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Research Article

An Iterative Scheme for Solving Systems of Nonlinear Fredholm Integrodifferential Equations

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Using fixed-point techniques and Faber-Schauder systems in adequate Banach spaces, we approximate the solution of a system of nonlinear Fredholm integrodifferential equations of the second kind.

1. Introduction

An important area of research interest is the study of systems of nonlinear Fredholm integrodifferential equations. A system of nonlinear Fredholm integrodifferential equations can be written in vectorial form as

$$\mathbf{X}'(t) = \mathbf{F}(t, \mathbf{X}(t)) + \int_0^1 \mathbf{K}(t, s, \mathbf{X}(s))\, ds \quad (t, s \in [0,1]), \qquad \mathbf{X}(0) = \boldsymbol{\rho}, \tag{1}$$

where $\mathbf{X} = [x_1, \ldots, x_n]^T \in C([0,1], \mathbb{R}^n)$ is the solution to be calculated and $\mathbf{F} = [F_1, \ldots, F_n]^T \in C([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbf{K} = [K_1, \ldots, K_n]^T \in C([0,1]^2 \times \mathbb{R}^n, \mathbb{R}^n)$, and $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_n]^T \in \mathbb{R}^n$ are known.

Observe that, for $i = 1, \ldots, n$, the ith equation of the system (1) adopts the form

$$x_i'(t) = F_i(t, x_1(t), \ldots, x_n(t)) + \int_0^1 K_i(t, s, x_1(s), \ldots, x_n(s))\, ds, \tag{2}$$

with $x_i(0) = \rho_i$.

The system (1) is linear when for all $t, s \in [0,1]$ and $\mathbf{u} = [u_1, \ldots, u_n]^T \in \mathbb{R}^n$ we have that

$$\mathbf{K}(t, s, \mathbf{u}) = \begin{bmatrix} K_{11}(t,s) & \cdots & K_{1n}(t,s) \\ \vdots & \ddots & \vdots \\ K_{n1}(t,s) & \cdots & K_{nn}(t,s) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{with } K_{ij} \in C\left([0,1]^2, \mathbb{R}\right). \tag{3}$$

Many problems of physics and engineering lead to the solution of integro or integrodifferential equations or systems of such equations. In most cases, these cannot be solved by direct methods, and this, together with the powerful computer tools available, has led to the development of numerical methods that allow obtaining approximate solutions of these equations or systems of equations. In literature it is easy to find many of them.

Danfu and Xufeng [1] utilize the CAS wavelet operational matrix of integration for obtaining numerical solution of linear Fredholm integrodifferential equations. Jafarian and Measoomy Nia [2] offer an architecture of artificial neural networks (NNs) for finding approximate solution of linear Fredholm integral equations system of the second kind. In [3], Maleknejad et al. present a rationalized Haar functions method for solving linear Fredholm integrodifferential systems. In [4] Maleknejad and Tavassoli Kajani use the hybrid Legendre and block-pulse functions on interval [0, 1) to solve the systems of linear integrodifferential equations. In [5], a fully discrete version of a piecewise polynomial collocation method is constructed to solve initial or boundary value problems of linear Fredholm integrodifferential equations with weakly singular kernels. In [6], Pour-Mahmoud et al. extend the Tau method for the numerical solution of integrodifferential equations system (IDES). Yalçinbaş et al. [7] present a Legendre collocation matrix method to solve high-order linear Fredholm integrodifferential equations under the mixed conditions in terms of Legendre polynomials. Yusufoğlu in [8] introduce a numerical method for solving initial value problems for a system of integrodifferential equations (the main idea is based on the interpolations of unknown functions at distinct interpolation points). Yüzbaşı et al. [9] present a numerical matrix method based on collocation points for the approximate solution of the systems of high-order linear Fredholm integrodifferential equations with variable coefficients under mixed conditions in terms of the Bessel polynomials. Zarebnia and Ali Abadi [10] use the Sinc-collocation method to solve systems of nonlinear second-order integrodifferential equations. Berenguer et al. used in [11–14] von Neumann series, fixed-point techniques, and Faber-Schauder systems in Banach spaces to solve integro and integrodifferential equations.

In the present paper we approximate the solution of (1) and we extend the numerical approximation method given in [14]. This paper is organized as follows. In Section 2 we describe the proposed method and in Section 3 the convergence of the proposed method is investigated. In Section 4 some numerical examples are presented to show the efficiency of the proposed scheme. Finally, in Section 5, we end with some conclusions.

2. Description of the Proposed Method

We suppose that $\mathbf{F}$, $\mathbf{K}$ satisfy a global Lipschitz condition in its last variable; that is, there exist $L_F, L_K > 0$ such that for all $t, s \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\begin{aligned} \|\mathbf{F}(t,\mathbf{u}) - \mathbf{F}(t,\mathbf{v})\|_\infty &\le L_F \|\mathbf{u}-\mathbf{v}\|_\infty, \\ \|\mathbf{K}(t,s,\mathbf{u}) - \mathbf{K}(t,s,\mathbf{v})\|_\infty &\le L_K \|\mathbf{u}-\mathbf{v}\|_\infty. \end{aligned} \tag{4}$$

If we reformulate the system (1) in terms of an adequate operator, we can derive its unique solvability under a suitable condition. To be more precise, if $\mathscr{V} : C([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^n)$ is the operator given for each $\mathbf{X}$ as

$$\begin{aligned} \mathscr{V}(\mathbf{X})(\cdot) := \boldsymbol{\rho} &+ \int_0^{(\cdot)} \mathbf{F}(u,\mathbf{X}(u))\,du \\ &+ \int_0^{(\cdot)}\int_0^1 \mathbf{K}(u,s,\mathbf{X}(s))\,ds\,du, \end{aligned} \tag{5}$$

then solving (1) is equivalent to finding a fixed point $\mathbf{X}$ of the operator $\mathscr{V}$.

A direct calculation over $\mathscr{V}$ leads to

$$\|\mathscr{V}(\mathbf{Y}) - \mathscr{V}(\mathbf{Z})\| \le L\|\mathbf{Y}-\mathbf{Z}\| \tag{6}$$

for all $\mathbf{Y}, \mathbf{Z} \in C([0,1],\mathbb{R}^n)$, where $L := L_F + L_K$, with L_F and L_K being the Lipschitz constants of $\mathbf{F}$ and $\mathbf{K}$, respectively. Thus, according to the Banach fixed-point theorem (see [15]), (5) has one unique fixed point; equivalently, (1) has one and only one solution $\mathbf{X} \in C([0,1],\mathbb{R}^n)$ provided that $L < 1$. In addition, for each $\widetilde{\mathbf{X}} \in C([0,1],\mathbb{R}^n)$,

$$\left\|\mathscr{V}^m\left(\widetilde{\mathbf{X}}\right) - \mathbf{X}\right\| \le \frac{L^m}{1-L}\left\|\mathscr{V}\left(\widetilde{\mathbf{X}}\right) - \widetilde{\mathbf{X}}\right\| \tag{7}$$

and in particular $\mathbf{X} = \lim_m \mathscr{V}^m(\widetilde{\mathbf{X}})$.

Then, given $\widetilde{\mathbf{X}} = [\widetilde{x}_1, \dots, \widetilde{x}_n]^T \in C([0,1],\mathbb{R}^n)$, our next target is to obtain $\mathscr{V}^m(\widetilde{\mathbf{X}})$. We consider the functions $\boldsymbol{\varphi} \in C([0,1],\mathbb{R}^n)$ and $\boldsymbol{\xi} \in C([0,1]^2,\mathbb{R}^n)$ defined by

$$\begin{aligned} \boldsymbol{\varphi}(t) &:= \mathbf{F}\left(t,\widetilde{\mathbf{X}}(t)\right) = \left[\varphi_1(t),\dots,\varphi_n(t)\right]^T, \\ \boldsymbol{\xi}(t,s) &:= \mathbf{K}\left(t,s,\widetilde{\mathbf{X}}(s)\right) = \left[\xi_1(t,s),\dots,\xi_1(t,s)\right]^T. \end{aligned} \tag{8}$$

Observe that $\mathscr{V}(\widetilde{\mathbf{X}})(t) = [(\mathscr{V}(\widetilde{\mathbf{X}}))_1(t),\dots,(\mathscr{V}(\widetilde{\mathbf{X}}))_n(t)]^T$, where for all $i = 1,\dots,n$

$$\left(\mathscr{V}\left(\widetilde{\mathbf{X}}\right)\right)_i(t) = \rho_i + \int_0^t \varphi_i(u)\,du + \int_0^t\int_0^1 \xi_i(u,s)\,ds\,du. \tag{9}$$

Now we will make use of the usual Schauder basis $\{b_i\}_{i\ge1}$ in $C([0,1],\mathbb{R})$ and the usual Schauder basis $\{B_i\}_{i\ge1}$ for the Banach space $C([0,1]^2,\mathbb{R})$ (see [16, 17]), although the numerical method given works equally well by replacing it with any complete biorthogonal system in this space. We denote by $\{P_i\}_{i\ge1}$ and $\{Q_i\}_{i\ge1}$ the sequences of projections in $C([0,1],\mathbb{R})$ and $C([0,1]^2,\mathbb{R})$, respectively (see Section 3 in [11]).

Then, for all $t \in [0,1]$ and $i = 1,\dots,n$,

$$\begin{aligned} \left(\mathscr{V}\left(\widetilde{\mathbf{X}}\right)\right)_i(t) = \rho_i &+ \sum_{k\ge1}\lambda_{ik}\int_0^t b_k(u)\,du \\ &+ \sum_{k\ge1}\delta_{ik}\int_0^t\int_0^1 B_k(u,s)\,ds\,du, \end{aligned} \tag{10}$$

where $\{\lambda_{ik}\}_{k\ge1}$ and $\{\delta_{ik}\}_{k\ge1}$ are the sequences of scalars satisfying $\varphi_i = \sum_{k\ge1}\lambda_{ik}b_k$ and $\xi_i = \sum_{k\ge1}\delta_{ik}B_k$, where $\lambda_{i1} = \varphi_i(t_1)$ and $\delta_{i1} = \xi_i(t_1,t_1)$, and for $k \ge 2$ is $\lambda_{ik} = \varphi_i(t_k) - \sum_{l=1}^{k-1} b_l^*(\varphi_i)b_l(t_k)$ and $\delta_{ik} = \xi_i(t_p,t_q) - \sum_{l=1}^{k-1} B_l^*(\xi_i)B_l(t_p,t_q)$ with $\sigma(k) = (p,q)$.

In view of (10) we can calculate, at least in a theoretical way, $\mathscr{V}^m(\widetilde{\mathbf{X}})$. From a practical point of view, in general these calculations are not possible explicitly, since they are infinite sums. The idea of our numerical method is to truncate them by means of the projections of the Schauder bases $\{b_i\}_{i\ge1}$, $\{B_i\}_{i\ge1}$ and approximate the solution in this way. Specifically, we consider the sequence $\{\mathbf{X_r}\}_{r\ge1}$ defined as follows. Let $\mathbf{X_0}(t) := \widetilde{\mathbf{X}}(t) = [x_{01}(t),\dots,x_{0n}(t)]^T \in C([0,1],\mathbb{R}^n)$, $t \in [0,1]$, and $\{i_1, i_2, \dots\}$, $\{j_1, j_2, \dots\}$ be subsets of natural numbers and $m \in \mathbb{N}$. Define inductively for $r \in \{1, \ldots, m\}$ and $t, s \in [0,1]$

$$\begin{aligned}\mathbf{X_r}(t) &= [x_{r1}(t), \ldots, x_{rn}(t)]^T \\ &:= \boldsymbol{\rho} + \left[\int_0^t P_{i_r}(\psi_{r-1,1}(u))\,du, \right. \\ &\qquad \left. \ldots, \int_0^t P_{i_r}(\psi_{r-1,n}(u))\,du\right]^T \\ &\quad + \left[\int_0^t \int_0^1 Q_{j_r^2}(\phi_{r-1,1}(u,s))\,ds\,du, \right. \\ &\qquad \left. \ldots, \int_0^t \int_0^1 Q_{j_r^2}(\phi_{r-1,n}(u,s))\,ds\,du\right]^T,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathbf{\Psi_{r-1}}(t) &= [\psi_{r-1,1}(t), \ldots, \psi_{r-1,n}(t)]^T \\ &:= \mathbf{F}(t, \mathbf{X_{r-1}}(t)), \\ \psi_{r-1,i}(t) &= F_i(t, \mathbf{X_{r-1}}(t)) \quad \text{for } i = 1, \ldots, n, \\ \mathbf{\Phi_{r-1}}(t,s) &= [\phi_{r-1,1}(t,s), \ldots, \phi_{r-1,n}(t,s)]^T \\ &:= \mathbf{K}(t, s, \mathbf{X_{r-1}}(s)), \\ \phi_{r-1,i}(t,s) &= K_i(t, s, \mathbf{X_{r-1}}(s)) \quad \text{for } i = 1, \ldots, n.\end{aligned} \tag{12}$$

Observe that for $i = 1, \ldots, n$

$$\begin{aligned}x_{ri}(t) &= \rho_i + \int_0^t P_{i_r}(\psi_{r-1,i}(u))\,du \\ &\quad + \int_0^t \int_0^1 Q_{j_r^2}(\phi_{r-1,i}(u,s))\,ds\,du.\end{aligned} \tag{13}$$

3. Convergence of the Scheme

This section is devoted to provide a convergence analysis for the numerical scheme $\{\mathbf{X_r}\}_{r\geq 1}$. To analyze the convergence we employ the following two results.

Theorem 1. *Let $\mathbf{F} \in C^1([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbf{K} \in C^1([0,1]^2 \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\mathbf{F}, \mathbf{K}$ and $\partial F_i/\partial t$, $\partial F_i/\partial u_j$, $\partial K_i/\partial t$, $\partial K_i/\partial s$, and $\partial K_i/\partial u_j$ for each $i, j = 1, \ldots, n$ satisfy a global Lipschitz condition in the last variables. Then, maintaining the notation above, the sequences $\{\psi'_{r-1,i}\}_{r\geq 1}$, $\{\partial\phi_{r-1,i}/\partial t\}_{r\geq 1}$, and $\{\partial\phi_{r-1,i}/\partial s\}_{r\geq 1}$, with $i = 1, \ldots, n$, are bounded.*

Proof. Let us fix $i = 1, \ldots, n$ and write $\partial F_i/\partial \mathbf{u} = [\partial F_i/\partial u_1, \ldots, \partial F_i/\partial u_n]^T$, $\partial K_i/\partial \mathbf{u} = [\partial K_i/\partial u_1, \ldots, \partial K_i/\partial u_n]^T$. Making use of definitions (11), it follows that, for all $r \geq 1$, $i = 1, \ldots, n$, and $t, s \in [0,1]$,

$$\begin{aligned}\psi'_{r-1,i}(t) &= \frac{\partial F_i}{\partial t}(t, \mathbf{X_{r-1}}(t)) + \frac{\partial F_i}{\partial \mathbf{u}}(t, \mathbf{X_{r-1}}(t)) \cdot \mathbf{X'_{r-1}}(t), \\ \frac{\partial \phi_{r-1,i}}{\partial t}(t,s) &= \frac{\partial K_i}{\partial t}(t, s, \mathbf{X_{r-1}}(s)), \\ \frac{\partial \phi_{r-1,i}}{\partial s}(t,s) &= \frac{\partial K_i}{\partial s}(t, s, \mathbf{X_{r-1}}(s)) \\ &\quad + \frac{\partial K_i}{\partial \mathbf{u}}(t, s, \mathbf{X_{r-1}}(s)) \cdot \mathbf{X'_{r-1}}(s),\end{aligned} \tag{14}$$

where "$\cdot$" stands for the usual inner product in $\mathbb{R}^n$.

For all $r \geq 1$ and $t, s \in [0,1]$, we have

$$\begin{aligned}\|\mathbf{\Psi_{r-1}}(t)\|_\infty &= \|\mathbf{F}(t, \mathbf{X_{r-1}}(t))\|_\infty \\ &\leq \|\mathbf{F}(t, \mathbf{X_{r-1}}(t)) - \mathbf{F}(t, \mathbf{0})\|_\infty + \|\mathbf{F}(t, \mathbf{0})\|_\infty \\ &\leq L_F \|\mathbf{X_{r-1}}(t)\|_\infty + R_F,\end{aligned} \tag{15}$$

with $R_F := \max_{t\in[0,1]} \|\mathbf{F}(t, \mathbf{0})\|_\infty$ and L_F being the Lipschitz constant of $\mathbf{F}$ and analogous

$$\begin{aligned}\|\mathbf{\Phi_{r-1}}(t,s)\|_\infty &= \|\mathbf{K}(t, s, \mathbf{X_{r-1}}(s))\|_\infty \\ &\leq \|\mathbf{K}(t, s, \mathbf{X_{r-1}}(s)) - \mathbf{K}(t, s, \mathbf{0})\|_\infty \\ &\quad + \|\mathbf{K}(t, s, \mathbf{0})\|_\infty \\ &\leq L_K \|\mathbf{X_{r-1}}(s)\|_\infty + R_K,\end{aligned} \tag{16}$$

with $R_K := \max_{(t,s)\in[0,1]^2} \|\mathbf{K}(t, s, \mathbf{0})\|_\infty$ and L_K being the Lipschitz constant of $\mathbf{K}$.

Now we will show that the sequence $\{\mathbf{X_r}\}_{r\geq 1}$ is bounded.

From the monotonicity of the Schauder bases $\{b_i\}_{i\geq 1}$, $\{B_i\}_{i\geq 1}$ and the recursive application of this inequality and the following one,

$$\begin{aligned}\|\mathbf{X_r}(t)\|_\infty &\leq \|\boldsymbol{\rho}\|_\infty + \int_0^t \|\mathbf{\Psi_{r-1}}(u)\|_\infty du \\ &\quad + \int_0^t \int_0^1 \|\mathbf{\Phi}_{r-1}(u,s)\|_\infty ds\,du,\end{aligned} \tag{17}$$

we have

$$\begin{aligned}\|\mathbf{X_r}(t)\|_\infty &\leq \|\boldsymbol{\rho}\|_\infty + \int_0^t (L_F \|\mathbf{X_{r-1}}(u)\|_\infty + R_F)\,du \\ &\quad + \int_0^t \int_0^1 (L_K \|\mathbf{X_{r-1}}(s)\|_\infty + R_K)\,ds\,du \\ &\leq \Gamma + L \|\mathbf{X_{r-1}}\|,\end{aligned} \tag{18}$$

with $L := L_F + L_K$, $\Gamma := \|\boldsymbol{\rho}\|_\infty + R$ and $R := R_F + R_K$. Applying it inductively, we arrive at

$$\|\mathbf{X_r}\| \leq \Gamma \sum_{k=1}^{r} L^{k-1} + L^r \|\mathbf{X_0}\|, \tag{19}$$

for all $r \geq 1$, and therefore the sequence $\{\mathbf{X_r}\}_{r\geq 1}$ is bounded.

Since the sequence $\{\mathbf{X_r}\}_{r\geq 1}$ in (15) and (16) is bounded it follows that $\{\boldsymbol{\Psi}_{r-1}\}_{r\geq 1}$ and $\{\boldsymbol{\Phi}_{r-1}\}_{r\geq 1}$ are uniformly bounded.

For $i = 1, \ldots, n$, we have

$$\begin{aligned} &\left|\frac{\partial F_i}{\partial t}(t, \mathbf{X_{r-1}}(t))\right| \\ &\quad \leq \left|\frac{\partial F_i}{\partial t}(t, \mathbf{X_{r-1}}(t)) - \frac{\partial F_i}{\partial t}(t, \mathbf{0})\right| + \left|\frac{\partial F_i}{\partial t}(t, \mathbf{0})\right| \\ &\quad \leq L_{F_i}\|\mathbf{X_{r-1}}\| + R_i, \end{aligned} \tag{20}$$

with $R_i := \max_{t\in[\alpha,\alpha+\beta]} |(\partial F_i/\partial t)(t, \mathbf{0})|$ and L_{F_i} as the Lipschitz constant of $\partial F_i/\partial t$.

Meanwhile,

$$\begin{aligned} &\left\|\frac{\partial F_i}{\partial \mathbf{u}}(t, \mathbf{X_{r-1}}(t))\right\|_\infty \\ &\quad \leq \left\|\frac{\partial F_i}{\partial \mathbf{u}}(t, \mathbf{X_{r-1}}(t)) - \frac{\partial F_i}{\partial \mathbf{u}}(t, \mathbf{0})\right\|_\infty + \left\|\frac{\partial F_i}{\partial \mathbf{u}}(t, \mathbf{0})\right\|_\infty \\ &\quad \leq L_{F_i}^* \|\mathbf{X_{r-1}}\| + R_i^*, \end{aligned} \tag{21}$$

with $R_i^* := \max_{j=1,\ldots,n} \max_{t\in[\alpha,\alpha+\beta]} |(\partial F_i/\partial u_j)(t, \mathbf{0})|$ and $L_{F_i}^*$ as the maximum of the Lipschitz constants for each $\partial F_i/\partial u_j$, $j = 1, \ldots, n$.

Therefore, $\{(\partial F_i/\partial t)(t, \mathbf{X_{r-1}}(t))\}_{r\geq 1}$ and $\{(\partial F_i/\partial \mathbf{u})(t, \mathbf{X_{r-1}}(t))\}_{r\geq 1}$ are bounded.

Next, we will show that the sequence $\{\mathbf{X'_{r-1}}\}_{r\geq 1}$ is bounded.

Given $r \geq 1$, taking into account the definition of $\mathbf{X_r}$, we have for all $t \in [0, 1]$ that

$$\begin{aligned} \mathbf{X'_r}(t) &= \left[P_{i_r}(\psi_{r-1,1}(t)), \ldots, P_{i_r}(\psi_{r-1,n}(t))\right]^T \\ &\quad + \left[\int_0^1 Q_{j_r^2}(\phi_{r-1,1}(t, s))\, ds, \right. \\ &\qquad \left. \ldots, \int_0^1 Q_{j_r^2}(\phi_{r-1,n}(t, s))\, ds\right]^T. \end{aligned} \tag{22}$$

In view of the monotonicity of the Schauder bases $\{b_i\}_{i\geq 1}$ and $\{B_i\}_{i\geq 1}$ and (15), (16), and (19), we obtain

$$\begin{aligned} \|\mathbf{X'_r}\| &\leq \|\boldsymbol{\Psi}_{\mathbf{r-1}}\| + \|\boldsymbol{\Phi}_{\mathbf{r-1}}\| \\ &\leq L_F \|\mathbf{X_{r-1}}\| + R_F + L_K \|\mathbf{X_{r-1}}\| + R_K \\ &\leq \|\mathbf{X_{r-1}}\| (L_F + L_K) + R_F + R_K \\ &\leq \Gamma \sum_{k=1}^{r} L^k + L^r \|\mathbf{X_0}\| + R. \end{aligned} \tag{23}$$

Therefore, the sequence $\{\mathbf{X'_r}\}_{r\geq 1}$ is also bounded.

We will prove that the sequences $\{(\partial K_i/\partial t)(t, s, \mathbf{X_{r-1}}(s))\}_{r\geq 1}$, $\{(\partial K_i/\partial s)(t, s, \mathbf{X_{r-1}}(s))\}_{r\geq 1}$, and $\{(\partial K_i/\partial \mathbf{u})(t, s, \mathbf{X_{r-1}}(s))\}_{r\geq 1}$ are bounded.

For $i = 1, \ldots, n$, we have

$$\begin{aligned} &\left|\frac{\partial K_i}{\partial t}(t, s, \mathbf{X_{r-1}}(s))\right| \\ &\quad \leq \left|\frac{\partial K_i}{\partial t}(t, s, \mathbf{X_{r-1}}(s)) - \frac{\partial K_i}{\partial t}(t, s, \mathbf{0})\right| + \left|\frac{\partial K_i}{\partial t}(t, s, \mathbf{0})\right| \\ &\quad \leq M_{1i}\|\mathbf{X_{r-1}}\| + N_{1i}, \end{aligned} \tag{24}$$

with $N_{1i} := \max_{(t,s)\in[0,1]^2} |(\partial K_i/\partial t)(t, s, \mathbf{0})|$ and M_{1i} as the Lipschitz constant of $\partial K_i/\partial t$.

By repeating the previous argument we obtain

$$\left|\frac{\partial K_i}{\partial s}(t, s, \mathbf{X_{r-1}}(s))\right| \leq M_{2i}\|\mathbf{X_{r-1}}\| + N_{2i}, \tag{25}$$

with $N_{2i} := \max_{(t,s)\in[0,1]^2} |(\partial K_i/\partial s)(t, s, \mathbf{0})|$ and M_{2i} as the Lipschitz constant of $\partial K_i/\partial s$.

Therefore, the sequences $\{(\partial K_i/\partial t)(t, s, \mathbf{X_{r-1}}(s))\}_{r\geq 1}$ and $\{(\partial K_i/\partial s)(t, s, \mathbf{X_{r-1}}(s))\}_{r\geq 1}$ are bounded.

Meanwhile,

$$\begin{aligned} &\left\|\frac{\partial K_i}{\partial \mathbf{u}}(t, s, \mathbf{X_{r-1}}(s))\right\|_\infty \\ &\quad \leq \left\|\frac{\partial K_i}{\partial \mathbf{u}}(t, s, \mathbf{X_{r-1}}(s)) - \frac{\partial K_i}{\partial \mathbf{u}}(t, s, \mathbf{0})\right\|_\infty + \left\|\frac{\partial K_i}{\partial \mathbf{u}}(t, s, \mathbf{0})\right\|_\infty \\ &\quad \leq M_{3i}\|\mathbf{X_{r-1}}\| + N_{3i}, \end{aligned} \tag{26}$$

with $N_{3i} := \max_{j=1,\ldots,n} \max_{(t,s)\in[0,1]^2} |(\partial K_i/\partial u_j)(t, s, \mathbf{0})|$ and M_{3i} as the maximum of the Lipschitz constants for each $\partial K_i/\partial u_j$, $j = 1, \ldots, n$. Therefore, $\{(\partial K_i/\partial \mathbf{u})(t, s, \mathbf{X_{r-1}}(s))\}$ is bounded.

In view of the identities (14), we have that the sequences, $\{\psi'_{r-1,i}\}_{r\geq 1}$, $\{\partial\phi_{r-1,i}/\partial t\}_{r\geq 1}$, and $\{\partial\phi_{r-1,i}/\partial s\}_{r\geq 1}$, with $i = 1, \ldots, n$, are bounded. □

For a dense subset $\{t_i\}_{i\geq 1}$ of distinct points in $[0, 1]$, let T_i be the set $\{t_j, 1 \leq j \leq i\}$ ordered in an increasing way for $i \geq 2$. Let ΔT_i denote the maximum distance between two consecutive points of T_i.

Theorem 2. *With the previous notation and the same hypothesis as in Theorem 1, for all $r \geq 1$, there are $\eta_r, \tau_r > 0$ and $i_r, j_r \geq 2$ such that*

$$\begin{aligned} &\left\|\left[\psi_{r-1,1} - P_{i_r}(\psi_{r-1,1}), \ldots, \psi_{r-1,n} - P_{i_r}(\psi_{r-1,n})\right]^T\right\| \\ &\qquad \leq \eta_r \Delta T_{i_r}, \\ &\left\|\left[\phi_{r-1,1} - Q_{j_r^2}(\phi_{r-1,1}), \ldots, \phi_{r-1,n} - Q_{j_r^2}(\phi_{r-1,n})\right]^T\right\| \\ &\qquad \leq \tau_r \Delta T_{j_r}. \end{aligned} \tag{27}$$

Proof. The announced estimation follows from the inequalities obtained in Propositions 4 and 5 in [11], respectively, and applying Theorem 1. □

In the result below we show that the sequence defined in (11) approximates the solution of (1).

Theorem 3. *With the same hypothesis as in Theorem 1, suppose that $\mathscr{V} : C([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^n)$ is the integral operator (5), $\widetilde{\mathbf{X}} \in C([0,1],\mathbb{R}^n)$, and that $\{\mathbf{X_r}\}_{r\geq 1}$ is the sequence defined by (11). Let us also assume that $m \in \mathbb{N}$, $i_r, j_r \geq 2$, and $\{\varepsilon_1,\ldots,\varepsilon_m\}$ is a set of positive numbers such that for all $r \in \{1,\ldots,m\}$ we have*

$$\Delta T_{i_r} \leq \frac{\varepsilon_r}{2\eta_r}, \qquad \Delta T_{j_r} \leq \frac{\varepsilon_r}{2\tau_r}. \tag{28}$$

Then

$$\left\| \mathscr{V}(\mathbf{X_{r-1}}) - \mathbf{X_r} \right\| \leq \varepsilon_r. \tag{29}$$

Moreover, if $\mathbf{X}$ is the exact solution of the integral equation (1), then the error $\|\mathbf{X} - \mathbf{X_m}\|$ is given by

$$\|\mathbf{X} - \mathbf{X_m}\| \leq \frac{L^m}{1-L}\left\|\mathscr{V}\widetilde{\mathbf{X}} - \widetilde{\mathbf{X}}\right\| + \sum_{r=1}^{m} L^{m-r}\varepsilon_r. \tag{30}$$

Proof. For $m \geq 1$, from (7), we have

$$\left\|\mathbf{X} - \mathscr{V}^m\widetilde{\mathbf{X}}\right\| \leq \frac{L^m}{1-L}\left\|\mathscr{V}\widetilde{\mathbf{X}} - \widetilde{\mathbf{X}}\right\|. \tag{31}$$

First we deal with proving (29). For all $r \in \{1,\ldots,m\}$ and $t \in [0,1]$, Theorem 2 gives

$$\begin{aligned}
&\left\|\mathscr{V}\mathbf{X_{r-1}}(t) - \mathbf{X_r}(t)\right\|_\infty \\
&= \Bigg\| \boldsymbol{\rho} + \left[\int_0^t \psi_{r-1,1}(u)\,du,\ldots,\int_0^t \psi_{r-1,n}(u)\,du\right]^T \\
&\quad + \left[\int_0^t\int_0^1 \phi_{r-1,1}(u,s)\,ds\,du,\ldots,\int_0^t\int_0^1 \phi_{r-1,n}(u,s)\,ds\,du\right]^T \\
&\quad - \Bigg(\boldsymbol{\rho} + \left[\int_0^t P_{i_r}(\psi_{r-1,1}(u))\,du,\ldots,\int_0^t P_{i_r}(\psi_{r-1,n}(u))\,du\right]^T \\
&\quad + \left[\int_0^t\int_0^1 Q_{j_r^2}(\phi_{r-1,1}(u,s))\,ds\,du,\ldots,\int_0^t\int_0^1 Q_{j_r^2}(\phi_{r-1,n}(u,s))\,ds\,du\right]^T\Bigg)\Bigg\|_\infty \\
&\leq \left\| \left[\int_0^t \left(\psi_{r-1,1}(u) - P_{i_r}(\psi_{r-1,1}(u))\right)du,\ldots,\int_0^t \left(\psi_{r-1,n}(u) - P_{i_r}(\psi_{r-1,n}(u))\right)du\right]^T\right\|_\infty \\
&\quad + \left\| \left[\int_0^t\int_0^1 \left(\phi_{r-1,1}(u,s) - Q_{j_r^2}(\phi_{r-1,1}(u,s))\right)ds\,du,\ldots,\int_0^t\int_0^1 \left(\phi_{r-1,n}(u,s) - Q_{j_r^2}(\phi_{r-1,n}(u,s))\right)ds\,du\right]^T\right\|_\infty \\
&\leq \int_0^t \left\|\left[\psi_{r-1,1}(u) - P_{i_r}(\psi_{r-1,1}(u)),\ldots,\psi_{r-1,n}(u) - P_{i_r}(\psi_{r-1,n}(u))\right]^T\right\|_\infty du \\
&\quad + \int_0^t\int_0^1 \left\|\left[\phi_{r-1,1}(u,s) - Q_{j_r^2}(\phi_{r-1,1}(u,s)),\ldots,\phi_{r-1,n}(u,s) - Q_{j_r^2}(\phi_{r-1,n}(u,s))\right]^T\right\|_\infty ds\,du \\
&\leq \eta_r \Delta T_{i_r} + \tau_r \Delta T_{j_r} \leq \varepsilon_r.
\end{aligned} \tag{32}$$

And, in turn, applying (29) and recursively (6), we obtain

$$\begin{aligned}
\left\|\mathscr{V}^m(\widetilde{\mathbf{X}}) - \mathbf{X_m}\right\| &= \left\|\mathscr{V}^m(\mathbf{X_0}) - \mathbf{X_m}\right\| \\
&\leq \sum_{r=1}^m \left\|\mathscr{V}^{m-r+1}(\mathbf{X_{r-1}}) - \mathscr{V}^{m-r}(\mathbf{X_r})\right\| \\
&= \sum_{r=1}^m \left\|\mathscr{V}^{m-r}\mathscr{V}(\mathbf{X_{r-1}}) - \mathscr{V}^{m-r}(\mathbf{X_r})\right\| \\
&\leq \sum_{r=1}^m L^{m-r}\left\|\mathscr{V}(\mathbf{X_{r-1}}) - \mathbf{X_r}\right\| \\
&\leq \sum_{r=1}^m L^{m-r}\varepsilon_r.
\end{aligned} \tag{33}$$

Finally, using the triangle inequality,

$$\|\mathbf{X} - \mathbf{X_m}\| \leq \left\|\mathbf{X} - \mathscr{V}^m\widetilde{\mathbf{X}}\right\| + \left\|\mathscr{V}^m(\widetilde{\mathbf{X}}) - \mathbf{X_m}\right\|, \tag{34}$$

the proof is complete in view of (31) and (33). □

Observe that under the hypotheses of Theorem 3, by inequality (30), we have

$$\|\mathbf{X} - \mathbf{X_m}\| \leq \frac{L^m}{1-L}\left\|\mathscr{V}\widetilde{\mathbf{X}} - \widetilde{\mathbf{X}}\right\| + \frac{1-L^m}{1-L}\max_{r\geq 1}\{\varepsilon_r\}. \tag{35}$$

Therefore, given $\varepsilon > 0$, there exists $m \geq 1$ such that $\|\mathbf{X}-\mathbf{X_m}\| < \varepsilon$ for sufficiently small ε_r, since the points of the partition can be chosen in such a way that ΔT_{i_r} and ΔT_{j_r} become so close to zero as we desire and the first sum on the right hand side approach zero when m increases.

Remark 4. If we consider an interval $[a,b]$, then $L = (b-a)L_F + (b-a)^2 L_k$ and the bound obtained in Theorem 3 for $\|\mathscr{V}\mathbf{X}_{\mathbf{r-1}}(t) - \mathbf{X}_{\mathbf{r}}(t)\|_\infty$ is given by

$$\|\mathscr{V}\mathbf{X}_{\mathbf{r-1}}(t) - \mathbf{X}_{\mathbf{r}}(t)\|_\infty \le \eta_r \Delta T_{i_r}(b-a) + \tau_r \Delta T_{j_r}(b-a)^2 \le \varepsilon_r, \tag{36}$$

when

$$\Delta T_{i_r} \le \frac{\varepsilon_r}{2\eta_r(b-a)}, \qquad \Delta T_{j_r} \le \frac{\varepsilon_r}{2\tau_r(b-a)^2}. \tag{37}$$

4. Numerical Examples

We now turn our attention to the application of the method presented in this paper for the numerical solution of six test problems. In order to construct the Schauder basis, we consider the subset $\{t_i\}_{i\ge1}$ defined by $t_1 = 0$, $t_2 = 1$ and for $n \in \mathbb{N} \cup \{0\}$, $t_{i+1} = (2k+1)/2^{n+1}$, if $i = 2^n + k + 1$, where $0 \le k < 2^n$ are integers. To define the sequence $\{\mathbf{X_r}\}_{r\ge1}$, we take $\mathbf{X_0}(t) = \boldsymbol{\rho}$ and $i_r = j_r = j$ (for all $r \ge 1$). We include, for different values of j, the absolute errors committed in some representative points of $[0,1]$ when we approximate the exact solution $\mathbf{X}(t)$ by the iteration $\mathbf{X_r}(t)$, where r is shown in each table. The algorithms associated with the numerical methods were performed using Mathematica 7. In Examples 1, 2, and 3, $\mathbf{X}(t) = x(t)$ and $\mathbf{X_r}(t) = x_r(t)$. In the other examples, $\mathbf{X}(t) = [x_1(t), x_2(t)]^T$ and $\mathbf{X_r}(t) = [x_{r1}(t), x_{r2}(t)]^T$.

Example 1. Consider the Fredholm integrodifferential equation appearing in [1]:

$$x'(t) = -2\pi\sin(2\pi t) - \frac{1}{2}\sin(4\pi t) + \int_0^1 \sin(4\pi t + 2\pi s)\, y(s)\, ds, \tag{38}$$
$$x(0) = 1,$$

whose exact solution is $x(t) = \cos(2\pi t)$. Numerical results obtained for this problem when we apply the method described in this paper and the results obtained in [1] are given in Table 1.

Example 2. Consider the Fredholm integrodifferential equation:

$$x'(t) = f(t) + \frac{1}{125}\int_0^5 stx(s)\, ds, \qquad x(0) = \frac{1}{125}, \tag{39}$$

where $f(t)$ is chosen so that the exact solution is given by $x(t) = e^{-5t}/250$. The numerical results are given in Table 2.

Table 1: Absolute errors for Example 1.

t	$j = 33$ $\|x_4(t) - x(t)\|$	Method in [1] with $k = 4, M = 1$
0.1	4.93×10^{-4}	2.40×10^{-3}
0.2	8.90×10^{-4}	5.07×10^{-3}
0.3	4.00×10^{-3}	6.25×10^{-3}
0.4	5.70×10^{-3}	3.87×10^{-3}
0.5	6.40×10^{-3}	1.74×10^{-2}
0.6	3.72×10^{-3}	1.58×10^{-2}
0.7	7.78×10^{-4}	8.41×10^{-3}
0.8	8.96×10^{-4}	9.65×10^{-3}
0.9	9.40×10^{-4}	9.49×10^{-3}

Table 2: Absolute errors for Example 2.

t	$j = 33$ $\|x_3(t) - x(t)\|$	t	$j = 33$ $\|x_3(t) - x(t)\|$
0.125	1.91×10^{-3}	2.625	5.93×10^{-3}
0.250	2.41×10^{-3}	2.750	6.11×10^{-3}
0.375	3.51×10^{-3}	2.825	6.29×10^{-3}
0.500	3.83×10^{-3}	3.00	6.49×10^{-3}
0.625	4.02×10^{-3}	3.125	6.69×10^{-3}
0.750	4.15×10^{-3}	3.250	6.91×10^{-3}
0.825	4.25×10^{-3}	3.375	7.13×10^{-3}
1.00	4.34×10^{-3}	3.500	7.35×10^{-3}
1.125	4.43×10^{-3}	3.625	7.59×10^{-3}
1.250	4.50×10^{-3}	3.750	7.84×10^{-3}
1.375	4.60×10^{-3}	3.875	8.09×10^{-3}
1.500	4.69×10^{-3}	4	8.35×10^{-3}
1.625	4.81×10^{-3}	4.125	8.62×10^{-3}
1.750	4.91×10^{-3}	4.250	8.90×10^{-3}
1.875	5.50×10^{-3}	4.375	9.19×10^{-3}
2	5.51×10^{-3}	4.500	9.48×10^{-3}
2.125	5.53×10^{-3}	4.625	9.78×10^{-3}
2.250	5.54×10^{-3}	4.750	1.01×10^{-2}
2.375	5.56×10^{-3}	4.875	1.04×10^{-2}
2.500	5.57×10^{-3}	5	1.07×10^{-2}

Example 3. Consider the Fredholm integrodifferential equation:

$$x'(t) = -e^{-t} + \frac{1}{10}\left(-2 + e^{-2} + \cos(2)\right) + \cos(t) + \int_0^2 \frac{x(s)}{10} ds, \tag{40}$$
$$x(0) = 1,$$

whose exact solution is $x(t) = e^{-t} + \sin(t)$. The numerical results are given in Table 3.

Example 4. Consider now the following system of Fredholm integrodifferential equations with the exact solutions $x_1(t) = \cos(t)$ and $x_2(t) = t$:

$$x_1'(t) = -\frac{1}{12} - \frac{1}{5}t^2\sin(1)x_2(s) - \sin(t) + \int_0^1 \left(\frac{t^3}{5}x_1(s) + \frac{s^2}{3}x_2(s)\right) ds,$$

TABLE 3: Absolute errors for Example 3.

t	$j = 33$ $\lvert x_5(t) - x(t)\rvert$	t	$j = 33$ $\lvert x_5(t) - x(t)\rvert$
0.125	1.00×10^{-4}	1.125	7.05×10^{-4}
0.250	1.95×10^{-4}	1.250	7.54×10^{-4}
0.375	2.85×10^{-4}	1.375	7.97×10^{-4}
0.500	3.69×10^{-4}	1.500	8.34×10^{-4}
0.625	4.48×10^{-4}	1.625	8.64×10^{-4}
0.750	5.21×10^{-4}	1.750	8.88×10^{-4}
0.825	5.89×10^{-4}	1.875	9.06×10^{-4}
1.00	6.50×10^{-4}	2	9.19×10^{-4}

TABLE 4: Absolute errors for Example 4.

t	$j = 9$ $\lvert x_{51}(t) - x_1(t)\rvert$	$\mathbf{X}_5 = [x_{51}, x_{52}]^T$ $\lvert x_{52}(t) - x_2(t)\rvert$	$j = 17$ $\lvert x_{51}(t) - x_1(t)\rvert$	$\mathbf{X}_5 = [x_{51}, x_{52}]^T$ $\lvert x_{52}(t) - x_2(t)\rvert$	$j = 33$ $\lvert x_{51}(t) - x_1(t)\rvert$	$\mathbf{X}_5 = [x_{51}, x_{52}]^T$ $\lvert x_{52}(t) - x_2(t)\rvert$
0.125	$1.7E-4$	$4.8E-5$	$4.4E-5$	$1.2E-5$	$1.1E-5$	$3.2E-6$
0.250	$3.7E-4$	$9.7E-5$	$9.3E-5$	$2.4E-5$	$2.3E-5$	$6.3E-6$
0.375	$5.9E-4$	$1.4E-4$	$1.4E-4$	$3.6E-5$	$3.7E-5$	$9.5E-6$
0.5	$8.2E-4$	$1.9E-4$	$2.0E-4$	$4.8E-5$	$5.2E-5$	$1.2E-5$
0.625	$1.0E-3$	$2.3E-4$	$2.7E-4$	$5.9E-5$	$6.8E-5$	$1.5E-5$
0.750	$1.3E-3$	$2.7E-4$	$3.3E-4$	$7.0E-5$	$8.5E-5$	$1.8E-5$
0.875	$1.6E-3$	$3.1E-4$	$4.0E-4$	$8.0E-5$	$1.0E-4$	$2.0E-5$
1	$1.9E-3$	$3.5E-4$	$4.7E-4$	$8.9E-5$	$1.2E-4$	$2.3E-5$

$$\begin{aligned} x_2'(t) &= \frac{20}{21} - \frac{1}{5} t x_2(t) \sin(1) \\ &\quad + \int_0^1 \left(\frac{t^2}{5} x_1(s) + \frac{s}{7} x_2(s) \right) ds, \\ x_1(0) &= 1, \\ x_2(0) &= 0. \end{aligned} \tag{41}$$

The numerical results are given in Table 4.

Example 5. Consider now the following system of Fredholm integrodifferential equations with the exact solutions $x_1(t) = t^2$ and $x_2(t) = t$:

$$\begin{aligned} x_1'(t) &= 2t - \frac{1}{5} x_1(t)(-\cos(1) + \sin(t)) \\ &\quad + \int_0^1 \frac{t^2 s}{5} \sin(x_2(s))\, ds, \\ x_2'(t) &= 1 - \frac{1}{48} x_2(t)(\pi - \log(4)) + \int_0^1 \frac{ts}{6} \operatorname{arctg}(x_1(s))\, ds, \\ x_1(0) &= 0, \\ x_2(0) &= 0. \end{aligned} \tag{42}$$

The numerical results are given in Table 5.

Example 6. Consider now the following system of Fredholm integrodifferential equations with the exact solutions $x_1(t) = \sin(t)$ and $x_2(t) = \cos(t)$:

$$\begin{aligned} x_1'(t) &= x_2(t) + \frac{1}{10}(-1 + \cos(2) - \sin(2)) \\ &\quad + \frac{1}{10} \int_0^2 (x_1(s) + x_2(s))\, ds, \\ x_2'(t) &= -x_1(t) + \frac{1}{15}(1 - \cos(2) - \sin(2)) \\ &\quad + \frac{1}{15} \int_0^2 (x_2(s) - x_1(s))\, ds, \\ x_1(0) &= 0, \\ x_2(0) &= 1. \end{aligned} \tag{43}$$

The numerical results are given in Table 6.

5. Conclusion

In this paper we have successfully approximated the solution of systems of nonlinear Fredholm integrodifferential equations. To this end, we have used the Banach fixed-point theorem and the Schauder basis. Moreover, the convergence of the proposed scheme is analyzed and some illustrative examples were included to demonstrate the validity and applicability of the method. The approximating functions x_r and $[x_{r1}, x_{r2}]^T$ are the sum of integrals of piecewise univariate

Table 5: Absolute errors for Example 5.

t	$j = 9$ $\lvert x_{41}(t) - x_1(t)\rvert$	$\mathbf{X}_4 = [x_{41}, x_{42}]^T$ $\lvert x_{42}(t) - x_2(t)\rvert$	$j = 17$ $\lvert x_{41}(t) - x_1(t)\rvert$	$\mathbf{X}_4 = [x_{41}, x_{42}]^T$ $\lvert x_{42}(t) - x_2(t)\rvert$	$j = 33$ $\lvert x_{41}(t) - x_1(t)\rvert$	$\mathbf{X}_4 = [x_{41}, x_{42}]^T$ $\lvert x_{42}(t) - x_2(t)\rvert$
0.125	$3.4E-7$	$3.1E-6$	$6.1E-8$	$8.1E-7$	$1.1E-8$	$2.3E-7$
0.250	$2.1E-6$	$1.2E-5$	$4.4E-7$	$3.2E-6$	$7.3E-8$	$9.4E-7$
0.375	$6.5E-6$	$2.7E-5$	$1.4E-6$	$7.2E-6$	$2.2E-7$	$2.1E-6$
0.5	$1.5E-5$	$4.9E-5$	$3.3E-6$	$1.3E-5$	$4.8E-7$	$3.9E-6$
0.625	$2.9E-5$	$7.7E-5$	$6.3E-6$	$2.1E-5$	$8.3E-7$	$6.2E-6$
0.750	$4.9E-5$	$1.1E-4$	$1.1E-5$	$2.9E-5$	$1.2E-6$	$9.2E-6$
0.875	$7.7E-5$	$1.5E-4$	$1.6E-5$	$4.1E-5$	$1.5E-6$	$1.2E-5$
1	$1.1E-4$	$1.9E-4$	$2.4E-5$	$5.3E-5$	$1.6E-6$	$1.7E-5$

Table 6: Absolute errors for Example 6.

t	$j = 17$ $\lvert x_{31}(t) - x_1(t)\rvert$	$\mathbf{X}_3 = [x_{31}, x_{32}]^T$ $\lvert x_{32}(t) - x_2(t)\rvert$	$j = 33$ $\lvert x_{31}(t) - x_1(t)\rvert$	$\mathbf{X}_3 = [x_{31}, x_{32}]^T$ $\lvert x_{32}(t) - x_2(t)\rvert$	$j = 65$ $\lvert x_{31}(t) - x_1(t)\rvert$	$\mathbf{X}_3 = [x_{31}, x_{32}]^T$ $\lvert x_{32}(t) - x_2(t)\rvert$
0.125	$1.1E-3$	$4.8E-5$	$8.5E-4$	$8.3E-6$	$9.3E-5$	$9.8E-7$
0.250	$2.2E-3$	$1.1E-4$	$9.2E-4$	$2.1E-5$	$1.8E-4$	$1.1E-6$
0.375	$3.4E-3$	$2.0E-4$	$2.9E-3$	$4.0E-5$	$3.3E-4$	$1.5E-6$
0.5	$4.5E-3$	$3.1E-4$	$4.1E-3$	$6.3E-5$	$5.4E-4$	$1.8E-6$
0.625	$5.7E-3$	$4.3E-4$	$5.6E-3$	$9.0E-5$	$7.8E-4$	$4.3E-6$
0.750	$6.8E-3$	$5.7E-4$	$6.7E-3$	$1.2E-4$	$9.2E-4$	$8.5E-6$
0.875	$8.0E-3$	$7.3E-4$	$7.9E-3$	$1.5E-4$	$3.5E-3$	$1.3E-5$
1	$9.2E-3$	$9.0E-4$	$8.8E-3$	$1.9E-4$	$5.3E-3$	$1.9E-5$
1.125	$1.0E-2$	$1.0E-3$	$9.7E-3$	$2.3E-4$	$7.9E-3$	$2.6E-5$
1.250	$1.1E-2$	$1.2E-3$	$1.0E-2$	$2.8E-4$	$8.2E-3$	$3.3E-5$
1.375	$1.2E-2$	$1.4E-3$	$1.1E-2$	$3.2E-4$	$9.0E-3$	$4.1E-5$
1.5	$1.4E-2$	$1.6E-3$	$1.3E-2$	$3.7E-4$	$9.9E-3$	$4.9E-5$
1.625	$1.5E-2$	$1.8E-3$	$1.5E-2$	$4.1E-4$	$1.0E-2$	$5.7E-5$
1.750	$1.6E-2$	$2.0E-3$	$1.6E-2$	$4.6E-4$	$1.2E-2$	$6.4E-5$
1.875	$1.8E-2$	$2.2E-3$	$1.7E-2$	$5.1E-4$	$1.5E-2$	$7.2E-5$
2	$1.9E-2$	$2.4E-3$	$1.8E-2$	$5.5E-4$	$1.7E-2$	$7.9E-5$

and bivariate polynomials of degree 2 and the calculation of the coefficients of such polynomials just requires linear combinations of several evaluations of the basic functions at sufficient number of points.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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