Research Article

Fejér and Hermite-Hadamard Type Inequalities for Harmonically Convex Functions

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We establish a Fejér type inequality for harmonically convex functions. Our results are the generalizations of some known results. Moreover, some properties of the mappings in connection with Hermite-Hadamard and Fejér type inequalities for harmonically convex functions are also considered.

1. Introduction

Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$; then

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}.$$  

(1)

Inequality (1) is known in the literature as the Hermite-Hadamard inequality. Fejér [1] established the following weighted generalization of inequality (1).

Theorem 1. If $f: [a, b] \rightarrow \mathbb{R}$ is a convex function, then the following inequality holds:

$$f\left(\frac{a + b}{2}\right) \int_a^b p(x) \, dx \leq \int_a^b f(x) p(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b p(x) \, dx,$$  

(2)

where $p: [a, b] \rightarrow \mathbb{R}$ is positive, integrable, and symmetric with respect to $x = (a + b)/2$.

Some generalizations, refinements, variations, and improvements of inequalities (1) and (2) were investigated by Wu [2], Chen and Liu [3], Sarikaya and Ogunmez [4], and Xiao et al. [5], respectively.

In [6], Dragomir proposed an interesting Hermite-Hadamard type inequality which refines the left hand side of inequality (1) as follows.

Theorem 2 (see [6]). Let $f$ be a convex function defined on $[a, b]$. Then $H$ is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, one has

$$f\left(\frac{a + b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b - a} \int_a^b f(x) \, dx,$$  

(3)

where

$$H(t) = \frac{1}{b - a} \int_a^b f\left(tx + (1 - t)\frac{a + b}{2}\right) \, dx.$$  

(4)

An analogous result for convex functions which refines the right hand side of inequality (1) was obtained by Yang and Hong in [7] as follows.

Theorem 3 (see [7]). Let $f$ be a convex function defined on $[a, b]$. Then $F$ is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, one has

$$\frac{1}{b - a} \int_a^b f(x) \, dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2},$$  

(5)
where
\[
F(t) = \frac{1}{2(b-a)} \int_a^b \left[ f\left(\frac{1+t}{2}\right) a + f\left(\frac{1-t}{2}\right) x \right] \, dx.
\]

(6)

Yang and Tseng in [8] established the following Fejér type inequalities, which is the generalization of inequalities (3) and (5) as well as the refinement of the Fejér inequality (2).

**Theorem 4** (see [8]). If \( f \) is convex on \([a, b]\), \( p : [a, b] \rightarrow \mathbb{R} \) is positive, integrable, and symmetric about \( x = (a+b)/2 \). Then \( P \) and \( Q \) are convex, increasing on \([0, 1]\), and for all \( t \in [0, 1] \), one has

\[
f\left(\frac{a+b}{2}\right) \int_a^b p(x) \, dx = P(0) \leq P(t) \leq P(1) = \int_a^b f(x) \, dx,
\]

\[
Q(0) \leq Q(t) \leq Q(1) = f(a) + f(b)
\]

(7)

where

\[
P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) \, dx,
\]

\[
Q(t) = \frac{1}{2} \int_a^b \left[ f\left(\frac{1+t}{2}\right) a + f\left(\frac{1-t}{2}\right) x \right] p\left(\frac{x+a}{2}\right) \, dx + f\left(\frac{1+t}{2}\right) b + f\left(\frac{1-t}{2}\right) x \right] p\left(\frac{x+b}{2}\right) \, dx.
\]

(8)

In [9, 10], İşcan and Wu gave the definition of harmonic convexity as follows.

**Definition 5.** Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically convex if

\[
f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x), \quad (10)
\]

for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (10) is reversed, then \( f \) is said to be harmonically concave.

The following Hermite-Hadamard inequality for harmonically convex functions holds true.

**Theorem 6** (see [9]). Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a,b) \), then one has

\[
f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{p(x)}{x^2} \, dx \leq \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

(11)

In [10], İşcan and Wu established the following Hermite-Hadamard inequalities for harmonically convex functions via the Riemann-Liouville fractional integral.

**Theorem 7** (see [10]). Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a function such that \( f \in L(a,b) \), where \( a, b \in I \) with \( a < b \). If \( f \) is a harmonically convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{2ab}{a+b}\right) \frac{\Gamma(\alpha+1)}{2} \left\{ f_1^a (f \circ g) \left(\frac{1}{b}\right) + f_1^b (f \circ g) \left(\frac{1}{a}\right) \right\} \leq \frac{f(a) + f(b)}{2},
\]

(12)

where \( \alpha > 0 \) and \( g(x) = \frac{1}{x} \).

The following Hermite-Hadamard inequality for harmonically convex functions holds true.

**Theorem 8.** Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a,b) \), then one has

\[
f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{p(x)}{x^2} \, dx \leq \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

(14)

where \( p : [a, b] \rightarrow \mathbb{R} \) is nonnegative and integrable and satisfies

\[
p\left(\frac{ab}{x} \right) = p\left(\frac{ab}{a+b-x} \right).
\]

(15)
Proof. Since \( f \) is a harmonically convex function on \([a, b]\), we have, for all \( x, y \in [a, b] \),
\[
f\left(\frac{2xy}{x+y}\right) \leq \frac{f(y) + f(x)}{2}. \tag{16}
\]
Choosing \( x = ab/(tb + (1-t)a) \) and \( y = ab/(ta + (1-t)b) \), we have
\[
f\left(\frac{2ab}{a+b}\right) \leq \frac{f(ab/(tb + (1-t)a)) + f(ab/(ta + (1-t)b))}{2} \tag{17}
\]
\[
\leq \frac{f(a) + f(b)}{2}.
\]
Since \( p \) is nonnegative and satisfies the condition of (15), we obtain
\[
f\left(\frac{2ab}{a+b}\right) p\left(\frac{ab}{tb + (1-t)a}\right)
\leq \left(\frac{f(ab/(tb + (1-t)a)) p\left(\frac{ab}{tb + (1-t)a}\right)}{2} + f\left(\frac{ab}{ta + (1-t)b}\right) p\left(\frac{ab}{ta + (1-t)b}\right)\right) \times 2^{-1}
\leq \frac{f(a) + f(b)}{2} p\left(\frac{ab}{tb + (1-t)a}\right). \tag{18}
\]
Integrating both sides of the above inequalities with respect to \( t \) over \([0, 1]\), we obtain
\[
f\left(\frac{2ab}{a+b}\right) \int_0^1 p\left(\frac{ab}{tb + (1-t)a}\right) dt
\leq \int_0^1 \left(\frac{f(ab/(tb + (1-t)a)) p\left(\frac{ab}{tb + (1-t)a}\right)}{2} + f\left(\frac{ab}{ta + (1-t)b}\right) p\left(\frac{ab}{ta + (1-t)b}\right)\right) dt \times 2^{-1}
\leq \frac{f(a) + f(b)}{2} \int_0^1 p\left(\frac{ab}{tb + (1-t)a}\right) dt. \tag{19}
\]
The proof of Theorem 8 is completed. \( \square \)

Remark 9. Putting \( p(x) = 1 \) in Theorem 8, we obtain inequality (11).

Remark 10. Choosing
\[
p(x) = \frac{\alpha}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{\left(\frac{1}{x} - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - \frac{1}{x}\right)^{\alpha-1}\right\}, \tag{20}
\]
\[
(\alpha > 0, 0 < a < b),
\]
in Theorem 8, it is easy to observe that \( p(ab/x) = p(ab/(a + b - x)) \).
where \( g(x) = 1/x \), which implies that inequality (14) can be transformed to inequality (12) under an appropriate selection of \( p(x) \).

Remark 11. In Theorem 8, taking \( p(ab/x) = \omega(x) \), where \( 0 < a < b \), \( \omega(x) \) is nonnegative, integrable, and symmetric with respect to \( x = (a + b)/2 \). Then inequality (14) becomes

\[
\int_a^b f \left( \frac{2ab}{a + b + tx} \right) \omega(x) dx \\
\leq \frac{f(a) + f(b)}{2} \int_a^b \omega(x) dx.
\]

### 3. Some Mappings in connection with Hermite-Hadamard and Fejér Inequalities for Harmonically Convex Functions

**Lemma 12.** Let \( f: I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \), and let

\[
h(t) = \frac{1}{2} f \left( \frac{2ab}{a + b - t} \right) + \frac{1}{2} f \left( \frac{2ab}{a + b + t} \right),
\]

t \in [0, b - a]. Then \( h \) is convex, increasing on \([0, b - a]\), and for all \( t \in [0, b - a] \),

\[
f \left( \frac{2ab}{a + b} \right) \leq h(t) \leq \frac{f(a) + f(b)}{2}.
\]

**Proof.** Firstly, for \( x, y \in [0, b - a] \), we have

\[
h(tx + (1-t)y) = \frac{1}{2} f \left( \frac{2ab}{a + b - (tx + (1-t)y)} \right) + \frac{1}{2} f \left( \frac{2ab}{a + b + (tx + (1-t)y)} \right)
\]

\[
= \frac{1}{2} f \left( \frac{2ab}{t(a + b - x) + (1-t)(a + b - y)} \right) + \frac{1}{2} f \left( \frac{2ab}{t(a + b + x) + (1-t)(a + b + y)} \right)
\]

\[
\leq \frac{t}{2} f \left( \frac{2ab}{a + b - x} \right) + \frac{1-t}{2} f \left( \frac{2ab}{a + b - y} \right) + \frac{t}{2} f \left( \frac{2ab}{a + b + x} \right) + \frac{1-t}{2} f \left( \frac{2ab}{a + b + y} \right)
\]

\[
= th(x) + (1-t)h(y),
\]

and hence \( h \) is convex on \([0, b - a]\).

Next, if \( t \in [0, b - a] \), it follows from the harmonic convexity of \( f \) that

\[
h(t) = \frac{1}{2} f \left( \frac{2ab}{a + b - t} \right) + \frac{1}{2} f \left( \frac{2ab}{a + b + t} \right)
\]

\[
\geq f \left( \frac{2ab}{(1/2)(a + b - t) + (1/2)(a + b + t)} \right)
\]

\[
= f \left( \frac{2ab}{a + b} \right).
\]

It is easy to observe that

\[
h(t) = \frac{1}{2} f \left( \frac{2ab}{a + b - t} \right) + \frac{1}{2} f \left( \frac{2ab}{a + b + t} \right)
\]

\[
\leq \frac{1}{2} f \left( \frac{2ab}{b-a} \right) + \frac{1}{2} f \left( \frac{2ab}{b-a} \right)
\]

\[
= \frac{f(a) + f(b)}{2}.
\]

Thus inequality (24) holds.

Finally, for \( 0 < t_1 < t_2 \leq b - a \), since \( h \) is convex, it follows from (24) that

\[
\frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq \frac{h(t_1) - h(0)}{t_1 - 0}
\]

\[
= \frac{h(t_1) - f(2ab/(a + b))}{t_1} \geq 0,
\]

and hence, \( h(t_2) \geq h(t_1) \), which means that \( h \) is increasing on \([0, b - a]\). This completes the proof of Lemma 12.

**Theorem 13.** Let \( f: I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) and \( H \) is defined by

\[
H(t) = \frac{1}{2} f \left( \frac{2ab}{a + b - tx} \right) dx
\]

\[
+ \frac{1}{2} f \left( \frac{2ab}{a + b + tx} \right) dx
\]

\[
= \frac{1}{b-a} \int_a^b f \left( \frac{ab}{(1-t)(a + b)/2 + tx} \right) dx,
\]

then \( H \) is convex and increasing on \([0, 1]\), and

\[
f \left( \frac{2ab}{a + b} \right) = H(0) \leq H(t) \leq H(1)
\]

\[
= \frac{ab}{b-a} \int_a^b f \left( \frac{x}{x^2} \right) dx.
\]
Proof. It follows from Lemma 12 that
\[ h(t) = \frac{1}{2} f\left( \frac{2ab}{a + b - t} \right) + \frac{1}{2} f\left( \frac{2ab}{a + b + t} \right) \quad (31) \]
is convex and increasing on \([0, b - a]\). Hence \(H(t)\) is convex and increasing on \([0, 1]\). Further, inequality (30) can be deduced from (24). Theorem 13 is proved.

**Theorem 14.** Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) and \( G \) is defined by
\[ G(t) = \frac{1}{2a(b - a)} \int_0^{b-a} f\left( \frac{2ab}{2a + (1-t) x} \right) dx + \frac{1}{2a(b - a)} \int_0^{b-a} f\left( \frac{2ab}{2b - (1-t) x} \right) dx \]
\[ = \frac{1}{2a(b - a)} \int_a^b f\left( \frac{2ab}{(1+t) a + (1-t) x} \right) dx + \frac{1}{2a(b - a)} \int_a^b f\left( \frac{2ab}{(1+t) b + (1-t) x} \right) dx, \quad (32) \]
then \( G \) is convex and increasing on \([0, 1]\), and
\[ \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx = G'(0) \leq G'(t) \leq G'(1) = \frac{f(a) + f(b)}{2}. \quad (33) \]

Proof. We note that if \( f \) is convex and \( g \) is linear, then the composition \( f \circ g \) is convex. It follows from Lemma 12 that
\[ h(t) = \frac{1}{2} f\left( \frac{2ab}{a + b - t} \right) + \frac{1}{2} f\left( \frac{2ab}{a + b + t} \right), \quad (34) \]
and \( k(t) = b - a - (1-t)x \) are increasing on \([0, b - a]\) and \([0, 1]\), respectively. Hence,
\[ h(k(t)) = f\left( \frac{2ab}{2a + (1-t) x} \right) + f\left( \frac{2ab}{2b - (1-t) x} \right) \quad (35) \]
is convex and increasing on \([0, 1]\). We infer that \( G \) is convex and increasing on \([0, 1]\). Furthermore, inequality (33) follows directly from (24). The proof of Theorem 14 is completed.

**Theorem 15.** Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) and \( Q \) is defined by
\[ Q(t) = \frac{1}{2ab} \int_0^{b-a} f\left( \frac{2ab}{2a + (1-t) x} \right) p\left( \frac{2ab}{2a + x} \right) dx + \frac{1}{2ab} \int_0^{b-a} f\left( \frac{2ab}{2b - (1-t) x} \right) p\left( \frac{2ab}{2b - x} \right) dx \]
\[ = \frac{1}{2ab} \int_a^b f\left( \frac{2ab}{(1+t) a + (1-t) x} \right) p\left( \frac{2ab}{x + a} \right) dx + \frac{1}{2ab} \int_a^b f\left( \frac{2ab}{(1+t) b + (1-t) x} \right) p\left( \frac{2ab}{x + b} \right) dx, \quad (39) \]
where \( p : [a, b] \rightarrow \mathbb{R} \) is nonnegative and integrable and satisfies the condition of (15), then \( Q \) is convex and increasing on \([0, 1]\), and
\[ \int_a^b \frac{f(x)}{x^2} p(x) dx = Q(0) \leq Q(t) \leq Q(1) = \frac{f(a) + f(b)}{2} \int_a^b \frac{p(x)}{x^2} dx. \quad (40) \]

Proof. By using the same method as in the proof of Theorem 14, we obtain from Lemma 12 that
\[ h(k(t)) = f\left( \frac{2ab}{2a + (1-t) x} \right) + f\left( \frac{2ab}{2b - (1-t) x} \right), \quad (41) \]
is convex and increasing on \([0, 1]\). Since \( p(2ab/(2a + x)) \) is nonnegative and satisfies \( p(2ab/(2a + x)) = p(2ab/(2b - x)) \),
we deduce that $Q(t)$ is convex and increasing on $[0, 1]$. Inequality (40) follows from (24) and the identity
\[
\frac{1}{2ab} \int_{a}^{b} p\left(\frac{2ab}{2a+x}\right) dx = \frac{1}{2ab} \int_{0}^{b-a} p\left(\frac{2ab}{2b-x}\right) dx
\]
\[
= \frac{1}{2} \left[ \frac{1}{2ab} \int_{0}^{b-a} p\left(\frac{2ab}{2a+x}\right) dx + \frac{1}{2ab} \int_{0}^{b-a} p\left(\frac{2ab}{2b-x}\right) dx \right]
\]
\[
= \frac{1}{2} \int_{a}^{b} p\left(\frac{x}{2}\right) dx.
\]
(42)

This completes the proof of Theorem 16. □

Remark 17. If we put
\[
p(x) = \alpha \left(\frac{ab}{b-a}\right)^{\alpha} \left\{ \left(\frac{1}{x} - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - \frac{1}{x}\right)^{\alpha-1} \right\},
\]
(43)
in inequalities (37) and (40), respectively, we obtain the refined versions of inequality (12).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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