Research Article

Some Results on Characterizations of Matrix Partial Orderings

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Some characterizations of the left-star, right-star, and star partial orderings between matrices of the same size are obtained. Based on those results, several characterizations of the star partial ordering between EP matrices are given. At last, one characterization of the sharp partial ordering between group matrices is obtained.

1. Introduction

In this paper we use the following notation. Let $\mathbb{C}^{m \times n}$ be the set of complex $m \times n$ matrices. For any matrix $A \in \mathbb{C}^{m \times n}$, A^*, $\mathfrak{R}(A)$, and $r(A)$ denote the conjugate transpose, the range, and the rank of A, respectively. The symbol I_n denotes the $n \times n$ identity matrix, and 0 denotes a zero matrix of appropriate size. The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m \times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the four matrix equations

$$
\begin{aligned}
&(1) \quad AXA = A, \\
&(2) \quad XAX = X, \\
&(3) \quad (AX)^* = AX, \\
&(4) \quad (XA)^* = XA,
\end{aligned}
\tag{1}
$$

and A^- denotes any solution to the matrix equation $AXA = A$ with respect to X; $A\{1\}$ denotes the set of A^-; that is, $A\{1\} = \{X \mid AXA = A\}$. Moreover, $A^\#$ denotes the group inverse of A with $r(A^2) = r(A)$, that is, the unique solution to

$$
\begin{aligned}
&(1) \quad AXA = A, \\
&(2) \quad XAX = X, \\
&(5) \quad AX = XA.
\end{aligned}
\tag{2}
$$

It is well known that $A^\#$ exists if and only if $r(A^2) = r(A)$, where case A is also called a group matrix. A matrix A is EP if and only if A is a group matrix with $A^\# = A^\dagger$. The symbols $\mathbb{C}_{GP}^n$ and $\mathbb{C}_{EP}^n$ stand for the subset of $\mathbb{C}^{n \times n}$ consisting of group matrices and EP matrices, respectively (see, e.g., [1, 2] for details).

Five matrix partial orderings defined in $\mathbb{C}^{m \times n}$ are considered in this paper. The first of them is the minus partial ordering defined by Hartwig [3] and Nambooripad [4] independently in 1980:

$$
A \leq B \iff A^- A = A^- B, \qquad AA^= = BA^=, \tag{3}
$$

where $A^-, A^= \in A\{1\}$. In [3] it was shown that

$$
A \leq B \iff r(B - A) = r(B) - r(A). \tag{4}
$$

The rank equality indicates why the minus partial ordering is also called the rank-subtractivity partial ordering. In the same paper [3] it was also shown that

$$
A \overset{*}{\leq} B \iff r\begin{bmatrix} A \\ B \end{bmatrix} = r\begin{bmatrix} A & B \end{bmatrix} = r(B), \qquad AB^- A = A, \tag{5}
$$

where $B^- \in B\{1\}$.

The second partial ordering of interest is the star partial ordering introduced by Drazin [5], which is determined by

$$
A \overset{*}{\leq} B \iff A^\dagger A = A^\dagger B, \qquad AA^\dagger = BA^\dagger. \tag{6}
$$

It is well known that

$$
A \overset{*}{\leq} B \iff A^* A = A^* B, \qquad AA^* = BA^*. \tag{7}
$$

In 1991, Baksalary and Mitra [6] defined the left-star and right-star partial orderings characterized as

$$A \overset{*}{\leq} B \iff A^*A = A^*B, \qquad \mathfrak{R}(A) \subseteq \mathfrak{R}(B),$$
$$A \leq^* B \iff AA^* = BA^*, \qquad \mathfrak{R}(A^*) \subseteq \mathfrak{R}(B^*). \tag{8}$$

The last partial ordering we will deal with in this paper is the sharp partial ordering, introduced by Mitra [7] in 1987, and is defined in the set $\mathbb{C}_{GP}^n$ by

$$A \overset{\#}{\leq} B \iff A^\#A = A^\#B, \qquad AA^\# = BA^\#. \tag{9}$$

A detailed discussion of partial orderings and their applications can be found in [1, 8–10].

It is well known that rank of matrix is an important tool in matrix theory and its applications, and many problems are closely related with the ranks of some matrix expressions under some restrictions (see [11–15] for details). Our aim in this paper is to characterize the left-star, right-star, star, and sharp partial orderings by applying rank equalities. In the following, when A is considered below B with respect to one partial ordering, then the partial ordering should entail the assumption $r(A) > r(B) \geq 1$.

2. The Star Partial Ordering

Let A and B be $m \times n$ complex matrices with ranks a and b, respectively. Let $A \overset{*}{\leq} B$. Then there exist unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ such that

$$U^*AV = \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad U^*BV = \begin{pmatrix} D_a & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

where both the $a \times a$ matrix D_a and the $(b-a) \times (b-a)$ matrix D are real, diagonal, and positive definite (see [16, Theorem 2]). In [1, Theorem 5.2.8], it was also shown that

$$A \overset{*}{\leq} B \iff A^\dagger A = B^\dagger A, \qquad AA^\dagger = AB^\dagger. \tag{11}$$

In [17], Wang obtained the following characterizations of the left-star and right-star partial orderings for matrices:

$$A \overset{*}{\leq} B \iff r \begin{bmatrix} B^*B & A^*A \\ B & A \end{bmatrix} = r(B), \tag{12}$$

$$A \leq^* B \iff r \begin{bmatrix} BB^* & AA^* \\ B^* & A^* \end{bmatrix} = r(B), \tag{13}$$

$$A \overset{*}{\leq} B \iff r \begin{bmatrix} B^*B & A^*A \\ B & A \end{bmatrix} = r(B),$$
$$r \begin{bmatrix} BB^* & AA^* \\ B^* & A^* \end{bmatrix} = r(B). \tag{14}$$

Theorem 1. *Let* $A, B \in \mathbb{C}^{m \times n}$. *Then*

(i)

$$A \overset{*}{\leq} B \iff r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix} = r(B); \tag{15}$$

(ii)

$$A \leq^* B \iff r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B & A \end{bmatrix} = r(B); \tag{16}$$

(iii)

$$A \overset{*}{\leq} B \iff r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix} = r(B); \tag{17}$$

(iv)

$$A \overset{*}{\leq} B \iff r \begin{bmatrix} BB^\dagger & AA^\dagger \\ BB^* & AA^* \\ B^* & A^* \end{bmatrix} = r(B). \tag{18}$$

Proof. From

$$\begin{aligned} r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix} &\geq r \left(\begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \right) \\ &= r \begin{bmatrix} B & A \\ B^*B & A^*A \end{bmatrix} \\ &\geq r \left(\begin{bmatrix} B & A \\ B^*B & A^*A \end{bmatrix} \begin{bmatrix} B^\dagger & 0 \\ 0 & A^\dagger \end{bmatrix} \right) \\ &= r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix}, \end{aligned} \tag{19}$$

we have

$$r \begin{bmatrix} B^*B & A^*A \\ B & A \end{bmatrix} = r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix}. \tag{20}$$

Applying (12) gives (i).

In the same way, applying

$$r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B & A \end{bmatrix} = r \begin{bmatrix} BB^* & AA^* \\ B^* & A^* \end{bmatrix} \tag{21}$$

and (13) gives (ii).

If

$$r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix} = r(B), \tag{22}$$

then

$$r \begin{bmatrix} BB^* & AA^* \\ B^* & A^* \end{bmatrix} = r(B), \qquad r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix} = r(B). \tag{23}$$

Applying (i), (ii), and (14), we obtain $A \overset{*}{\leq} B$. Conversely, if $A \overset{*}{\leq} B$, by using (11) and (14), we have $A^\dagger A - B^\dagger A = 0$, and

$$
r \begin{bmatrix} B^*B & A^*A \\ B & A \end{bmatrix} = r \begin{bmatrix} 0 & A^\dagger A - B^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix}
$$

$$
= r \left(\begin{bmatrix} I_n & 0 & B^\dagger \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & A^\dagger A - B^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix} \right)
$$

$$
= r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix},
$$

$$
r(B) = r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B^*B & A^*A \\ B & A \end{bmatrix}.
$$

$$(24)$$

Hence, we have (iii).

Similarly, applying $A \overset{*}{\leq} B$, (11), and (14), we obtain $AA^\dagger - AB^\dagger = 0$, $AB^\dagger = (AB^\dagger)^* = (B^*)^\dagger A^*$, and

$$
r \begin{bmatrix} BB^* & AA^* \\ B^* & A^* \end{bmatrix} = r \begin{bmatrix} 0 & AA^\dagger - AB^\dagger \\ BB^* & AA^* \\ B^* & A^* \end{bmatrix}
$$

$$
= r \left(\begin{bmatrix} I_m & 0 & (B^*)^\dagger \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & AA^\dagger - AB^\dagger \\ BB^* & AA^* \\ B^* & A^* \end{bmatrix} \right)
$$

$$
= r(B).
$$

$$(25)$$

Then, we obtain (iv). $\qquad\square$

In [9, Theorem 2.1], Benítez et al. deduce the characterizations of the left-star, right-star, and star partial orderings for matrices, when at least one of the two involved matrices is EP. When both $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are EP matrices, [1, Theorems 5.4.15 and 5.4.2] give the following results:

$$
A \overset{*}{\leq} B \Longleftrightarrow A \leq B, \quad AB^* \text{ and } B^*A \text{ are Hermitian.}
$$
$$(26)$$
$$
A \overset{*}{\leq} B \Longleftrightarrow (AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger = A^{\dagger 2}.
$$

In addition, it was also shown that $A \overset{*}{\leq} B$ if and only if A and B have the form

$$
A = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \qquad B = U \begin{bmatrix} T & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad (27)
$$

where $T \in \mathbb{C}^{r(A)\times r(A)}$ is nonsingular, $K \in \mathbb{C}^{(r(B)-r(A))\times(r(B)-r(A))}$ is nonsingular, and $U \in \mathbb{C}^{n\times n}$ is unitary (see [1, Theorem 5.4.1]).

Based on these results, we consider the characterizations of the star partial ordering for matrices in the set of $\mathbb{C}^n_{EP}$.

Theorem 2. *Let* $A, B \in \mathbb{C}^n_{EP}$, $r(B) \geq r(A)$. *Then*

(v)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B & A \\ B^2 & A^2 \end{bmatrix} = r(B); \qquad (28)
$$

(vi)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B & B^2 \\ A & A^2 \end{bmatrix} = r(B). \qquad (29)
$$

Proof. By $A, B \in \mathbb{C}^n_{EP}$, it is obvious that $AA^\dagger = A^\dagger A$ and $BB^\dagger = B^\dagger B$. Then

$$
r \begin{bmatrix} B & A \\ B^2 & A^2 \end{bmatrix} = r(B) \Longleftrightarrow r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B & A \end{bmatrix} = r(B). \qquad (30)
$$

Hence, we have (v).

The proof of (vi) is similar to that of (v). $\qquad\square$

Theorem 3. *Let* $A, B \in \mathbb{C}^n_{EP}$. *Then*

(vii)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B & A \\ B^* & A^* \end{bmatrix} = r(B); \qquad (31)
$$

(viii)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B^*B & A^*A \\ B & A \\ B^* & A^* \end{bmatrix} = r(B); \qquad (32)
$$

(ix)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B & BA \\ B & AB \\ A & AB \end{bmatrix} = r(B); \qquad (33)
$$

(x)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B & BA^\dagger \\ B & A^\dagger B \\ A & A^\dagger B \end{bmatrix} = r(B); \qquad (34)
$$

(xi)

$$
A \overset{*}{\leq} B \Longleftrightarrow r \begin{bmatrix} B & BA^* \\ B & A^*B \\ A & A^*B \end{bmatrix} = r(B). \qquad (35)
$$

Proof. By $A, B \in \mathbb{C}^n_{EP}$, it is obvious that $AA^\dagger = A^\dagger A$ and $BB^\dagger = B^\dagger B$. Applying (i), (ii), and the rank equality in (vii) we obtain

$$
r \begin{bmatrix} B^\dagger B & A^\dagger A \\ B & A \end{bmatrix} = r(B), \qquad r \begin{bmatrix} BB^\dagger & AA^\dagger \\ B^* & A^* \end{bmatrix} = r(B); \qquad (36)
$$

that is, $A \overset{*}{\leq} B$. Conversely, suppose that $A \overset{*}{\leq} B$. Applying $A - AA^{\dagger}B = 0$ and $B^*BB^{\dagger} = B^*$, we obtain

$$r(B) = r\begin{bmatrix} BB^{\dagger} & AA^{\dagger} \\ B & A \end{bmatrix} = r\begin{bmatrix} BB^{\dagger} & AA^{\dagger} \\ B & A \\ 0 & A^* - B^*AA^{\dagger} \end{bmatrix}$$

$$= r\begin{bmatrix} BB^{\dagger} & AA^{\dagger} \\ B & A \\ B^* & A^* \end{bmatrix}. \tag{37}$$

Applying (11), we obtain $B^*BB^{\dagger}B = B^*B$ and $B^*BA^{\dagger}A = A^*A$ and also $(B^*B)^{\dagger}B^*B = B^{\dagger}B$ and $(B^*B)^{\dagger}A^*A = A^{\dagger}A$. Then

$$r\begin{bmatrix} BB^{\dagger} & AA^{\dagger} \\ B & A \\ B^* & A^* \end{bmatrix} = r\begin{bmatrix} B^{\dagger}B & A^{\dagger}A \\ B & A \\ B^* & A^* \end{bmatrix}$$

$$\geq r\left(\begin{bmatrix} B^*B & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} B^{\dagger}B & A^{\dagger}A \\ B & A \\ B^* & A^* \end{bmatrix}\right)$$

$$= r\begin{bmatrix} B^*B & A^*A \\ B & A \\ B^* & A^* \end{bmatrix}$$

$$\geq r\left(\begin{bmatrix} (B^*B)^{\dagger} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} B^*B & A^*A \\ B & A \\ B^* & A^* \end{bmatrix}\right)$$

$$= r\begin{bmatrix} B^{\dagger}B & A^{\dagger}A \\ B & A \\ B^* & A^* \end{bmatrix}; \tag{38}$$

that is,

$$r\begin{bmatrix} BB^{\dagger} & AA^{\dagger} \\ B & A \\ B^* & A^* \end{bmatrix} = r\begin{bmatrix} B^*B & A^*A \\ B & A \\ B^* & A^* \end{bmatrix}. \tag{39}$$

Hence, we have (viii).

Suppose that $A \overset{*}{\leq} B$. Since $A, B \in \mathbb{C}_{EP}^n$, applying (27), it is easy to check the rank equality in (ix). Conversely, under the rank equality in (ix), we have

$$r\begin{bmatrix} B & BA \\ B & AB \end{bmatrix} = r\begin{bmatrix} B & BA \\ 0 & AB - BA \end{bmatrix} = r(B) \Longrightarrow AB = BA,$$

$$r\begin{bmatrix} B & BA \\ A & AB \end{bmatrix} = r\begin{bmatrix} B & 0 \\ A & AB - AA^2 \end{bmatrix} = r(B) \Longrightarrow AB = A^2. \tag{40}$$

Since A is EP, there exists a unitary matrix $U_1 \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $T \in \mathbb{C}^{r(A)\times r(A)}$ such that

$$A = U_1\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}U_1^*. \tag{41}$$

Correspondingly denote $P^{-1}BP$ by

$$B = U_1\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}U_1^*, \tag{42}$$

where $B_1 \in \mathbb{C}^{r(A)\times r(A)}$. It follows that

$$\begin{bmatrix} TB_1 & TB_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1T & 0 \\ B_3T & 0 \end{bmatrix}, \qquad \begin{bmatrix} TB_1 & TB_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{43}$$

Since T is a unitary matrix,

$$B_1 = T, \qquad B_2 = 0, \qquad B_3 = 0. \tag{44}$$

Thus

$$B = U\begin{bmatrix} T & 0 \\ 0 & B_4 \end{bmatrix}U^*. \tag{45}$$

Since B is EP, B_4 is EP, and there exists a unitary matrix $U_2 \in \mathbb{C}^{(n-r(A))\times(n-r(A))}$ and a nonsingular matrix $K \in \mathbb{C}^{(r(B)-r(A))\times(r(B)-r(A))}$ such that

$$B_4 = U_2\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}U_2^*. \tag{46}$$

Write

$$U = U_1\begin{bmatrix} 0 & 0 \\ 0 & U_2 \end{bmatrix}. \tag{47}$$

Then A and B have the form

$$A = U\begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^*, \qquad B = U\begin{bmatrix} T & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{bmatrix}U^*. \tag{48}$$

Applying (27), we have $A \overset{*}{\leq} B$.

The proofs of (x) and (xi) are similar to that of (ix). □

3. The Sharp Partial Ordering

Let $A, B \in \mathbb{C}_{GP}^n$ with ranks a and b, respectively. It is well known that

$$A \leq^{\#} B \Longleftrightarrow A^2 = AB = BA. \tag{49}$$

In addition, $A \leq^{\#} B$ if and only if A and B can be written as

$$A = P\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1}, \qquad B = P\begin{bmatrix} E & 0 & 0 \\ 0 & E' & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1}, \tag{50}$$

where $E \in \mathbb{C}^{a\times a}$ is nonsingular, $E' \in \mathbb{C}^{(b-a)\times(b-a)}$ is nonsingular, and $P \in \mathbb{C}^{n\times n}$ is nonsingular (see [18]).

In Theorem 4, we give one characterization of the sharp partial ordering by using one rank equality.

Theorem 4. *Let $A, B \in \mathbb{C}_{GP}^n$. Then*

$$A \leq^{\#} B \Longleftrightarrow r\begin{bmatrix} A & BA \\ AB & ABA \end{bmatrix} = r(ABA). \tag{51}$$

Proof. Let A have the core-nilpotent decomposition (see [19, Exercise 5.10.12])

$$A = P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \tag{52}$$

with nonsingular matrices $\Sigma \in \mathbb{C}^{r(A) \times r(A)}$ and $P \in \mathbb{C}^{n \times n}$. Correspondingly denote $P^{-1}BP$ by

$$P^{-1}BP = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \tag{53}$$

where $B_1 \in \mathbb{C}^{r(A) \times r(A)}$. It follows that

$$r(ABA) = r(\Sigma B_1 \Sigma),$$

$$r\begin{bmatrix} A & BA \\ AB & ABA \end{bmatrix} = r\begin{bmatrix} \Sigma & 0 & B_1\Sigma \\ 0 & 0 & B_3\Sigma \\ \Sigma B_1 & \Sigma B_2 & \Sigma B_1\Sigma \end{bmatrix}$$

$$= r\begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & B_3\Sigma \\ 0 & \Sigma B_2 & \Sigma B_1\Sigma - \Sigma B_1\Sigma^{-1}B_1\Sigma \end{bmatrix}$$

$$= r(\Sigma) + r\begin{bmatrix} 0 & B_3\Sigma \\ \Sigma B_2 & \Sigma B_1\Sigma - \Sigma B_1\Sigma^{-1}B_1\Sigma \end{bmatrix}. \tag{54}$$

Applying (54) to the rank equality in (51), we obtain

$$r\begin{bmatrix} 0 & B_3\Sigma \\ \Sigma B_2 & \Sigma B_1\Sigma - \Sigma B_1\Sigma^{-1}B_1\Sigma \end{bmatrix} + r(\Sigma) = r(\Sigma B_1 \Sigma). \tag{55}$$

Hence $r(\Sigma B_1 \Sigma) = r(\Sigma)$, $\Sigma B_2 = 0$, $B_3 \Sigma = 0$, and $\Sigma B_1\Sigma = \Sigma B_1\Sigma^{-1}B_1\Sigma$. Since $\Sigma \in \mathbb{C}^{r(A) \times r(A)}$ is invertible and $B_1 \in \mathbb{C}^{r(A) \times r(A)}$, it follows immediately that

$$r(B_1) = r(\Sigma), \qquad B_3 = 0, \qquad B_2 = 0, \qquad B_1 = \Sigma. \tag{56}$$

Therefore

$$B = P \begin{bmatrix} \Sigma & 0 \\ 0 & B_4 \end{bmatrix} P^{-1}. \tag{57}$$

Applying

$$A^2 = P \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$= P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P \begin{bmatrix} \Sigma & 0 \\ 0 & B_4 \end{bmatrix} P^{-1}$$

$$= AB$$

$$= P \begin{bmatrix} \Sigma & 0 \\ 0 & B_4 \end{bmatrix} P^{-1} P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$= BA, \tag{58}$$

and (49), we obtain that $A \leq^{\#} B$.

Conversely, it is a simple matter. $\square$

Conflict of Interests

Acknowledgments

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