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Research Article

The Yang-Laplace Transform for Solving the IVPs with Local Fractional Derivative

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The IVPs with local fractional derivative are considered in this paper. Analytical solutions for the homogeneous and nonhomogeneous local fractional differential equations are discussed by using the Yang-Laplace transform.

1. Introduction

In recent years, the ordinary and partial differential equations have found applications in many problems in mathematical physics [1, 2]. Initial value problems (IVPs) for ordinary and partial differential equations have been developed by some authors in [3–6]. There are analytical methods and numerical methods for solving the differential equations, such as the finite element method [6], the harmonic wavelet method [7–9], the Adomian decomposition method [10–12], the homotopy analysis method [13, 14], the homotopy decomposition method [15, 16], the heat balance integral method [17, 18], the homotopy perturbation method [19], the variational iteration method [20], and other methods [21].

Recently, owing to limit of classical and fractional differential equations, the local fractional differential equations have been applied to describe nondifferentiable problems for the heat and wave in fractal media [22, 23], the structure relation in fractal elasticity [24], and Fokker-Planck equation in fractal media [25]. Some methods were utilized to solve the local fractional differential equations. For example, the local fractional variation iteration method was used to solve the heat conduction in fractal media [26, 27]. The local fractional decomposition method for solving the local fractional diffusion and heat-conduction equations was considered in [28, 29]. The local fractional series expansion method for solving the Schrödinger equation with the local fractional derivative was presented [30]. The Yang-Laplace transform structured in 2011 [22] was suggested to deal with local fractional differential equations [31, 32]. The coupling method for variational iteration method within Yang-Laplace transform for solving the heat conduction in fractal media was proposed in [33].

In this paper, our aim is to use the Yang-Laplace transform to solve IVPs with local fractional derivative. The structure of the paper is as follows. In Section 2, some definitions and properties for the Yang-Laplace transform are given. Section 3 is devoted to the solutions for the homogeneous and nonhomogeneous IVPs with local fractional derivative. Finally, conclusions are presented in Section 4.

2. Yang-Laplace Transform

In this section we show some definitions and properties for the Yang-Laplace transform.

The local fractional integral operator is defined as [22, 23, 26–33]

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)\,(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f\left(t_j\right)\left(\Delta t_j\right)^\alpha, \end{aligned} \tag{1}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_j, \ldots\}$, $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a, b]$.

As the inverse operator of (1), the local fractional derivative operator is given by [22, 23, 26–33]

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (2)$$

with $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$.

The Yang-Laplace transform is expressed by [22, 31–33]

$$\widetilde{L}_\alpha \{f(x)\} = f_s^{\widetilde{L},\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \quad 0 < \alpha \le 1, \quad (3)$$

where $f(x)$ is a local fractional continuous function.

The inverse Yang-Laplace transform reads as [22, 31–33]

$$f(x) = \widetilde{L}_\alpha^{-1}\left\{f_s^{L,\alpha}(s)\right\} = \frac{1}{(2\pi)^\alpha} \times \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{\widetilde{L},\alpha}(s) (ds)^\alpha, \quad (4)$$

where $s^\alpha = \beta^\alpha + i^\alpha \infty^\alpha$ and $\mathrm{Re}(s^\alpha) = \beta^\alpha$.

Some properties for Yang-Laplace transform are presented as follows [21, 22, 22–33]:

$$\widetilde{L}_\alpha \{af(x) + bg(x)\} = a\widetilde{L}_\alpha \{f(x)\} + b\widetilde{L}_\alpha \{g(x)\}, \quad (5)$$

$$\widetilde{L}_\alpha \left\{f^{(n\alpha)}(x)\right\} = s^{n\alpha}\widetilde{L}_\alpha \{f(x)\} - \sum_{k=1}^{n} s^{(k-1)\alpha} f^{(n-k)\alpha}(0), \quad (6)$$

$$\lim_{x \to 0} f(x) = \lim_{s \to \infty} s^\alpha F(s), \quad (7)$$

$$\lim_{x \to \infty} f(x) = \lim_{s \to 0} s^\alpha F(s), \quad (8)$$

$$\widetilde{L}_\alpha \{f(ax)\} = \frac{1}{a^\alpha} f_s^{L,\alpha}\left(\frac{s}{a}\right), \quad a > 0, \quad (9)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha} f(x)\right\} = (-1)^k \frac{d^{k\alpha} f_s^{L,\alpha}(s)}{ds^{k\alpha}}, \quad (10)$$

$$\widetilde{L}_\alpha \{f(x - c)\} = f_s^{L,\alpha}(s) E_\alpha(-c^\alpha s^\alpha), \quad (11)$$

$$\widetilde{L}_\alpha \{f(x) E_\alpha(c^\alpha x^\alpha)\} = f_s^{L,\alpha}(s - c), \quad (12)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha} E_\alpha(c^\alpha x^\alpha)\right\} = \frac{\Gamma(1 + k\alpha)}{(s - c)^{(k+1)\alpha}}, \quad (13)$$

$$\widetilde{L}_\alpha \{\sin_\alpha(c^\alpha x^\alpha)\} = \frac{c^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (14)$$

$$\widetilde{L}_\alpha \{\cos_\alpha(c^\alpha x^\alpha)\} = \frac{s^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (15)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha}\right\} = \frac{\Gamma(1 + k\alpha)}{s^{(k+1)\alpha}}. \quad (16)$$

3. IVPs with Local Fractional Derivatives

In this section we handle the homogeneous and non-homogeneous IVPs with local fractional derivative.

3.1. Homogeneous IVPs with Local Fractional Derivative

Example 1. The homogeneous IVPs with local fractional derivative are expressed by

$$\frac{d^{2\alpha} y}{d^{2\alpha} x} - \frac{d^\alpha y}{d^\alpha x} + 2y = 0. \quad (17)$$

The initial boundary conditions are presented as

$$y(0) = 1, \qquad y^{(\alpha)}(0) = 0. \quad (18)$$

From (6) we have

$$\widetilde{L}_\alpha \left\{y^{(\alpha)}(x)\right\} = s^\alpha \widetilde{L}_\alpha \{y(x)\} - y(0), \quad (19)$$

$$\widetilde{L}_\alpha \left\{y^{(2\alpha)}(x)\right\} = s^{2\alpha} \widetilde{L}_\alpha \{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0). \quad (20)$$

Hence, making use of (19) and (20), (19) can be written as

$$s^{2\alpha}\widetilde{L}_\alpha \{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0) - \left\{s^\alpha \widetilde{L}_\alpha \{y(x)\} - y(0)\right\} + 2\widetilde{L}_\alpha \{y(x)\} = 0. \quad (21)$$

Hence, we obtain

$$\widetilde{L}_\alpha \{y(x)\} = \frac{1}{s^\alpha + 2} y(0) = \frac{1}{s^\alpha + 2}. \quad (22)$$

So, making use of (13), we get the solution of (17):

$$y(x) = E_\alpha(-2x^\alpha). \quad (23)$$

The solution of (17) for $\alpha = \ln 2 / \ln 3$ is shown in Figure 1.

Example 2. Let us consider the homogeneous IVPs with local fractional derivative in the form

$$\frac{d^{4\alpha} y}{d^{4\alpha} x} - y = 0 \quad (24)$$

subject to initial boundary conditions

$$y(0) = 0, \qquad y^{(\alpha)}(0) = 0, \qquad y^{(2\alpha)}(0) = 0, \qquad y^{(3\alpha)}(0) = 1. \quad (25)$$

From (6) we have

$$\widetilde{L}_\alpha \left\{y^{(4\alpha)}(x)\right\} = s^{4\alpha}\widetilde{L}_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^\alpha y^{(2\alpha)}(0) - f^{(3\alpha)}(0), \quad (26)$$

so that

$$s^{4\alpha}\widetilde{L}_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^\alpha y^{(2\alpha)}(0) - f^{(3\alpha)}(0) - \widetilde{L}_\alpha \{y(x)\} = 0. \quad (27)$$

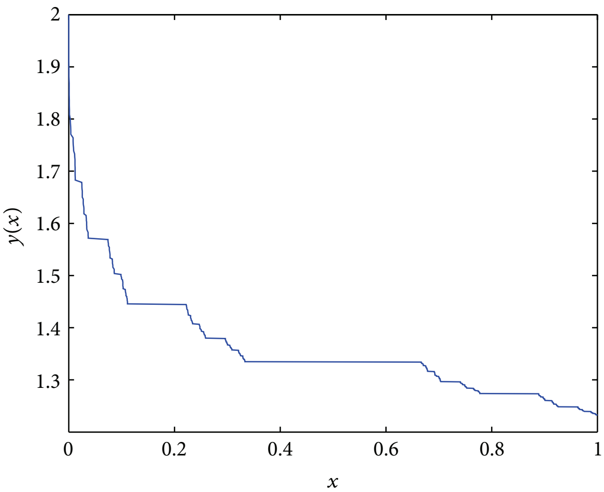

FIGURE 1: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

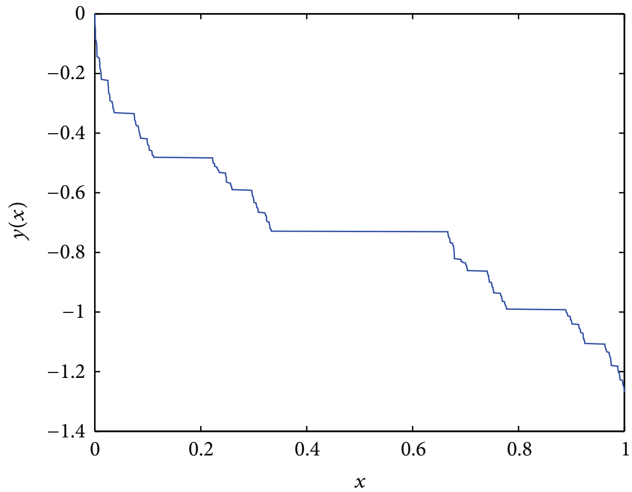

FIGURE 2: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

Hence, (27) can be written as

$$s^{4\alpha}\widetilde{L}_\alpha\{y(x)\} - 1 - \widetilde{L}_\alpha\{y(x)\} = 0, \tag{28}$$

which leads to

$$\widetilde{L}_\alpha\{y(x)\} = \frac{1}{s^{4\alpha}-1}. \tag{29}$$

Therefore, we get

$$\begin{aligned} y(x) &= \widetilde{L}_\alpha^{-1}\left\{\frac{1}{s^{4\alpha}-1}\right\} \\ &= \widetilde{L}_\alpha^{-1}\left\{\frac{1}{2}\left(\frac{1}{2}\frac{1}{s^\alpha-1} - \frac{1}{2}\frac{1}{s^\alpha+1} - \frac{1}{s^{2\alpha}+1}\right)\right\} \\ &= \frac{1}{4}E_\alpha(-x^\alpha) - \frac{1}{4}E_\alpha(x^\alpha) - \frac{1}{2}\sin_\alpha(x^\alpha). \end{aligned} \tag{30}$$

The exact solution of (24) for $\alpha = \ln 2/\ln 3$ is shown in Figure 2.

3.2. Nonhomogeneous IVPs with Local Fractional Derivative

Example 3. We now consider the non-homogeneous IVPs with local fractional derivative

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} - y = \sin_\alpha(x^\alpha) \tag{31}$$

subject to initial boundary conditions

$$y(0) = 0, \qquad y^{(\alpha)}(0) = 1. \tag{32}$$

By using (6), we have

$$\begin{aligned} \widetilde{L}_\alpha\left\{y^{(2\alpha)}(x)\right\} &= s^{2\alpha}\widetilde{L}_\alpha\{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0), \\ \widetilde{L}_\alpha\{\sin_\alpha(x^\alpha)\} &= \frac{1}{s^{2\alpha}+1} \end{aligned} \tag{33}$$

so that

$$\widetilde{L}_\alpha\{y(x)\} = \frac{3}{4}\left(\frac{1}{s^\alpha-1} - \frac{1}{s^\alpha+1}\right) - \frac{1}{2}\frac{1}{s^{2\alpha}+1}. \tag{34}$$

So,

$$y(x) = \frac{3}{4}E_\alpha(-x^\alpha) - \frac{3}{4}E_\alpha(x^\alpha) - \frac{1}{2}\sin_\alpha(x^\alpha). \tag{35}$$

The exact solution of (31) for $\alpha = \ln 2/\ln 3$ is shown in Figure 3.

Example 4. The non-homogeneous IVPs with local fractional derivative are

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} + y = E_\alpha(x^\alpha). \tag{36}$$

The initial boundary conditions are

$$y(0) = 1, \qquad y^{(\alpha)}(0) = 0. \tag{37}$$

In view of (6), we give

$$\widetilde{L}_\alpha\{y(x)\} = \frac{1}{(s^\alpha+1)(s^{2\alpha}+1)} + \frac{s^\alpha}{s^{2\alpha}+1}. \tag{38}$$

So, we obtain

$$\begin{aligned} y(x) &= \cos_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha(x-t)^\alpha \sin_\alpha(t^\alpha)(dt)^\alpha \\ &= \cos_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)} \\ &\quad \times \int_0^x E_\alpha(t^\alpha)(\sin_\alpha(x^\alpha)\cos_\alpha(t^\alpha) \\ &\qquad -\cos_\alpha(x^\alpha)\sin_\alpha(t^\alpha))(dt)^\alpha \end{aligned}$$

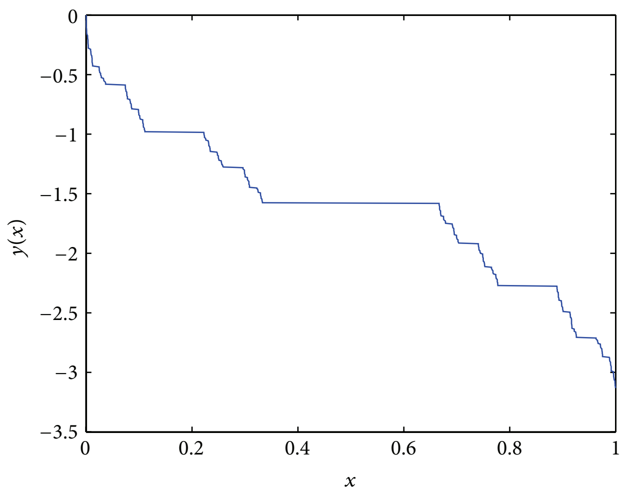

Figure 3: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

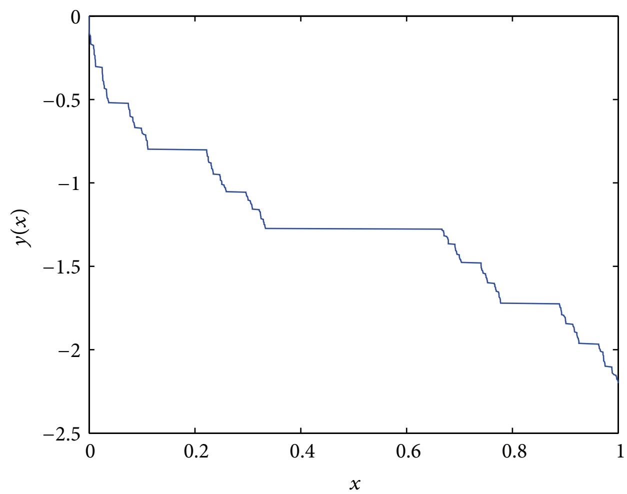

Figure 4: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

$$\begin{aligned}
&= \cos_\alpha(x^\alpha) \\
&\quad + \sin_\alpha(x^\alpha)\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha(t^\alpha)\cos_\alpha(t^\alpha)(dt)^\alpha\right\} \\
&\quad - \cos_\alpha(x^\alpha)\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha(t^\alpha)\sin_\alpha(t^\alpha)(dt)^\alpha\right\} \\
&= \cos_\alpha(x^\alpha) \\
&\quad + \frac{\sin_\alpha(x^\alpha)\{E_\alpha(x^\alpha)[\cos_\alpha(x^\alpha)+\sin(x^\alpha)]-1\}}{2} \\
&\quad - \frac{\cos_\alpha(x^\alpha)\{E_\alpha(x^\alpha)[\sin_\alpha(x^\alpha)-\cos_\alpha(x^\alpha)]+1\}}{2} \\
&= \frac{1}{2}\left[\cos_\alpha(x^\alpha)-\sin_\alpha(x^\alpha)+E_\alpha(x^\alpha)\right].
\end{aligned} \tag{39}$$

The exact solution of (36) for $\alpha = \ln 2/\ln 3$ is shown in Figure 4.

4. Conclusions

In this work we have used the Yang-Laplace transform to handle the homogeneous and non-homogeneous IVPs with looselocal fractional derivative. Some illustrative examples of approximate solutions for local fractional IVPs are discussed. The nondifferentiable solutions for fractal dimension $\alpha = \ln 2/\ln 3$ are shown graphically. The obtained results illustrate that the Yang-Laplace transform is an efficient mathematical tool to solve the homogeneous and non-homogeneous IVPs with local fractional derivative.

Conflict of Interests

The authors declare that there is no conflicts of interests regarding publication of this paper.

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References

[1] N. S. Koshlyakov, M. M. Smirnov, and E. B. Gliner, *Differential Equations of Mathematical Physics*, North-Holland, New York, NY, USA, 1964.

[2] U. Tyn Myint, *Partial Differential Equations of Mathematical Physics*, Elsevier, New York, NY, USA, 1973.

[3] A. H. Stroud, "Initial value problems for ordinary differential equations," in *Numerical Quadrature and Solution of Ordinary Differential Equations*, pp. 207–303, Springer, New York, NY, USA, 1974.

[4] H. Rutishauser, "Initial value problems for ordinary differential equations," in *Lectures on Numerical Mathematics*, pp. 208–277, Birkhäuser, Boston, Mass, USA, 1990.

[5] J. A. Gatica, V. Oliker, and P. Waltman, "Singular nonlinear boundary value problems for second-order ordinary differential equations," *Journal of Differential Equations*, vol. 79, no. 1, pp. 62–78, 1989.

[6] S. F. Davis and J. E. Flaherty, "An adaptive finite element method for initial-boundary value problems for partial differential equations," *SIAM Journal on Scientific and Statistical Computing*, vol. 3, no. 1, pp. 6–27, 1982.

[7] C. Cattani and A. Kudreyko, "Harmonic wavelet method towards solution of the Fredholm type integral equations of the second kind," *Applied Mathematics and Computation*, vol. 215, no. 12, pp. 4164–4171, 2010.

[8] C. Cattani, "Harmonic wavelet solutions of the Schrödinger equation," *International Journal of Fluid Mechanics Research*, vol. 30, no. 5, pp. 463–472, 2003.

[9] C. Cattani, "Harmonic wavelets towards the solution of nonlinear PDE," *Computers & Mathematics with Applications*, vol. 50, no. 8-9, pp. 1191–1210, 2005.

[10] V. Daftardar-Gejji and H. Jafari, "Adomian decomposition: a tool for solving a system of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 2, pp. 508–518, 2005.

[11] J. S. Duan, R. Rach, and A. M. Wazwaz, "Solution of the model of beam-type micro- and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems," *International Journal of Non-Linear Mechanics*, vol. 49, pp. 159–169, 2013.

[12] C. Li and Y. Wang, "Numerical algorithm based on Adomian decomposition for fractional differential equations," *Computers & Mathematics with Applications*, vol. 57, no. 10, pp. 1672–1681, 2009.

[13] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.

[14] I. Hashim, O. Abdulaziz, and S. Momani, "Homotopy analysis method for fractional IVPs," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 674–684, 2009.

[15] A. Atangana and A. Secer, "The time-fractional coupled-Korteweg-de-Vries equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 947986, 8 pages, 2013.

[16] A. Atangana and A. Kılıçman, "Analytical solutions of boundary values problem of 2D and 3D poisson and biharmonic equations by homotopy decomposition method," *Abstract and Applied Analysis*, vol. 2013, Article ID 380484, 9 pages, 2013.

[17] J. Hristov, "Heat-balance integral to fractional (half-time) heat diffusion sub-model," *Thermal Science*, vol. 14, no. 2, pp. 291–316, 2010.

[18] J. Hristov, "Transient flow of a generalized second grade fluid due to a constant surface shear stress: an approximate integral-balance solution," *International Review of Chemical Engineering*, vol. 3, no. 6, pp. 802–809, 2011.

[19] S. T. Mohyud-Din and M. A. Noor, "Homotopy perturbation method for solving fourth-order boundary value problems," *Mathematical Problems in Engineering*, vol. 2007, Article ID 98602, 15 pages, 2007.

[20] M. A. Noor and S. T. Mohyud-Din, "Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 9, no. 2, pp. 141–156, 2008.

[21] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific Publishing, Boston, Mass, USA, 2012.

[22] X.-J. Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher, Hong Kong, 2011.

[23] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, NY, USA, 2012.

[24] A. Carpinteri, B. Chiaia, and P. Cornetti, "The elastic problem for fractal media: basic theory and finite element formulation," *Computers and Structures*, vol. 82, no. 6, pp. 499–508, 2004.

[25] K. M. Kolwankar and A. D. Gangal, "Local fractional Fokker-Planck equation," *Physical Review Letters*, vol. 80, no. 2, pp. 214–217, 1998.

[26] X. J. Yang and D. Baleanu, "Fractal heat conduction problem solved by local fractional variation iteration method," *Thermal Science*, vol. 17, no. 2, pp. 625–628, 2013.

[27] J. H. He and F. J. Liu, "Local fractional variational iteration method for fractal heat transfer in silk cocoon hierarchy," *Nonlinear Science Letters A*, vol. 4, no. 1, pp. 15–20, 2013.

[28] X.-J. Yang, D. Baleanu, and W. P. Zhong, "Approximate solutions for diffusion equations on cantor space-time," *Proceedings of the Romanian Academy A*, vol. 14, no. 2, pp. 127–133, 2013.

[29] X. J. Yang, D. Baleanu, M. P. Lazarevic, and M. S. Cajic, "Fractal boundary value problems for integral and differential equations with local fractional operators," *Thermal Science*, pp. 103–103, 2013.

[30] Y. Zhao, D.-F. Cheng, and X.-J. Yang, "Approximation solutions for local fractional Schrödinger equation in the one-dimensional Cantorian system," *Advances in Mathematical Physics*, vol. 2013, Article ID 291386, 5 pages, 2013.

[31] J.-H. He, "Asymptotic methods for solitary solutions and compactons," *Abstract and Applied Analysis*, vol. 2012, Article ID 916793, 130 pages, 2012.

[32] W. P. Zhong and F. Gao, "Application of the Yang-Laplace transforms to solution to nonlinear fractional wave equation with fractional derivative," in *Proceeding of the 3rd International Conference on Computer Technology and Development*, pp. 209–213, 2011.

[33] C. F. Liu, S. S. Kong, and S. J. Yuan, "Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem," *Thermal Science*, vol. 17, no. 3, pp. 715–721, 2013.