

Research Article

Picard Successive Approximation Method for Solving Differential Equations Arising in Fractal Heat Transfer with Local Fractional Derivative

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The Fourier law of one-dimensional heat conduction equation in fractal media is investigated in this paper. An approximate solution to one-dimensional local fractional Volterra integral equation of the second kind, which is derived from the transformation of Fourier flux equation in discontinuous media, is considered. The Picard successive approximation method is applied to solve the temperature field based on the given Mittag-Leffler-type Fourier flux distribution in fractal media. The nondifferential approximate solutions are given to show the efficiency of the present method.

1. Introduction

Engineering problems can be mathematically described by differential equations. Many initial and boundary value problems associated with differential equations can be transformed into problems of solving some approximate integral equations. Heat transfer is described by theory of integral equations. Integral equation arising in heat transfer with smooth condition is valid for continuous media [1–4]. The common methods for solving the equations of heat transfer are purely mathematical are among them; the finite difference techniques (FDT) [5], the regression analysis (RA) [6], the Adomian decomposition method (ADM) [7], the combined Laplace-Adomian method (CLAM) [8], the homotopy analysis method (HAM) [9, 10], the differential transformation method (DTM) [11], the spline-wavelets techniques (SWT) [12], the boundary element method (BEM) [13], the heat-balance integral method (HBIM) [14, 15], the variational iteration method (VIM) [16], the local fractional variational

iteration method (LFVIM) [17], and the Picard successive approximation method (PSAM) [18].

On the other hand, the nanoscale heat problem can be characterized as fractal behaviors. As usual, the materials are called the Cantor materials. Heat transfer in fractal media with nonsmooth conditions is a hot topic. For example, the heat transfer equations in a medium with fractal geometry [19] and fractal domains [20] were considered. The local fractional transient heat conduction equations based upon the Fourier law within local fractional derivative arising in heat transfer from discontinuous media were presented in [21–24].

Fractional calculus was successfully used to deal with the real world problems [25–30]. There is its limit that the operators do not deal with the local fractional continuous functions (nondifferential functions). Hence, the local fractional Fourier flux [21] is not handled by using some approaches from the classical and fractional operators. This paper focuses on analytical solution to local fractional Fourier flux in fractal

media by using Picard’s successive approximation method [18]. This paper is organized as follows. In Section 2, we give notations to local fractional derivative and integrals and investigate the heat transfer in fractal media. Section 3 is devoted to Picard’s successive approximation method based upon local fractional integrals. Analysis solution is shown in Section 4. Conclusions are in Section 5.

2. Heat Transfer in Fractal Media with Local Fractional Derivative

In order to study the non-differential solution for the heat problem in fractal media with local fractional derivative, we here begin with the Fourier flux equation in discontinuous media.

The temperature field reads as [21]

$$T(x, y, z, \tau) = f(x, y, z, \tau) \quad \text{at } \tau > \tau_0 \text{ and in } \Omega, \quad (1)$$

where $f(x, y, z, \tau)$ is local fractional continuous at fractal domain Ω .

For a given temperature field T , a local fractional temperature gradient [21] can be written as follows:

$$\nabla^\alpha T = \frac{\partial^\alpha T}{\partial u_1^\alpha} e_1 + \frac{\partial^\alpha T}{\partial u_2^\alpha} e_2 + \frac{\partial^\alpha T}{\partial u_3^\alpha} e_3, \quad (2)$$

where the local fractional partial derivative is defined by [21–24]

$$\begin{aligned} f_x^{(\alpha)}(x_0, y) &= \left. \frac{\partial^\alpha f(x, y)}{\partial x^\alpha} \right|_{x=x_0} \\ &= \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x, y) - f(x_0, y))}{(x - x_0)^\alpha}, \end{aligned} \quad (3)$$

where $\Delta^\alpha (f(x, y) - f(x_0, y)) \cong \Gamma(1 + \alpha)\Delta(f(x, y) - f(x_0, y))$.

Here, the local fractional derivative is defined on the fractal set like a Cantor set. For example, when we consider the Cantor set, we can find the local fractional derivative of discontinuous function T (however, T is a local fractional continuous function).

We consider the heat flux per unit fractal area \vec{q} is proportional to the temperature gradient in fractal medium. Fourier law of heat conduction in fractal medium with local fractional derivative is expressed by [21]

$$\vec{q}(x, y, z, t) = -K^{2\alpha} \nabla^\alpha T(x, y, z, t), \quad (4)$$

where $K^{2\alpha}$ denotes the thermal conductivity of the fractal material, and it is related to fractal dimensions of materials. It is shown that the fractal dimensions of materials are an important characteristic value. Here, we consider the fractal Fourier flow, which is discontinuous; however, it is found that it is local fractional continuous. Like classical Fourier flow, its thermal conductivity is an approximate value for fractal one when $\alpha = 1$.

Fourier law of one-dimensional heat conduction equation in fractal media reads as [21]

$$q(x, t) = -K^{2\alpha} \frac{d^\alpha T(x, t)}{dx^\alpha}, \quad \text{at } \tau > \tau_0 \text{ and in } A, \quad (5)$$

where $K^{2\alpha}$ denotes the thermal conductivity of the fractal materials.

When $\tau = \tau_0$, from (5) we have

$$q(x) = -K^{2\alpha} \frac{d^\alpha T(x)}{dx^\alpha}, \quad (6)$$

at $\tau > \tau_0$ and in A , where $K^{2\alpha}$ is the thermal conductivity of the fractal materials. Namely, T is a bi-Lipschitz mapping, and shows the fractal characteristic behavior [21].

Local fractional heat conduction equation with heat generation in fractal media can be written as [21]

$$K^{2\alpha} \nabla^{2\alpha} T + g - \rho_\alpha c_\alpha \frac{\partial^\alpha T}{\partial t^\alpha} = 0 \quad \text{at } \tau > \tau_0 \text{ and in } \Omega. \quad (7)$$

Local fractional heat conduction equation with no heat generation in fractal media is suggested as [21, 22]

$$K^{2\alpha} \nabla^{2\alpha} T - \rho_\alpha c_\alpha \frac{\partial^\alpha T}{\partial t^\alpha} = 0 \quad \text{at } \tau > \tau_0 \text{ and in } \Omega, \quad (8)$$

where $\nabla^{2\alpha}$ is a local fractional Laplace operator [21].

3. The Method

In this section, we discuss the Picard successive approximation method. Meanwhile, we transfer the Fourier law of one-dimensional heat conduction equation in fractal media into the local fractional Volterra integral equation of the second kind.

3.1. Picard’s Successive Approximation Method. This method is first proposed in [18]. Here we will give a short introduction to Picard’s successive approximation method within the local fractional calculus.

In this method, we set

$$u_0(x) = f(x). \quad (9)$$

We give the first approximation $u_1(x)$ by

$$u_1(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1 + \alpha)} \int_0^x K(x, t) u_0(x) (dt)^\alpha, \quad (10)$$

where the local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined as follows [21–24]:

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{j=N-1} f(t_j) (\Delta t_j)^\alpha \end{aligned} \quad (11)$$

with $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \Delta t_2, \dots\}$, and $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$, $t_0 = a$, and $t_N = b$.

Here, we find that the equality $u_1(x)$ is a local fractional continuous function if $f(x)$, $K(x, t)$ and $u_0(x)$ are local fractional continuous functions.

Continuing in this manner, we have the infinite sequences of functions

$$u_0(x), u_1(x), u_2(x), \dots, u_n(x), \dots \quad (12)$$

such that the recurrence equations are given by

$$u_n(x) = f(x) - \frac{1}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_{n-1}(x) (dt)^\alpha, \quad (13)$$

$$(n = 1, 2, 3, \dots),$$

where $u_0(x)$ is equivalent to any selected function, which is the local fractional continuous function.

Hence, we have successive approximation as follows:

$$u_1(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) f(t) (dt)^\alpha,$$

$$\vdots \quad (14)$$

$$u_n(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_{n-1}(x) (dt)^\alpha.$$

Thus, at the limit, the solution $u(x)$ is written as

$$u(x) = \lim_{n \rightarrow \infty} u_n(x). \quad (15)$$

3.2. *An Alternative Method from Local Fractional Derivative to Local Fractional Volterra Integral Equations.* We directly observe that the local fractional differential equation of α order

$$K^{2\alpha} \frac{d^{\alpha T}}{dx^\alpha} = q(x) \quad (0 \leq x \leq b) \quad (16)$$

can be written immediately as the local fractional Volterra integral equation of the second kind in the form

$$T(x) = T(a) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x q(t) (dt)^\alpha, \quad (17)$$

where $\lambda = 1/K^2$.

The Mittag-Leffler type Fourier flux distribution in fractal media can be written as follows:

$$q(x) = q_t(t, x) = E_\alpha(x-t)^\alpha T(t) \quad (0 \leq t \leq b). \quad (18)$$

Making use of (18), we can get the local fractional Volterra integral equation of the second kind in the form:

$$T(x) = T(a) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha T(t) (dt)^\alpha. \quad (19)$$

4. Approximate Solutions for Local Fractional Volterra Integral Equation of the Second Kind

Let us assume that the zeroth approximation is

$$u_0(x) = 0. \quad (20)$$

Then the first approximation can be written as follows:

$$u_1(x) = f(x) = T(a). \quad (21)$$

Here we obtain the second approximation, which reads as

$$u_2(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha u_1(t) (dt)^\alpha. \quad (22)$$

Proceeding in this manner, we have the third approximation in the following form:

$$u_3(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha u_2(t) (dt)^\alpha. \quad (23)$$

Hence, continuing in this manner, we obtain

$$u_n(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x E_\alpha(x-t)^\alpha u_{n-1}(t) (dt)^\alpha$$

$$= f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)}$$

$$\times \int_0^x \left(1 + \dots + \frac{\lambda^{(n-1)\alpha} (x-t)^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} \right)$$

$$\times E_\alpha(x-t)^\alpha f(t) (dt)^\alpha. \quad (24)$$

Taking the limit, we have

$$u(x) = \lim_{n \rightarrow \infty} u_n(x)$$

$$= T(a) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)}$$

$$\times \int_0^x E_\alpha[(x-t)^\alpha (1+\lambda^\alpha)] T(a) (dt)^\alpha \quad (25)$$

$$= T(a) \left\{ \frac{2\lambda^\alpha}{1+\lambda^\alpha} E_\alpha[x^\alpha (1+\lambda^\alpha)] \right\}$$

$$= T(a) \left\{ \frac{2}{K^{2\alpha} + 1} E_\alpha \left[x^\alpha \left(1 + \frac{1}{K^{2\alpha}} \right) \right] \right\},$$

where the term $E_\alpha[x^\alpha(1+\lambda^\alpha)]$ is a Mittag-Leffler type Fourier flux distribution in fractal media, which is related to the fractal coarse-grained mass function [21, 24]. When $K = 1$, we get $u(x) = T(a)E_\alpha[2x^\alpha]$. The nondifferentiable solution of (25) for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 1$ is shown in Figure 1; the non-differentiable solution of (25) for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 2$ is shown in Figure 2; the non-differentiable solution of (25) for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 3$ is shown in Figure 3; the non-differentiable solution of (25) for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 4$ is shown in Figure 4.

5. Conclusions

This work studied the Fourier law of one-dimensional heat conduction equation in fractal media. Mittag-Leffler type

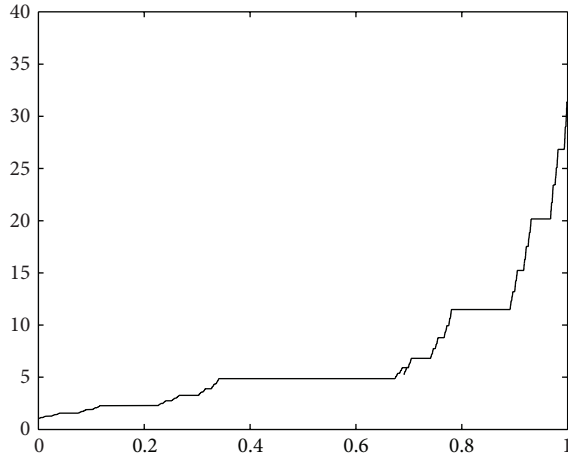


FIGURE 1: The nondifferentiable solution for Mittag-Leffler type Fourier flux distribution for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 1$.

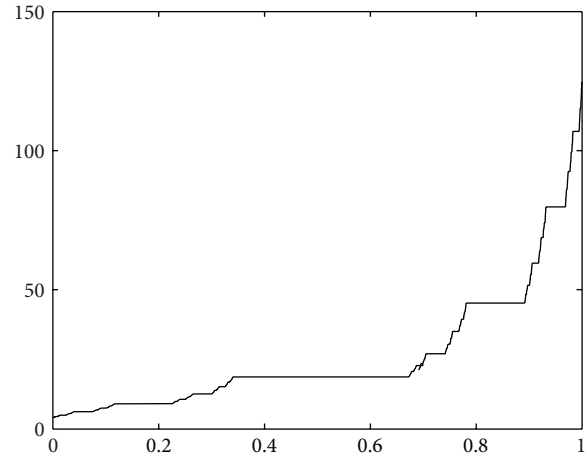


FIGURE 4: The nondifferentiable solution for Mittag-Leffler type Fourier flux distribution for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 4$.

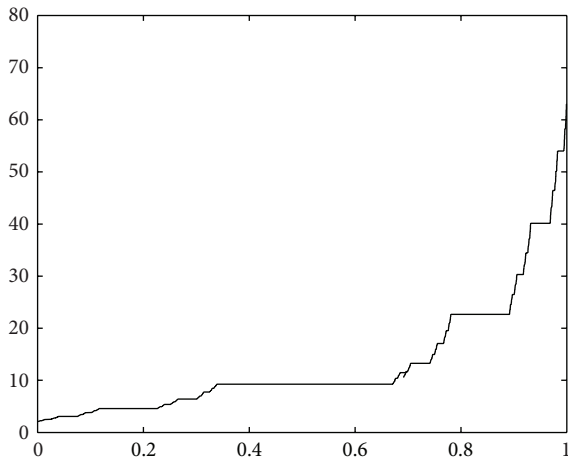


FIGURE 2: The nondifferentiable solution for Mittag-Leffler type Fourier flux distribution for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 2$.

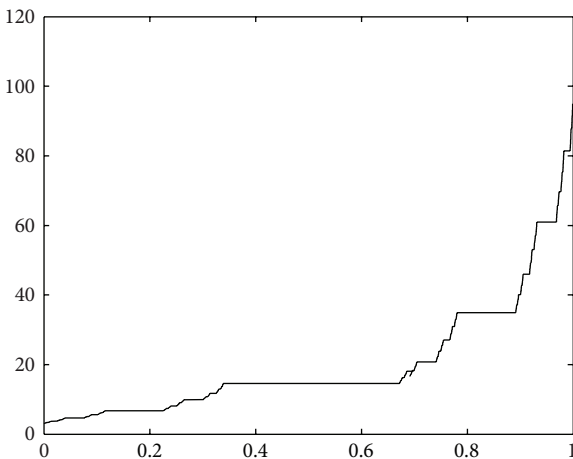


FIGURE 3: The nondifferentiable solution for Mittag-Leffler type Fourier flux distribution for parameters $K = 1$, $\alpha = \ln 2/\ln 3$, and $T(a) = 3$.

Fourier flux distribution in fractal media with temperature field effect was considered. An approximation solution for the local fractional Volterra integral equation of the second kind derived from Fourier law of one-dimensional heat conduction equation for heat conduction in discontinuous media was studied by using Picard's successive approximation method. The non-differential approximate solutions were given to show the efficiency of the present method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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