Research Article

Complex Factorizations of the Lucas Sequences via Matrix Methods

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Firstly, we show a connection between the first Lucas sequence and the determinants of some tridiagonal matrices. Secondly, we derive the complex factorizations of the first Lucas sequence by computing those determinants with the help of Chebyshev polynomials of the second kind. Furthermore, we also obtain the complex factorizations of the second Lucas sequence by the similar matrix method using Chebyshev polynomials of the first kind.

1. Introduction

Given two nonzero integers P and Q satisfying $P^2 - 4Q \neq 0$. *The first Lucas sequence* $\{U_n(P, Q)\}_{n \geq 0}$ and *the second Lucas sequence* $\{V_n(P, Q)\}_{n \geq 0}$ are defined by the recurrence relations

$$U_0(P, Q) = 0,$$

$$U_1(P, Q) = 1, \tag{1}$$

$$U_n(P, Q) = PU_{n-1}(P, Q) - QU_{n-2}(P, Q), \quad n \geq 2,$$

$$V_0(P, Q) = 2,$$

$$V_1(P, Q) = P, \tag{2}$$

$$V_n(P, Q) = PV_{n-1}(P, Q) - QV_{n-2}(P, Q), \quad n \geq 2,$$

respectively.

By assigning P and Q some special values, we will see some well-known Lucas sequences, which are important historically and for their own sake. The numbers $\{U_n(1, -1)\}_{n \geq 0}$ are called the *Fibonacci numbers* while the numbers $\{V_n(1, -1)\}_{n \geq 0}$ are called the *Lucas numbers*, the numbers $\{U_n(2, -1)\}_{n \geq 0}$ and $\{V_n(2, -1)\}_{n \geq 0}$ are the *Pell numbers* and the *Pell-Lucas numbers*, $\{U_n(1, -2)\}_{n \geq 0}$ and $\{V_n(1, -2)\}_{n \geq 0}$ are *Jacobsthal numbers* and *Jacobsthal-Lucas numbers*, respectively, $\{U_n(3, 2)\}_{n \geq 0}$ are *Mersenne numbers*, and so on.

There is a long tradition of using matrix methods to study Lucas sequences [2–5]. In 2003, Cahill et al. obtained complex factorizations for Fibonacci numbers and Lucas numbers by using the determinants of two slightly different sequences of tridiagonal matrices [2]. They used the $n \times n$ tridiagonal matrix $M(n)$ with entries $m_{k,k} = 1$ ($1 \leq k \leq n$), and $m_{k-1,k} = m_{k,k-1} = \mathrm{i}$ ($2 \leq k \leq n$) to prove that

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2\mathrm{i} \cos \frac{\pi k}{n}\right), \quad n \geq 2, \tag{3}$$

where F_n is the nth Fibonacci number and $\mathrm{i} = \sqrt{-1}$, and also used the $n \times n$ tridiagonal matrix $S(n)$ with entries $s_{1,1} = 1/2$, $s_{k,k} = 1$ ($2 \leq k \leq n$) and $s_{k-1,k} = s_{k,k-1} = \mathrm{i}$ ($2 \leq k \leq n$) to prove that

$$L_n = \prod_{k=1}^{n} \left(1 - 2\mathrm{i} \cos \frac{\pi (k - (1/2))}{n}\right), \quad n \geq 1, \tag{4}$$

where L_n is the nth Lucas number and $\mathrm{i} = \sqrt{-1}$. In 2011, Burcu Bozkurt et al. obtained the complex factorization of the second Lucas sequence $\{V_n(P, -1)\}_{n \geq 0}$ [4]:

$$V_n = \prod_{k=1}^{n} \left(P - 2\mathrm{i} \cos \frac{\pi (k - (1/2))}{n}\right), \quad n \geq 1, \tag{5}$$

by using the $n \times n$ tridiagonal matrix $D(n)$ with entries $d_{1,1} = P/2$, $d_{k,k} = P$ ($2 \leq k \leq n$), and $d_{k-1,k} = d_{k,k-1} = \mathrm{i}$ ($2 \leq k \leq n$), where $\mathrm{i} = \sqrt{-1}$.

In this study, number theory and linear algebra (with the help of orthogonal polynomials) are similarly intertwined to yield the complex factorizations of the first and second Lucas sequences.

Now we give the following lemma which will be needed later.

Lemma 1 ([2]). *Let $\{H(n), n = 1, 2, \ldots\}$ be a sequence of tridiagonal matrices of the form*

$$H(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & \ddots & & \\ & & \ddots & \ddots & h_{n-1,n} & \\ & & & h_{n,n-1} & h_{n,n} \end{pmatrix}. \quad (6)$$

Then the successive determinants of $H(n)$ are given by the recursive formula:

$$|H(1)| = h_{1,1}, \qquad |H(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1},$$

$$|H(n)| = h_{n,n}|H(n-1)| \qquad (7)$$

$$- h_{n-1,n}h_{n,n-1}|H(n-2)|, \quad n \geq 3.$$

2. Complex Factorizations of the First Lucas Sequence

First of all, we introduce the tridiagonal matrix sequence $\{A(n), n = 1, 2, \ldots\}$ and express the first Lucas sequence $U_n(P, Q)$ by the determinant of $A(n-1)$ for $n \geq 2$. Then we use this connection to prove the explicit formula for $U_n(P, Q)$ which is a generalization of the *Binet Form* for $U_n(P, -1)$.

Theorem 2. *Let $\{A(n), n = 1, 2, \ldots\}$ be a sequence of tridiagonal matrices of the form*

$$A(n) = \begin{pmatrix} P & \alpha & & & & \\ \beta & P & \alpha & & & \\ & \beta & P & \ddots & & \\ & & \ddots & \ddots & \alpha & \\ & & & \beta & P \end{pmatrix} \quad (8)$$

with $\alpha\beta = Q$, where P and Q are nonzero integers satisfying $P^2 - 4Q \neq 0$. Then

$$U_n(P, Q) = |A(n-1)| \quad \text{for } n \geq 2, \quad (9)$$

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} \quad \text{for } n \geq 0, \quad (10)$$

where $|\cdot|$ denotes the determinant a and b are roots of $x^2 - Px + Q = 0$.

Proof. According to Lemma 1, successive determinants of $A(n)$ are given by the recursive formula:

$$|A(1)| = P, \qquad |A(2)| = P^2 - Q,$$

$$|A(n)| = P|A(n-1)| - Q|A(n-2)|, \quad n \geq 3. \quad (11)$$

Clearly, this is also the first Lucas sequence, starting with $U_2(P, Q)$. Therefore,

$$U_n(P, Q) = |A(n-1)| \quad \text{for } n \geq 2. \quad (12)$$

Since a and b are roots of $x^2 - Px + Q = 0$,

$$a + b = P, \qquad ab = Q. \quad (13)$$

Note that $U_0(P, Q) = 0 = (a^0 - b^0)/(a - b)$ and $U_1(P, Q) = 1 = (a^1 - b^1)/(a - b)$. In order to prove (10), it is sufficient to prove

$$|A(n-1)| = \frac{a^n - b^n}{a - b} \quad \text{for } n \geq 2 \quad (14)$$

by (9). Now we prove (14) by induction on n. When $n = 2, 3$, we have

$$|A(1)| = P = a + b = \frac{a^2 - b^2}{a - b},$$

$$|A(2)| = P^2 - Q = (a + b)^2 - ab = a^2 + ab + b^2 = \frac{a^3 - b^3}{a - b}. \quad (15)$$

Next we assume (14) holds for $3 \leq n \leq k$. Then when $n = k+1$, we have

$$|A(k)| = P|A(k-1)| - Q|A(k-2)|$$

$$= (a + b)|A(k-1)| - ab|A(k-2)|$$

$$= (a + b)\frac{a^k - b^k}{a - b} - ab\frac{a^{k-1} - b^{k-1}}{a - b} \quad (16)$$

$$= \frac{a^{k+1} - ab^k + a^k b - b^{k+1} - a^k b + ab^k}{a - b}$$

$$= \frac{a^{k+1} - b^{k+1}}{a - b}.$$

Hence, (14) is proved.   □

Now we give two complex factorizations and one explicit formula of the first Lucas sequence by using the determinants of tridiagonal matrices and Chebyshev polynomials of the second kind.

Theorem 3. *The first Lucas sequence $\{U_n(P, Q)\}_{n \geq 0}$ satisfies*

$$U_n(P, Q) = \prod_{k=1}^{n-1}\left(P + 2\sqrt{Q}\cos\frac{k\pi}{n}\right), \quad n \geq 2, \quad (17)$$

$$U_n(P, Q) = \prod_{k=1}^{n-1}\left(P - 2\sqrt{Q}\cos\frac{k\pi}{n}\right), \quad n \geq 2, \quad (18)$$

$$U_n(P, Q) = \left(\sqrt{Q}\right)^{n-1}\frac{\sin\left(n\arccos\left(P/2\sqrt{Q}\right)\right)}{\sin\left(\arccos\left(P/2\sqrt{Q}\right)\right)}, \quad n \geq 1, \quad (19)$$

where $\sqrt{Q} \equiv \sqrt{-Q}\,i$ and $i = \sqrt{-1}$ if Q is a negative integer.

Proof. Choosing $\alpha = \beta = \sqrt{Q}$ in $\{A(n), n = 1, 2, \ldots\}$, where $\sqrt{Q}$ represents $\sqrt{-Q}\mathrm{i}$ when Q is negative, we obtain a sequence of tridiagonal matrices $\{B(n), n = 1, 2, \ldots\}$ of the form

$$B(n) = \begin{pmatrix} P & \sqrt{Q} & & & & \\ \sqrt{Q} & P & \sqrt{Q} & & & \\ & \sqrt{Q} & P & \ddots & & \\ & & \ddots & \ddots & \sqrt{Q} & \\ & & & \sqrt{Q} & P \end{pmatrix}. \qquad (20)$$

Then

$$U_{n+1}(P, Q) = |B(n)| \quad \text{for } n \geq 1 \qquad (21)$$

by Theorem 2. Considering that the determinant of a matrix can be found by taking the product of its eigenvalues, we will compute the spectrum of $B(n)$ in order to find an alternate formulation for $|B(n)|$.

We now introduce another sequence of tridiagonal matrices $\{G(n), n = 1, 2, \ldots\}$, where $G(n)$ is the $n \times n$ tridiagonal matrix with $g_{k,k} = 0$ ($1 \leq k \leq n$) and $g_{k-1,k} = g_{k,k-1} = 1$ ($2 \leq k \leq n$). That is,

$$G(n) = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 \end{pmatrix}. \qquad (22)$$

Note that $B(n) = PI_n + \sqrt{Q}G(n)$, where I_n is the $n \times n$ identity matrix. Let λ_k, $k = 1, 2, \ldots, n$, be the eigenvalues of $G(n)$ with associated eigenvectors x_k. Then for each $1 \leq j \leq n$, we have

$$\begin{aligned} B(n) x_j &= \left(PI_n + \sqrt{Q}G(n) \right) x_j \\ &= PI_n x_j + \sqrt{Q}G(n) x_j \\ &= Px_j + \sqrt{Q}\lambda_j x_j \\ &= \left(P + \sqrt{Q}\lambda_j \right) x_j. \end{aligned} \qquad (23)$$

Therefore, $P + \sqrt{Q}\lambda_k$, $k = 1, 2, \ldots, n$ are the eigenvalues of $B(n)$. Hence,

$$|B(n)| = \prod_{k=1}^{n} \left(P + \sqrt{Q}\lambda_k \right), \quad n \geq 1. \qquad (24)$$

Next we compute λ_k's by recalling that each λ_k is a zero of the characteristic polynomial $p_n(\lambda) = |\lambda I_n - G(n)|$. Notice that

$$\lambda I_n - G(n) = \begin{pmatrix} \lambda & -1 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & \lambda & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & \lambda \end{pmatrix}; \qquad (25)$$

we use Lemma 1 to obtain a recursive formula for the characteristic polynomials of $\{G(n), n = 1, 2, \ldots\}$:

$$\begin{aligned} p_1(\lambda) &= \lambda, \qquad p_2(\lambda) = \lambda^2 - 1, \\ p_n(\lambda) &= \lambda p_{n-1}(\lambda) - p_{n-2}(\lambda), \quad n \geq 3. \end{aligned} \qquad (26)$$

This family of characteristic polynomials can be transformed into another family $\{U_n(x), n = 1, 2, \ldots\}$ by the transformation $\lambda \equiv 2x$ as follows:

$$\begin{aligned} U_1(x) &= 2x, \qquad U_2(x) = 4x^2 - 1, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x), \quad n \geq 3, \end{aligned} \qquad (27)$$

and this family $\{U_n(x), n = 1, 2, \ldots\}$ is the set of Chebyshev polynomials of the second kind. It is well known [6] that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as follows:

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \qquad (28)$$

From (28), we can easily see that the roots of $U_n(x) = 0$ are $\theta_k = k\pi/(n+1)$, $k = 1, 2, \ldots, n$, or equally, $x_k = \cos\theta_k = \cos(k\pi/(n+1))$, $k = 1, 2, \ldots, n$. So we get the eigenvalues of $G(n)$ as follows:

$$\lambda_k = 2\cos\frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n. \qquad (29)$$

Combining (21), (24), and (29), we finally have

$$U_{n+1}(P, Q) = \prod_{k=1}^{n} \left(P + 2\sqrt{Q}\cos\frac{k\pi}{n+1} \right), \quad n \geq 1, \qquad (30)$$

which is equivalent to (17). Moreover, by taking $k = n + 1 - t$, we get

$$\begin{aligned} U_{n+1}(P, Q) &= \prod_{t=1}^{n} \left(P + 2\sqrt{Q}\cos\left(\pi - \frac{t\pi}{n+1}\right) \right) \\ &= \prod_{t=1}^{n} \left(P - 2\sqrt{Q}\cos\frac{t\pi}{n+1} \right), \quad n \geq 1, \end{aligned} \qquad (31)$$

which is identical to (18).

From (25), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$
H(n, x) = \begin{pmatrix} 2x & -1 & & & & \\ -1 & 2x & -1 & & & \\ & -1 & 2x & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2x \end{pmatrix}, \qquad (32)
$$

where $H(n, x)$ is $n \times n$. Note that $B(n) = -\sqrt{Q}H(n, -P/2\sqrt{Q})$. So we obtain

$$
|B(n)| = \left(-\sqrt{Q}\right)^n \left| H\left(n, -\frac{P}{2\sqrt{Q}}\right) \right| = \left(-\sqrt{Q}\right)^n U_n\left(-\frac{P}{2\sqrt{Q}}\right). \tag{33}
$$

Since the Chebyshev polynomial of the second kind $U_n(x)$ is an even or odd function, involving only even or odd powers of x, as n is even or odd, the right side of (33) can be represented as

$$
\left(-\sqrt{Q}\right)^n U_n\left(-\frac{P}{2\sqrt{Q}}\right) = \left(\sqrt{Q}\right)^n U_n\left(\frac{P}{2\sqrt{Q}}\right). \tag{34}
$$

Combining (21), (33), (34), and (28), we derive

$$
U_{n+1}(P, Q) = \left(\sqrt{Q}\right)^n \frac{\sin\left((n+1)\arccos\left(P/2\sqrt{Q}\right)\right)}{\sin\left(\arccos\left(P/2\sqrt{Q}\right)\right)}, \quad n \geq 1, \tag{35}
$$

which is identical to (19) since it also holds for $n = 0$:

$$
U_1(P, Q) = 1 = \left(\sqrt{Q}\right)^0 \frac{\sin\left(\arccos\left(P/2\sqrt{Q}\right)\right)}{\sin\left(\arccos\left(P/2\sqrt{Q}\right)\right)}. \tag{36}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4. For $P = 1$, $Q = -1$, (18) and (19) are exactly the formulas for Fibonacci numbers in [2].

3. Complex Factorizations of the Second Lucas Sequence

The complex factorization of $\{V_n(P, -1)\}_{n \geq 0}$ was derived in [4], and we generalize the formula to $\{V_n(P, Q)\}_{n \geq 0}$ for any integer Q in the following theorem.

Theorem 5. *The second Lucas sequence* $\{V_n(P, Q)\}_{n \geq 0}$ *satisfies*

$$
V_n(P, Q) = \prod_{k=1}^{n} \left(P + 2\sqrt{Q} \cos \frac{(k - (1/2))\pi}{n} \right), \quad n \geq 1, \tag{37}
$$

$$
V_n(P, Q) = \prod_{k=1}^{n} \left(P - 2\sqrt{Q} \cos \frac{(k - (1/2))\pi}{n} \right), \quad n \geq 1, \tag{38}
$$

$$
V_n(P, Q) = 2\left(\sqrt{Q}\right)^n \cos\left(n \arccos\left(\frac{P}{2\sqrt{Q}}\right) \right), \quad n \geq 1, \tag{39}
$$

where $\sqrt{Q} \equiv \sqrt{-Q}\,i$ *and* $i = \sqrt{-1}$ *if* Q *is a negative integer.*

Proof. In order to obtain (37), we introduce a sequence of tridiagonal matrices $\{F(n), n = 1, 2, \ldots\}$ of the form

$$
F(n) = \begin{pmatrix} \dfrac{P}{2} & \sqrt{Q} & & & & \\ \sqrt{Q} & P & \sqrt{Q} & & & \\ & \sqrt{Q} & P & \ddots & & \\ & & \ddots & \ddots & \sqrt{Q} & \\ & & & \sqrt{Q} & P \end{pmatrix}, \qquad (40)
$$

where $\sqrt{Q}$ represents $\sqrt{-Q}i$ when Q is negative. Then

$$
V_n(P, Q) = 2 |F(n)| \tag{41}
$$

by Theorem 1 of [4]. Unlike the derivation in Theorem 3, we will not compute the spectrum of $F(n)$ directly. Instead, since $|I_n + e_1 e_1^T| = 2$, we have

$$
|F(n)| = \frac{1}{2} \left| \left(I_n + e_1 e_1^T\right) F(n) \right|, \tag{42}
$$

where I_n is the $n \times n$ identity matrix and e_j is the jth column of I_n. Moreover, we can express the right-hand side of (42) as follows:

$$
\frac{1}{2} \left| \left(I_n + e_1 e_1^T\right) F(n) \right| = \frac{1}{2} \left| P I_n + \sqrt{Q}\left(G(n) + e_1 e_2^T\right) \right|, \tag{43}
$$

where $G(n)$ is the $n \times n$ matrix defined in Theorem 3. Let μ_k, $k = 1, 2, \ldots, n$ be the eigenvalues of $G(n) + e_1 e_2^T$ with associated eigenvectors y_k. Then for each $1 \leq j \leq n$, we get

$$
\begin{aligned}
&\left(P I_n + \sqrt{Q}\left(G(n) + e_1 e_2^T\right) \right) y_j \\
&= P I_n y_j + \sqrt{Q}\left(G(n) + e_1 e_2^T\right) y_j \\
&= P y_j + \sqrt{Q}\mu_j y_j \\
&= \left(P + \sqrt{Q}\mu_j \right) y_j.
\end{aligned} \tag{44}
$$

Therefore, $P + \sqrt{Q}\mu_k$, $k = 1, 2, \ldots, n$, are the eigenvalues of $P I_n + \sqrt{Q}(G(n) + e_1 e_2^T)$. Hence,

$$
\left| \left(I_n + e_1 e_1^T\right) F(n) \right| = \prod_{k=1}^{n} \left(P + \sqrt{Q}\mu_k \right), \quad n \geq 1. \tag{45}
$$

Next we compute μ_k's by recalling that each μ_k is a zero of the characteristic polynomial $q_n(\mu) = |\mu I_n - (G(n) + e_1 e_2^T)|$. Since $|(I_n - (1/2)e_1 e_1^T)| = 1/2$, we can represent the characteristic polynomial as follows:

$$q_n(\mu) = 2\left|\left(I_n - \frac{1}{2}e_1 e_1^T\right)\left(\mu I_n - \left(G(n) + e_1 e_2^T\right)\right)\right|. \quad (46)$$

Notice that

$$\left(I_n - \frac{1}{2}e_1 e_1^T\right)\left(\mu I_n - \left(G(n) + e_1 e_2^T\right)\right)$$

$$= \begin{pmatrix} \frac{\mu}{2} & -1 & & & \\ -1 & \mu & -1 & & \\ & -1 & \mu & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \mu \end{pmatrix}. \quad (47)$$

We use Lemma 1 to obtain a recursive formula for the characteristic polynomials:

$$q_1(\mu) = \frac{\mu}{2}, \qquad q_2(\mu) = \frac{\mu^2}{2} - 1,$$
$$q_n(\mu) = \mu q_{n-1}(\mu) - q_{n-2}(\mu), \quad n \geq 3. \quad (48)$$

This family of characteristic polynomials can be transformed into another family $\{T_n(x), n = 1, 2, \ldots\}$ by taking $\mu = 2x$:

$$T_1(x) = x, \qquad T_2(x) = 2x^2 - 1,$$
$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), \quad n \geq 3. \quad (49)$$

In fact, this family $\{T_n(x), n = 1, 2, \ldots\}$ is the set of Chebyshev polynomials of the first kind. It is well known [6] that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as

$$T_n(x) = \cos n\theta. \quad (50)$$

It is easy to see that the roots of $T_n(x) = 0$ are given by $\theta_k = (k - (1/2))\pi/n$, $k = 1, 2, \ldots, n$, or $x_k = \cos\theta_k = \cos(k - (1/2))\pi/n$, $k = 1, 2, \ldots, n$. Thus, we derive

$$\mu_k = 2\cos\frac{(k - (1/2))\pi}{n}, \quad k = 1, 2, \ldots, n. \quad (51)$$

Combining (41), (42), (45), and (51), we finally have

$$V_n(P, Q) = \prod_{k=1}^{n}\left(P + 2\sqrt{Q}\cos\frac{(k - (1/2))\pi}{n}\right), \quad n \geq 1. \quad (52)$$

Furthermore, by taking $k = n + 1 - t$, we obtain

$$V_n(P, Q) = \prod_{t=1}^{n}\left(P + 2\sqrt{Q}\cos\left(\pi - \frac{(t - (1/2))\pi}{n}\right)\right)$$
$$= \prod_{t=1}^{n}\left(P - 2\sqrt{Q}\cos\frac{(t - (1/2))\pi}{n}\right), \quad n \geq 1. \quad (53)$$

Equations (37) and (38) are proved.

Now we think of Chebyshev polynomials of the first kind from (47) as being generated by determinants of successive matrices of the form

$$K(n, x) = \begin{pmatrix} x & -1 & & & & \\ -1 & 2x & -1 & & & \\ & -1 & 2x & \ddots & & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2x \end{pmatrix}, \quad (54)$$

where $K(n, x)$ is $n \times n$. Note that $F(n) = -\sqrt{Q}K(n, -P/2\sqrt{Q})$. So we get

$$|F(n)| = \left(-\sqrt{Q}\right)^n\left|K\left(n, -\frac{P}{2\sqrt{Q}}\right)\right| = \left(-\sqrt{Q}\right)^n T_n\left(-\frac{P}{2\sqrt{Q}}\right). \quad (55)$$

Since the Chebyshev polynomial of the first kind $T_n(x)$ is an even or odd function, involving only even or odd powers of x, according as n is even or odd, the right side of (55) can be represented as

$$\left(-\sqrt{Q}\right)^n T_n\left(-\frac{P}{2\sqrt{Q}}\right) = \left(\sqrt{Q}\right)^n T_n\left(\frac{P}{2\sqrt{Q}}\right). \quad (56)$$

Combining (41), (55), (56), and (50), we derive

$$V_n(P, Q) = 2\left(\sqrt{Q}\right)^n\cos\left(n\arccos\left(\frac{P}{2\sqrt{Q}}\right)\right), \quad n \geq 1. \quad (57)$$

Equation (39) is completed. $\square$

Remark 6. For $P = 1, Q = -1$, (38) and (39) are exactly the formulas for Lucas numbers in [2].

Remark 7. For $Q = -1$, (38) and (39) are exactly the formulas for $\{V_n(P, -1)\}$ in [4].

4. Examples

Finally, we give some examples.

Example 1. Choosing $P = 2, Q = -1$, we can get the complex factorization of Pell numbers by (17). Let P_n be the nth Pell number and $i = \sqrt{-1}$. Then the fourth Pell number is

$$P_4 = U_4(2, -1) = \prod_{k=1}^{3}\left(2 + 2i\cos\frac{k\pi}{4}\right)$$
$$= \left(2 + 2i\cos\frac{\pi}{4}\right)\left(2 + 2i\cos\frac{2\pi}{4}\right)$$
$$\times \left(2 + 2i\cos\frac{3\pi}{4}\right) \quad (58)$$
$$= \left(2 + \sqrt{2}i\right)(2 + 0)\left(2 - \sqrt{2}i\right)$$
$$= 12.$$

Example 2. Choosing $P = 1, Q = -2$, we can get the complex factorization of Jacobsthal numbers by (18). Let J_n be the nth Jacobsthal number and $i = \sqrt{-1}$. Then the third Jacobsthal number is

$$
\begin{aligned}
J_3 = U_3\,(1, -2) &= \prod_{k=1}^{2} \left(1 - 2\sqrt{2}i \cos \frac{k\pi}{3} \right) \\
&= \left(1 - 2\sqrt{2}i \cos \frac{\pi}{3} \right)\left(1 - 2\sqrt{2}i \cos \frac{2\pi}{3} \right) \\
&= \left(1 - \sqrt{2}i \right)\left(1 + \sqrt{2}i \right) \\
&= 3.
\end{aligned}
\tag{59}
$$

Example 3. Choosing $P = 2, Q = -1$, we can get the complex factorization of Pell-Lucas numbers by (37). Let p_n be the nth Pell-Lucas number and $i = \sqrt{-1}$. Then the third Pell-Lucas number is

$$
\begin{aligned}
p_3 = V_3\,(2, -1) &= \prod_{k=1}^{3} \left(2 + 2i \cos \frac{(k - (1/2))\,\pi}{3} \right) \\
&= \left(2 + 2i \cos \frac{\pi}{6} \right)\left(2 + 2i \cos \frac{3\pi}{6} \right)\left(2 + 2i \cos \frac{5\pi}{6} \right) \\
&= \left(2 + \sqrt{3}i \right)(2 + 0)\left(2 - \sqrt{3}i \right) \\
&= 14.
\end{aligned}
\tag{60}
$$

Example 4. Choosing $P = 1, Q = -2$, we can get the complex factorization of Jacobsthal-Lucas numbers by (38). Let j_n be the nth Jacobsthal-Lucas number and $i = \sqrt{-1}$. Then the third Jacobsthal-Lucas number is

$$
\begin{aligned}
j_3 = V_3\,(1, -2) &= \prod_{k=1}^{3} \left(1 - 2\sqrt{2}i \cos \frac{(k - (1/2))\,\pi}{3} \right) \\
&= \left(1 - 2\sqrt{2}i \cos \frac{\pi}{6} \right)\left(1 - 2\sqrt{2}i \cos \frac{3\pi}{6} \right) \\
&\quad \times \left(1 - 2\sqrt{2}i \cos \frac{5\pi}{6} \right) \\
&= \left(1 - \sqrt{6}i \right)(1 - 0)\left(1 + \sqrt{6}i \right) \\
&= 7.
\end{aligned}
\tag{61}
$$

5. Conclusion

We use the tridiagonal matrices $B(n)$ and $F(n)$ to obtain the complex factorizations of the first and second Lucas sequences, respectively. It is possible to find other suitable matrices for different factorizations of Lucas sequences.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this article.

Acknowledgments

References

[1] P. Ribenboim, *My Numbers, My Friends*, Popular Lectures on Number Theory, Springer, Berlin, Germany, 2000.

[2] N. D. Cahill, J. R. D'Errico, and J. P. Spence, "Complex factorizations of the Fibonacci and Lucas numbers," *The Fibonacci Quarterly*, vol. 41, no. 1, pp. 13–19, 2003.

[3] N. D. Cahill and D. A. Narayan, "Fibonacci and Lucas numbers as tridiagonal matrix determinants," *The Fibonacci Quarterly*, vol. 42, no. 3, pp. 216–221, 2004.

[4] Ş. Burcu Bozkurt, F. Yılmaz, and D. Bozkurt, "On the complex factorization of the Lucas sequence," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1317–1321, 2011.

[5] J. Feng, "Fibonacci identities via the determinant of tridiagonal matrix," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5978–5981, 2011.

[6] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, CRC Press, 2003.