

$$D_a^{q-k} x(a) = b_k \quad (k = 1, 2, \ldots, m-1),$$

$$\lim_{t \to a^+} I_a^{m-q} x(t) = b_m,$$

(13)

where D_a^q denotes the Riemann-Liouville or Caputo differential operator of order q with $m - 1 < q \le m$, $m \ge 1$, and the operator I_a^{m-q} is the Riemann-Liouville fractional integral operator. The authors obtained some new oscillation criteria by the same method with [17].

In 2014, Wang et al. [26] extended some oscillation results from integer differential equation to the fractional differential equation:

$$D_a^\alpha x(t) + q(t) f(x(t)) = 0, \quad t \in [a, +\infty), \ a > 0, \quad (14)$$

where D_a^α denotes the standard Riemann-Liouville differential operator of order α with $0 < \alpha \le 1$, q is a positive

real-valued function, f is a continuous functional defined on $[0, +\infty) \to [0, +\infty)$ satisfying that

$$\frac{f(x)}{I^{2-\alpha}x} \geq K > 0, \tag{15}$$

and $I^{2-\alpha}$ denotes Riemann-Liouville integral operator. The authors obtained some new oscillation criteria by the method of Riccati transformation technique.

The main purpose of this paper is giving several oscillation theorems for the fractional differential equation:

$$\left[a(t)\left(p(t) + q(t)\left(D_-^\alpha x\right)(t)\right)^\gamma \right]'$$

$$- b(t) f\left(\int_t^\infty (s-t)^{-\alpha} x(s)\, ds \right) = 0, \tag{16}$$

$$\text{for } t \geq t_0 > 0,$$

where $\alpha \in (0,1)$ is a constant, $\gamma > 0$ is a quotient of odd positive integers, and $(D_-^\alpha x)$ is the Liouville right-sided fractional derivative of order α of x defined by

$$\left(D_-^\alpha x\right)(t) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (s-t)^{-\alpha} x(s)\, ds \tag{17}$$

for $t \in \mathbb{R}_+ := (0, \infty)$; here Γ is the gamma function defined by $\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds$ for $t \in \mathbb{R}_+$, and the following conditions are assumed to hold.

(A) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(\eta)/\eta^\gamma \geq K$ for a certain constant $K > 0$ and for all $\eta \neq 0$. a, b, and q are positive continuous functions on $[t_0, \infty)$ for a certain $t_0 > 0$, and p is a nonnegative continuous function on $[t_0, \infty)$ for a certain $t_0 > 0$. There exists $M > 0$, $q(t) \leq M$, for $t \in [t_0, \infty)$. And $\int_{t_0}^\infty (p(t)/q(t))dt < \infty$.

(B) $(p(t)/q(t))' \neq 0$, for $t \in [t_0, \infty)$.

By a solution x of (16) we mean a nontrivial function $x \in C(\mathbb{R}_+, \mathbb{R})$ such that $\int_t^\infty (s-t)^{-\alpha} x(s)ds \in C^1(\mathbb{R}_+, \mathbb{R})$ and $a(t)(p(t) + q(t)(D_-^\alpha x)(t))^\gamma \in C^1(\mathbb{R}_+, \mathbb{R})$, satisfying (16) for $t \geq t_0 > 0$. We consider only those solutions of (16) that satisfy $\sup\{|x(t)| : t > t_x\} > 0$ for any $t_x \geq t_0$. A solution x of (16) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (16) is said to be oscillatory if all its solutions are oscillatory.

Our results obtained here improve and extend the main results of [18]. In [18], the author studied the oscillation of (16), where $p(t) = 0$ and $q(t) = 1$. We are dealing with the oscillation theorems for (16).

For the sake of convenience, we remember

$$z(t) = p(t) + q(t)\left(D_-^\alpha x\right)(t). \tag{18}$$

2. Preliminaries and Lemmas

In this section, we present some useful preliminaries and lemmas, which will be used in the proof of our main results.

Definition 1 (see [28]). The Liouville right-sided fractional integral of order $\sigma > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$\left(I_-^\sigma y\right)(t) := \frac{1}{\Gamma(\sigma)} \int_t^\infty (s-t)^{\sigma-1} y(s)\, ds \tag{19}$$

for $t > 0$, provided that the right-hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2 (see [28]). The Liouville right-sided fractional derivative of order $\sigma > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$\left(D_-^\sigma y\right)(t) := (-1)^{\lceil \sigma \rceil} \frac{d^{\lceil \sigma \rceil}}{dt^{\lceil \sigma \rceil}} \left(I_-^{\lceil \sigma \rceil - \sigma} y\right)(t)$$

$$= \frac{(-1)^{\lceil \sigma \rceil}}{\Gamma(\lceil \sigma \rceil - \sigma)} \frac{d^{\lceil \sigma \rceil}}{dt^{\lceil \sigma \rceil}} \int_t^\infty (s-t)^{\lceil \sigma \rceil - \sigma - 1} y(s)\, ds,$$

$$\text{for } t > 0, \tag{20}$$

provided that the right-hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil \sigma \rceil := \min\{z \in \mathbb{Z} : z \geq \sigma\}$ is the ceiling function.

Lemma 3 (see [29]). *If A and B are nonnegative constants, then*

$$A^\beta + (\beta - 1) B^\beta - \beta A B^{\beta-1} \geq 0, \quad \beta > 1, \tag{21}$$

where the equality holds if and only if $A = B$.

Lemma 4 (see [18]). *Let x be a solution of (16) and*

$$G(t) := \int_t^\infty (s-t)^{-\alpha} x(s)\, ds, \quad \text{for } \alpha \in (0,1), \ t > 0. \tag{22}$$

Then

$$G'(t) = -\Gamma(1-\alpha)\left(D_-^\alpha x\right)(t), \quad \text{for } \alpha \in (0,1), \ t > 0. \tag{23}$$

The proof of Lemma 4 is the same as the proof of Lemma 2.1 in [18].

3. Main Results

In this section, we establish some new oscillation criteria for (16).

Theorem 5. *Assume that (A) holds, and*

$$\int_{t_0}^\infty a^{-1/\gamma}(t)\, dt = \infty. \tag{24}$$

Furthermore, assume that there exists a positive function $r \in C^1[t_0, \infty)$ such that

$$\limsup_{t \to \infty} \int_{t_0}^t \left[Kr(s)b(s) - \left(\frac{r_+'(s)}{r(s)}\right)^{\gamma+1} \frac{M^\gamma r(s) a(s)}{(\gamma+1)^{(\gamma+1)} \Gamma^\gamma(1-\alpha)} \right] ds = \infty, \tag{25}$$

where $r'_+(s) := \max\{r'(s), 0\}$. *Then every solution of* (16) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (16). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (16). Then there exists $t_1 \in [t_0, \infty)$ such that

$$x(t) > 0, \quad G(t) > 0, \quad \text{for } t \in [t_1, \infty), \qquad (26)$$

where $G(t)$ is defined as in (22). Therefore, it follows from (16) that

$$[a(t) z^\gamma(t)]' = b(t) f(G(t)) > 0, \quad \text{for } t \in [t_1, \infty). \tag{27}$$

Thus, $a(t) z^\gamma(t)$ is strictly increasing on $[t_1, \infty)$ and is eventually of one sign. Since $a(t) > 0$ for $t \in [t_0, \infty)$ and $\gamma > 0$ is a quotient of odd positive integers, we see that $z(t)$ is eventually of one sign. We first show

$$z(t) < 0, \quad \text{for } t \in [t_1, \infty). \tag{28}$$

Otherwise, there exists $t_2 \geq t_1$ such that $z(t_2) > 0$, and since $a(t) z^\gamma(t)$ is strictly increasing on $[t_1, \infty)$, it is clear that $a(t) z^\gamma(t) \geq a(t_2) z^\gamma(t_2) := c_1 > 0$ for $t \in [t_2, \infty)$. Therefore, we have

$$z(t) \geq c_1^{1/\gamma} a^{-1/\gamma}(t). \tag{29}$$

Due to $q(t) > 0$, from (18), we get

$$\frac{p(t)}{q(t)} + (D_-^\alpha x)(t) \geq \frac{c_1^{1/\gamma} a^{-1/\gamma}(t)}{q(t)} \geq \frac{c_1^{1/\gamma} a^{-1/\gamma}(t)}{M}. \tag{30}$$

Integrating both sides of last inequality from t_2 to t, from (23), we obtain

$$\int_{t_2}^t \left(\frac{p(s)}{q(s)} - \frac{G'(s)}{\Gamma(1-\alpha)} \right) ds \geq \int_{t_2}^t \frac{c_1^{1/\gamma} a^{-1/\gamma}(s)}{M} ds. \tag{31}$$

So, we get

$$G(t) \leq G(t_2) + \Gamma(1-\alpha)$$
$$\times \left(\int_{t_2}^t \frac{p(s)}{q(s)} ds - \frac{c_1^{1/\gamma}}{M} \int_{t_2}^t a^{-1/\gamma}(s) ds \right) \longrightarrow -\infty,$$
$$\text{as } t \longrightarrow \infty, \tag{32}$$

and this contradicts (26). Hence, we have that (28) holds.

From (A), (18), and (23), we have

$$z(t) = p(t) + q(t)(D_-^\alpha x)(t) = p(t) + q(t)\left(-\frac{G'(t)}{\Gamma(1-\alpha)} \right). \tag{33}$$

Therefore,

$$G'(t) = \Gamma(1-\alpha) \frac{p(t) - z(t)}{q(t)}$$
$$\geq -\Gamma(1-\alpha) \frac{z(t)}{q(t)} \geq -\Gamma(1-\alpha) \frac{z(t)}{M}. \tag{34}$$

Define the function $w(t)$ by a generalized Riccati transformation

$$w(t) = r(t) \frac{-a(t) z^\gamma(t)}{G^\gamma(t)}, \quad \text{for } t \in [t_1, \infty). \tag{35}$$

Then, we have $w(t) > 0$ for $t \in [t_0, \infty)$, and from (16), (34), (35), and (A), it follows that

$$w'(t) = r'(t) \frac{-a(t) z^\gamma(t)}{G^\gamma(t)} + r(t) \left[\frac{-a(t) z^\gamma(t)}{G^\gamma(t)} \right]'$$

$$= \frac{r'(t)}{r(t)} w(t) - r(t) \frac{b(t) f(G(t))}{G^\gamma(t)}$$
$$+ \gamma r(t) a(t) z^\gamma(t) \frac{G'(t)}{G^{\gamma+1}(t)}$$

$$\leq \frac{r'_+(t)}{r(t)} w(t) - \frac{\gamma r(t) a(t) \Gamma(1-\alpha)}{M} \frac{z^{\gamma+1}(t)}{G^{\gamma+1}(t)}$$
$$- \mathrm{Kr}(t) b(t) \tag{36}$$

$$= \frac{r'_+(t)}{r(t)} w(t) - \frac{\gamma \Gamma(1-\alpha)}{M[r(t) a(t)]^{1/\gamma}} w^{1+1/\gamma}(t)$$
$$- \mathrm{Kr}(t) b(t), \quad \text{for } t \in [t_1, \infty),$$

where $r'_+(t)$ is defined as in Theorem 5. Let

$$\beta = 1 + \frac{1}{\gamma}, \qquad A = \left(\frac{\gamma \Gamma(1-\alpha)}{M(r(t) a(t))^{1/\gamma}} \right)^{1/\beta} w(t),$$
$$B = \left(\frac{r'_+(t)}{r(t)} \right)^\gamma \frac{M^{\gamma/\beta}[r(t) a(t)]^{1/\beta}}{\beta^\gamma [\gamma \Gamma(1-\alpha)]^{\gamma/\beta}}. \tag{37}$$

From (21) and (36), we derive

$$w'(t) \leq -\mathrm{Kr}(t) b(t) + \left(\frac{r'_+(t)}{r(t)} \right)^{\gamma+1} \frac{M^\gamma r(t) a(t)}{(\gamma+1)^{(\gamma+1)} \Gamma^\gamma(1-\alpha)},$$
$$\text{for } t \in [t_1, \infty). \tag{38}$$

Integrating both sides of (38) from t_1 to t, we have

$$\int_{t_1}^t \left[\mathrm{Kr}(s) b(s) - \left(\frac{r'_+(s)}{r(s)} \right)^{\gamma+1} \frac{M^\gamma r(s) a(s)}{(\gamma+1)^{(\gamma+1)} \Gamma^\gamma(1-\alpha)} \right] ds$$
$$\leq w(t_1) - w(t) < w(t_1), \quad \text{for } t \in [t_1, \infty). \tag{39}$$

Letting $t \rightarrow \infty$, we get

$$\limsup_{t \rightarrow \infty} \int_{t_1}^{t} \left[\mathrm{Kr}(s) b(s) \right.$$

$$\left. -\left(\frac{r_+'(s)}{r(s)}\right)^{\gamma+1} \frac{M^{\gamma} r(s) a(s)}{(\gamma+1)^{(\gamma+1)} \Gamma^{\gamma}(1-\alpha)} \right] ds$$

$$\leq w(t_1) < \infty, \tag{40}$$

which contradicts (25). The proof is complete. $\square$

Theorem 6. *Suppose that (A) and (24) hold. Furthermore, suppose that there exists a positive function $r \in C^1[t_0, \infty)$, and a function $H \in C(\mathbb{I}, \mathbb{R})$, where $\mathbb{I} := \{(s, t) : s \geq t \geq t_0\}$, such that*

$$H(t, t) = 0, \quad \text{for } t \geq t_0,$$
$$H(s, t) > 0, \quad \text{for } (s, t) \in \mathbb{I}_0, \tag{41}$$

where $\mathbb{I}_0 := \{(s, t) : s > t \geq t_0\}$, and H has a nonpositive continuous partial derivate $H_t'(s, t) = \partial H(s, t)/\partial t$ on $\mathbb{I}_0$, with respect to the second variable, and satisfies

$$\limsup_{s \rightarrow \infty} \frac{1}{H(s, t_0)}$$

$$\times \int_{t_0}^{s-1} \left[\mathrm{Kr}(t) b(t) H(s, t) \right. \tag{42}$$

$$\left. -\frac{M^{\gamma} r(t) a(t) h_+^{\gamma+1}(s, t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s, t)]^{\gamma}} \right] dt = \infty,$$

where $h_+(s, t) = \max\{0, H_t'(s, t) + (r_+'(t)/r(t)) H(s, t)\}$ for $(s, t) \in \mathbb{I}_0$; here $r_+'(t)$ is defined as in Theorem 5. Then all solutions of (16) are oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (16). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (16). We proceed as in proof of Theorem 5 to get that (36) holds. Multiplying (36) by $H(s, t)$ and integrating from t_1 to $s - 1$, for $s \in [t_1 + 1, \infty)$, we derive

$$\int_{t_1}^{s-1} \mathrm{Kr}(t) b(t) H(s, t) dt$$

$$\leq -\int_{t_1}^{s-1} H(s, t) w'(t) dt + \int_{t_1}^{s-1} \frac{r_+'(t)}{r(t)} H(s, t) w(t) dt$$

$$-\int_{t_1}^{s-1} H(s, t) \frac{\gamma \Gamma(1-\alpha) w^{1+1/\gamma}(t)}{M[r(t) a(t)]^{1/\gamma}} dt,$$

$$\text{for } s \in [t_1 + 1, \infty). \tag{43}$$

From

$$-\int_{t_1}^{s-1} H(s, t) w'(t) dt$$

$$= [-H(s, t) w(t)]_{t=t_1}^{t=s-1} + \int_{t_1}^{s-1} H_t'(s, t) w(t) dt$$

$$\leq H(s, t_1) w(t_1) + \int_{t_1}^{s-1} H_t'(s, t) w(t) dt, \tag{44}$$

$$\text{for } s \in [t_1 + 1, \infty),$$

we have

$$\int_{t_1}^{s-1} \mathrm{Kr}(t) b(t) H(s, t) dt$$

$$\leq H(s, t_1) w(t_1)$$

$$+ \int_{t_1}^{s-1} \left\{ \left[H_t'(s, t) + \frac{r_+'(t)}{r(t)} H(s, t) \right] w(t) \right.$$

$$\left. -\frac{\gamma \Gamma(1-\alpha) H(s, t)}{M[r(t) a(t)]^{1/\gamma}} w^{1+1/\gamma}(t) \right\} dt \tag{45}$$

$$\leq H(s, t_1) w(t_1)$$

$$+ \int_{t_1}^{s-1} \left[h_+(s, t) w(t) \right.$$

$$\left. -\frac{\gamma \Gamma(1-\alpha) H(s, t)}{M[r(t) a(t)]^{1/\gamma}} w^{(1+\gamma)/\gamma}(t) \right] dt,$$

$$\text{for } s \in [t_1 + 1, \infty),$$

where $h_+(t)$ is defined as in Theorem 6. Let

$$\beta = 1 + \frac{1}{\gamma}, \quad A = \left[\frac{\gamma \Gamma(1-\alpha) H(s, t)}{M[r(t) a(t)]^{1/\gamma}} \right]^{1/\beta} w(t),$$

$$B = \left[\frac{h_+(s, t)}{\beta} \right]^{\gamma} \frac{M^{\gamma/\beta} [r(t) a(t)]^{1/\beta}}{[\gamma \Gamma(1-\alpha) H(s, t)]^{\gamma/\beta}}. \tag{46}$$

From (21) and (45), we get

$$\int_{t_1}^{s-1} \mathrm{Kr}(t) b(t) H(s, t) dt$$

$$\leq H(s, t_1) w(t_1) \tag{47}$$

$$+ \int_{t_1}^{s-1} \frac{M^{\gamma} r(t) a(t) h_+^{\gamma+1}(s, t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s, t)]^{\gamma}} dt.$$

From $H_t'(s, t) < 0$, for $s > t \geq t_0$, we have

$$0 < H(s, t_1) \leq H(s, t_0), \quad \text{for } s > t_1 \geq t_0, \tag{48}$$

and $0 < H(s, t) \leq H(s, t_0)$, for $s > t \geq t_0$; then

$$0 < \frac{H(s, t)}{H(s, t_0)} \leq 1, \quad \text{for } s > t \geq t_0. \tag{49}$$

Therefore, we get

$$\frac{1}{H(s,t_0)} \int_{t_0}^{s-1} \left[Kr(t) b(t) H(s,t) \right.$$

$$\left. - \frac{M^\gamma r(t) a(t) h_+^{\gamma+1}(s,t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s,t)]^\gamma} \right] dt$$

$$= \frac{1}{H(s,t_0)} \int_{t_0}^{t_1} \left[Kr(t) b(t) H(s,t) \right.$$

$$\left. - \frac{M^\gamma r(t) a(t) h_+^{\gamma+1}(s,t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s,t)]^\gamma} \right] dt$$

$$+ \frac{1}{H(s,t_0)} \int_{t_1}^{s-1} \left[Kr(t) b(t) H(s,t) \right.$$

$$\left. - \frac{M^\gamma r(t) a(t) h_+^{\gamma+1}(s,t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s,t)]^\gamma} \right] dt$$

$$\leqslant \frac{1}{H(s,t_0)} \int_{t_0}^{t_1} Kr(t) b(t) H(s,t) \, dt$$

$$+ \frac{1}{H(s,t_0)} H(s,t_1) w(t_1)$$

$$\leqslant \int_{t_0}^{t_1} Kr(t) b(t) \, dt + w(t_1), \quad \text{for } s \in [t_1+1,\infty).$$

$$\tag{50}$$

Letting $s \to \infty$, we get

$$\limsup_{s \to \infty} \frac{1}{H(s,t_0)}$$

$$\times \int_{t_0}^{s-1} \left[Kr(t) b(t) H(s,t) \right.$$

$$\left. - \frac{M^\gamma r(t) a(t) h_+^{\gamma+1}(s,t)}{(\gamma+1)^{\gamma+1} [\Gamma(1-\alpha) H(s,t)]^\gamma} \right] dt$$

$$\leqslant \int_{t_0}^{t_1} Kr(t) b(t) \, dt + w(t_1) < \infty,$$

$$\tag{51}$$

which is a contradiction to (42). The proof is complete. □

Next, we consider the condition of

$$\int_{t_0}^{\infty} a^{-1/\gamma}(t) \, dt < \infty, \tag{52}$$

which yields that (24) does not hold. Under this condition, we have the following results.

Theorem 7. *Suppose that (A), (B), and (52) hold, and there exists a positive function $r \in C^1[t_0,\infty)$ such that (25) holds. Furthermore, assume that, for every constant $T \geqslant t_0$,*

$$\int_T^{\infty} \left[\frac{1}{a(t)} \int_T^t b(s) ds \right]^{1/\gamma} dt = \infty. \tag{53}$$

Then every solution x of (16) is oscillatory or satisfies $\lim_{t\to\infty} G'(t) = 0$ or $\lim_{t\to\infty} G(t) = 0$, where $G(t)$ is defined as Lemma 4.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (16). Without loss of generality, assume that $x(t)$ is an eventually positive solution of (16). Proceeding as in the proof of Theorem 5, we get that (26) and (27) hold. Then there are two cases for the sign of $z(t)$.

When $z(t)$ is eventually negative, from the proof of Theorem 5, we get that every solution $x(t)$ of (16) is oscillatory.

Next, assume that $z(t)$ is eventually positive; then there exists $t_{21} \geqslant t_0$, such that $z(t) > 0$, for $t \geqslant t_{21}$. From (18) and (23), we get

$$z(t) = p(t) + q(t) (D_-^\alpha x)(t) = p(t)$$

$$+ q(t) \left(-\frac{G'(t)}{\Gamma(1-\alpha)} \right) > 0. \tag{54}$$

Therefore,

$$G'(t) < \Gamma(1-\alpha) \frac{p(t)}{q(t)}. \tag{55}$$

Since (B) holds and $\int_{t_0}^{\infty} (p(t)/q(t)) dt < \infty$, then we get

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} = 0. \tag{56}$$

Letting $t \to \infty$ in (55), we have

$$\lim_{t\to\infty} G'(t) \leqslant 0. \tag{57}$$

If $\lim_{t\to\infty} G'(t) < 0$, then there exists $t_{22} \geqslant t_1$ such that $G'(t) < 0$, for $t \geqslant t_{22}$. We set $t_2 := \max\{t_{21}, t_{22}\}$. Thus, we get $\lim_{t\to\infty} G(t) = C$, and $G(t) \geqslant C$, $t \geq t_2$.

We now prove $C = 0$. If not, that is, $C > 0$, then from (27), we derive

$$[a(t) z^\gamma(t)]' = b(t) f(G(t)) \geqslant Kb(t) G^\gamma(t) \geqslant KC^\gamma b(t),$$

$$\text{for } t \in [t_2,\infty). \tag{58}$$

Integrating both sides of (58) from t_2 to t, we have

$$a(t) z^\gamma(t) \geqslant a(t_2) z^\gamma(t_2) + KC^\gamma \int_{t_2}^t b(s) ds$$

$$> KC^\gamma \int_{t_2}^t b(s) ds, \quad \text{for } t \in [t_2,\infty). \tag{59}$$

Therefore,

$$z(t) > CK^{1/\gamma} \left[\frac{1}{a(t)} \int_{t_2}^{t} b(s)ds \right]^{1/\gamma}. \tag{60}$$

Hence, from (18), (A), and (23), we get

$$\frac{p(t)}{q(t)} - \frac{G'(t)}{\Gamma(1-\alpha)} > \frac{CK^{1/\gamma}}{q(t)} \left[\frac{1}{a(t)} \int_{t_2}^{t} b(s)\,ds \right]^{1/\gamma}$$
$$> \frac{CK^{1/\gamma}}{M} \left[\frac{1}{a(t)} \int_{t_2}^{t} b(s)\,ds \right]^{1/\gamma}. \tag{61}$$

Integrating both sides of (61) from t_2 to t, we have

$$\int_{t_2}^{t} \frac{p(s)}{q(s)}ds - \frac{G(t) - G(t_2)}{\Gamma(1-\alpha)}$$
$$> \frac{CK^{1/\gamma}}{M} \int_{t_2}^{t} \left[\frac{1}{a(v)} \int_{t_2}^{v} b(s)\,ds \right]^{1/\gamma} dv. \tag{62}$$

Then, we obtain

$$G(t) < \left\{ \left[\int_{t_2}^{t} \frac{p(s)}{q(s)}ds - \frac{CK^{1/\gamma}}{M} \right. \right.$$
$$\left. \times \int_{t_2}^{t} \left[\frac{1}{a(v)} \int_{t_2}^{v} b(s)\,ds \right]^{1/\gamma} dv \right\} \Gamma(1-\alpha) + G(t_2)$$
$$\longrightarrow -\infty, \quad \text{as } t \longrightarrow \infty. \tag{63}$$

This contradicts (26). Therefore, we have $C = 0$; that is,

$$\lim_{t \to \infty} G(t) = 0. \tag{64}$$

The proof is complete. □

Theorem 8. *Suppose that (A), (B), and (52) hold. Let $r(t)$ and $H(s,t)$ be defined as in Theorem 6 such that (42) holds. Furthermore, assume that, for every $T \geq t_0$, (53) holds. Then every solution x of (16) is oscillatory or satisfies $\lim_{t\to\infty} G'(t) = 0$ or $\lim_{t\to\infty} G(t) = 0$, where $G(t)$ is defined as in Lemma 4.*

Proof. Assume that x is a nonoscillatory solution of (16). Without loss of generality, assume that x is an eventually positive solution of (16). Proceeding as in the proof of Theorem 5, we get that (26) and (27) hold. Then there are two cases for the sign of $z(t)$.

When $z(t)$ is eventually negative, the proof is similar to that of Theorem 6. When $z(t)$ is eventually positive, the proof is similar to that of Theorem 7. Here we omitted it. □

Remark 9. From Theorems 5–8, we can get many different sufficient conditions for the oscillation of (16) with different choices of the functions r and H.

4. Examples

Example 10. Consider the differential equation with fractional-order derivatives:

$$\left[t^{\gamma-2} \left(e^{-t} + \frac{1}{t} \left(D_-^{\alpha} x \right)(t) \right)^{\gamma} \right]'$$
$$- \frac{1}{t^2} \left(\int_t^{\infty} (s-t)^{-\alpha} x(s)\,ds \right)^{\gamma} = 0, \quad t \geq 1, \tag{65}$$

where $\alpha \in (0,1)$, and $\gamma > 0$ is a quotient of odd positive integers.

Here, $a(t) = t^{\gamma-2}$, $b(t) = 1/t^2$, $p(t) = e^{-t}$, and $q(t) = 1/t$. Take $t_0 = 1$, $K = 1$, and $M = 1$. From

$$\int_{t_0}^{\infty} a^{-1/\gamma}(t)\,dt = \int_{t_0}^{\infty} \frac{1}{t^{1-(2/\gamma)}}dt = \infty,$$
$$\int_{t_0}^{\infty} \frac{p(t)}{q(t)}dt = \int_{t_0}^{\infty} \frac{e^{-t}}{1/t}dt = \int_{t_0}^{\infty} te^{-t}dt < \infty, \tag{66}$$

we see that (A) and (24) hold. Letting $r(s) = s$, we get

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[Kr(s) b(s) \right.$$
$$\left. - \left(\frac{r'_+(s)}{r(s)} \right)^{\gamma+1} \frac{M^{\gamma} r(s) a(s)}{(\gamma+1)^{(\gamma+1)} \Gamma^{\gamma}(1-\alpha)} \right] ds$$
$$= \limsup_{t \to \infty} \int_{t_0}^{t} \left[s \cdot \frac{1}{s^2} - \left(\frac{1}{s} \right)^{\gamma+1} \frac{s \cdot s^{\gamma-2}}{(\gamma+1)^{\gamma+1} \Gamma^{\gamma}(1-\alpha)} \right] ds$$
$$= \limsup_{t \to \infty} \int_{t_0}^{t} \left[\frac{1}{s} - \frac{1}{(\gamma+1)^{\gamma+1} \Gamma^{\gamma}(1-\alpha)} \cdot \frac{1}{s^2} \right] ds = \infty, \tag{67}$$

which satisfies condition (25). Therefore, by Theorem 5, every solution of (65) is oscillatory.

Example 11. Consider the differential equation with fractional-order derivatives:

$$\left[t^{\gamma+2} \left(\frac{1}{t^3} + \frac{1}{t} \left(D_-^{\alpha} x \right)(t) \right)^{\gamma} \right]'$$
$$- 6t \left(2 + \left(\int_t^{\infty} (s-t)^{-\alpha} x(s)\,ds \right)^2 \right)$$
$$\times \left(\int_t^{\infty} (s-t)^{-\alpha} x(s)\,ds \right)^{\gamma} = 0, \quad t \geq 2, \tag{68}$$

where $\alpha \in (0,1)$, and $\gamma > 0$ is a quotient of odd positive integers.

Here, $a(t) = t^{2+\gamma}$, $b(t) = 6t$, $p(t) = 1/t^3$, $q(t) = 1/t$, and $f(t) = (2 + t^2)t^\gamma$. Take $K = 2$, $M = 1$ and $t_0 = 2$, $f(u) = u$. From

$$\int_{t_0}^{\infty} a^{-1/\gamma}(t)\,dt = \int_{t_0}^{\infty} \frac{1}{t^{1+(2/\gamma)}}\,dt < \infty,$$

$$\left(\frac{p(t)}{q(t)}\right)' = \left(\frac{1/t^3}{1/t}\right)' = -\frac{2}{t^3} \neq 0, \qquad (69)$$

$$\int_{t_0}^{\infty} \frac{p(t)}{q(t)}\,dt = \int_{t_0}^{\infty} \frac{1}{t^2}\,dt < \infty,$$

we find that (A), (B), and (52) hold.

Take $r(s) \equiv 1$, we have

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[Kr(s)b(s) \right.$$

$$\left. - \left(\frac{r'_+(s)}{r(s)}\right)^{\gamma+1} \frac{M^\gamma r(s)a(s)}{(\gamma+1)^{(\gamma+1)}\Gamma^\gamma(1-\alpha)} \right] ds$$

$$= \limsup_{t \to \infty} \int_{t_0}^{t} 12s\,ds = \infty,$$

$$(70)$$

which satisfies condition (25). For every constant $T \geqslant t_0$, $t \in [2T, \infty)$, we obtain

$$\int_{T}^{\infty} \left[\frac{1}{a(t)} \int_{T}^{t} b(s)\,ds \right]^{1/\gamma} dt$$

$$= \int_{T}^{\infty} \left[\frac{1}{t^{\gamma+2}} \int_{T}^{t} 6s\,ds \right]^{1/\gamma} dt$$

$$= \int_{T}^{\infty} \left[\frac{3t^2 - 3T^2}{t^{2+\gamma}} \right]^{1/\gamma} dt \qquad (71)$$

$$\geqslant \int_{T}^{\infty} \frac{2^{1/\gamma}}{t}\,dt = \infty,$$

which implies that (53) holds. Therefore, by Theorem 7, every solution x of (68) is oscillatory or satisfies $\lim_{t \to \infty} G'(t) = 0$ or $\lim_{t \to \infty} G(t) = 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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