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Research Article

On the Deficiencies of Some Differential-Difference Polynomials

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The characteristic functions of differential-difference polynomials are investigated, and the result can be viewed as a differential-difference analogue of the classic Valiron-Mokhon'ko Theorem in some sense and applied to investigate the deficiencies of some homogeneous or nonhomogeneous differential-difference polynomials. Some special differential-difference polynomials are also investigated and these results on the value distribution can be viewed as differential-difference analogues of some classic results of Hayman and Yang. Examples are given to illustrate our results at the end of this paper.

1. Introduction

Throughout this paper, we use standard notations in the Nevanlinna theory (see, e.g., [1–3]). Let $f(z)$ be a meromorphic function. Here and in the following the word "meromorphic" means being meromorphic in the whole complex plane. We use normal notations $m(r,f)$, $T(r,f)$, $N(r,f)$, $N(r,1/f)$, $\sigma(f)$, $\lambda(f)$, and $\lambda(1/f)$. And we also use $\sigma_2(f)$ to denote the hyperorder of $f(z)$ and $\delta(\alpha,f)$ to denote the Nevanlinna deficiency of α with respect to $f(z)$. Moreover, we denote by $S(r,f)$ any real quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure.

Recently, with some establishments of difference analogues of the classic Nevanlinna theory (two typical and most important ones can be seen in [4–6]), there has been a renewed interest in the properties of complex difference expressions and meromorphic solutions of complex difference equations (see, e.g., [4–17]). By combining complex differentiates and complex differences, we proceed in this way in this paper.

It is well known that the following Valiron-Mokhon'ko Theorem, due to Valiron [18] and A. Z. Mokhon'ko and V. D. Mokhon'ko [19], is of essential importance in the theory of complex differential equations and functional equations.

Theorem A (see [2, 3]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},\tag{1}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),\tag{2}$$

where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Noting that the difference analogue of Theorem A may not hold, we have obtained a result of this type in [16] by adding some additional assumptions as follows.

Theorem B (see [16]). *Suppose that $P(z,f)$ is a difference polynomial of the form*

$$P(z,f)=\sum_{\lambda\in I}a_\lambda(z)f(z)^{i_0}f(z+c_1)^{i_1}\cdots f(z+c_n)^{i_n},\tag{3}$$

containing just one monomial of degree $d(P)$, and $f(z)$ is a transcendental meromorphic function of finite order. If $f(z)$ also satisfies $N(r,f)+N(r,1/f)=S(r,f)$, then we have

$$T(r,P(z,f)) = d(P)T(r,f)+S(r,f). \quad (4)$$

In this paper, we consider removing the assumption "$P(z,f)$ contains just one monomial of degree $d(P)$" in Theorem B and obtain a weaker result, which is also generalized into differential-difference case. The concrete result can be seen in Section 2.

Next, we recall a classic result concerning Picard's values of meromorphic functions and its derivatives, due to Hayman [20].

Theorem C (see [20]). *Let $f(z)$ be a transcendental entire function. Then*

(a) *for $n \geq 3$ and $a \neq 0$, $\Psi(z)=f'(z)-a(f(z))^n$ assumes all finite values infinitely often;*

(b) *for $n \geq 2$, $\Phi(z)=f'(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.*

Corresponding difference analogues of Theorem C can be seen in [12, 17].

Theorem D (see [12, 17]). *Let $f(z)$ be a transcendental entire function of finite order, and let c be a nonzero complex constant. Then*

(a) *for $n \geq 3$ and $a \neq 0$, $\Psi_1(z)=f(z+c)-af(z)^n$ assumes all finite complex values infinitely often;*

(b) *for $n \geq 2$, $\Phi_1(z)=f(z+c)f(z)^n$ assumes all finite complex values except possibly zero infinitely often.*

After Theorem C, many results have been obtained on the value distribution of differential polynomials. A typical one is as follows.

Theorem E (see [21, 22]). *Let f be a transcendental meromorphic function with $N(r,f)+N(r,1/f)=S(r,f)$, and let Ψ be a differential polynomial in f of the form*

$$\Psi(z)=\sum a(z)f(z)^{l_0}f'(z)^{l_1}\cdots f^{(k)}(z)^{l_k} \quad (5)$$

with no constant term. Furthermore, assume the degree, n, of Ψ is greater than one and $l_0<n$, $0 \leq l_i \leq n$, for all $i \neq 0$. Then $\delta(a,\Psi)<1$ for all $a \neq 0,\infty$. Moreover, if all the terms of Ψ have different degrees at least two, that is, Ψ is nonhomogeneous, then $\delta(a,\Psi) \leq 1-(1/2n)$ for all $a \neq \infty$.

We also consider deficiencies of difference polynomials of meromorphic functions of finite order in [16], which can be viewed as difference analogues of Theorem E, as well as generalizations of Theorem D.

In this paper, we proceed to investigate deficiencies of differential-difference polynomials of meromorphic functions. The concrete results can be seen in Section 3.

Examples are given in Section 4 to illustrate our results.

2. A Differential-Difference Analogue of Valiron-Mokhon'ko Theorem

In what follows, we will consider differential-difference polynomials. A differential-difference polynomial is a polynomial in $f(z)$, its shifts, its derivatives, and derivatives of its shifts (see [14]), that is, an expression of the form

$$\begin{aligned} P(z,f) &= \sum_{\lambda\in I} a_\lambda(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \cdots f^{(m)}(z)^{\lambda_{0,m}} \\ &\quad \times f(z+c_1)^{\lambda_{1,0}} f'(z+c_1)^{\lambda_{1,1}} \cdots f^{(m)}(z+c_1)^{\lambda_{1,m}} \\ &\quad \cdots f(z+c_n)^{\lambda_{n,0}} f'(z+c_n)^{\lambda_{n,1}} \cdots f^{(m)}(z+c_n)^{\lambda_{n,m}} \\ &= \sum_{\lambda\in I} a_\lambda(z) \prod_{i=0}^{n}\prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}, \end{aligned} \quad (6)$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0},\ldots,\lambda_{0,m},\lambda_{1,0},\ldots,\lambda_{1,m},\ldots,\lambda_{n,0},\ldots,\lambda_{n,m})$, and $c_0(=0)$ and $c_1,\ldots,c_n$ are distinct complex constants. And we assume that the meromorphic coefficients $a_\lambda(z)$, $\lambda \in I$ of $P(z,f)$ are of growth $S(r,f)$. We denote the degree of the monomial $\prod_{i=0}^{n}\prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}$ of $P(z,f)$ by $d(\lambda)=\sum_{i=0}^{n}\sum_{j=0}^{m}\lambda_{i,j}$. Then we denote the degree and the lower degree of $P(z,f)$ by

$$d(P)=\max_{\lambda\in I}\{d(\lambda)\}, \qquad d^*(P)=\min_{\lambda\in I}\{d(\lambda)\}, \quad (7)$$

respectively. In particular, we call $P(z,f)$ a homogeneous differential-difference polynomial if $d(P)=d^*(P)$. Otherwise, $P(z,f)$ is nonhomogeneous.

In the following, we assume $d(P) \geq 1$ and $P(z,f) \not\equiv P(z,0)$.

We prove a weaker differential-difference version of the classic Valiron-Mokhon'ko Theorem as follows.

Theorem 1. *Suppose that $f(z)$ is a transcendental meromorphic function, and $P(z,f)$ is a differential-difference polynomial of the form (6). If $f(z)$ also satisfies $\sigma_2(f)<1$ and*

$$N(r,f)+N\left(r,\frac{1}{f}\right)=S(r,f), \quad (8)$$

then one has

$$\begin{aligned} d^*(P)T(r,f)+S(r,f) &\leq T(r,P(z,f)) \\ &\leq d(P)T(r,f)+S(r,f). \end{aligned} \quad (9)$$

Remark 2. If $P(z,f)$ is a homogeneous differential-difference polynomial in addition, then

$$T(r,P(z,f))=d(P)T(r,f)+S(r,f). \quad (10)$$

Remark 3. Especially, assumption (8) can be replaced by the assumption "$\max\{\lambda(f),\lambda(1/f)\}<\sigma(f)$". In fact, if $f(z)$ satisfies $\max\{\lambda(f),\lambda(1/f)\}<\sigma(f)$, then $f(z)$ is of regular growth, and (8) holds consequently.

To prove Theorem 1, we need the following lemmas.

Lemma 4 (see [6]). *Let $f(z)$ be a nonconstant meromorphic function, $\varepsilon > 0$, and $c \in \mathbb{C}$. If $\zeta = \sigma_2(f) < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\zeta-\varepsilon}}\right) \tag{11}$$

for all r outside of a set of finite logarithmic measure.

Lemma 5 (see [6]). *Let $T : [0,+\infty) \to [0,+\infty)$ be a nondecreasing continuous function and let $s \in (0,+\infty)$. If the hyperorder of T is strictly less than one, that is, $\overline{\lim}_{r\to\infty}(\log_2 T(r)/\log r) = \zeta < 1$ and $\delta \in (0, 1-\zeta)$, then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right), \tag{12}$$

where r runs to infinity outside of a set of finite logarithmic measure.

It is shown in [23, p.66] and [7, Lemma 1] that the inequality

$$\begin{aligned}(1+o(1))\,T(r-|c|, f) &\le T(r, f(z+c)) \\ &\le (1+o(1))\,T(r+|c|, f)\end{aligned} \tag{13}$$

holds for $c \neq 0$ and $r \to \infty$. And from the proof, the above relation is also true for counting function. By combing Lemma 5 and these inequalities, we immediately deduce the following lemma.

Lemma 6. *Let $f(z)$ be a nonconstant meromorphic function of $\sigma_2(f) < 1$, and let c be a nonzero complex constant. Then one has*

$$\begin{aligned} T(r, f(z+c)) &= T(r,f) + S(r,f), \\ N(r, f(z+c)) &= N(r,f) + S(r,f), \\ N\left(r, \frac{1}{f(z+c)}\right) &= N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned} \tag{14}$$

Lemma 7. *Let $f(z)$ be a transcendental meromorphic function of $\sigma_2(f) < 1$, and let $P(z,f)$ be a differential-difference polynomial of the form (6); then we one has*

$$m(r, P(z,f)) \le d(P)\, m(r,f) + S(r,f). \tag{15}$$

Furthermore, if $f(z)$ also satisfies

$$N(r,f) = S(r,f), \tag{16}$$

then one has

$$T(r, P(z,f)) \le d(P)\, T(r,f) + S(r,f). \tag{17}$$

Proof. For $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, we define $g_{i,j}(z) = f^{(j)}(z+c_i)/f(z)$. We also define

$$\begin{aligned} g^*_{i,j}(z) &= \begin{cases} g_{i,j}(z), & \text{if } |g_{i,j}(z)| > 1; \\ 1, & \text{if } |g_{i,j}(z)| \le 1, \end{cases} \\ f^*(z) &= \begin{cases} f(z), & \text{if } |f(z)| > 1, \\ 1, & \text{if } |f(z)| \le 1. \end{cases} \end{aligned} \tag{18}$$

Thus,

$$\begin{aligned} |P(z,f)| &\le \sum_{\lambda\in I}\left(|a_\lambda(z)|\,|f(z)|^{d(\lambda)}\prod_{i=0}^{n}\prod_{j=0}^{m}|g_{i,j}(z)|^{\lambda_{i,j}}\right) \\ &\le \left(\sum_{\lambda\in I}|a_\lambda(z)|\prod_{i=0}^{n}\prod_{j=0}^{m}|g^*_{i,j}(z)|^{\lambda_{i,j}}\right)|f^*(z)|^{d(P)} \\ &\le \left(\sum_{\lambda\in I}|a_\lambda(z)|\prod_{i=0}^{n}\prod_{j=0}^{m}|g^*_{i,j}(z)|^{d(\lambda)}\right)|f^*(z)|^{d(P)} \\ &\le \left(\sum_{\lambda\in I}|a_\lambda(z)|\right)\left(\prod_{i=0}^{n}\prod_{j=0}^{m}|g^*_{i,j}(z)|\,|f^*(z)|\right)^{d(P)}. \end{aligned} \tag{19}$$

By the definitions of $f^*(z)$ and $g^*_{i,j}(z)$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, we have

$$\begin{aligned} m(r, f^*) &= m(r,f), \\ m(r, g^*_{i,j}) &= m(r, g_{i,j}), \quad i = 0, \ldots, n, \; j = 0, \ldots, m. \end{aligned} \tag{20}$$

It follows by (19) and (20) that

$$\begin{aligned} m(r, P(z,f)) &\le d(P)\, m(r, f^*) \\ &\quad + d(P)\sum_{i=0}^{n}\sum_{j=0}^{m} m(r, g^*_{i,j}) + S(r,f) \\ &= d(P)\, m(r,f) \\ &\quad + d(P)\sum_{i=0}^{n}\sum_{j=0}^{m} m(r, g_{i,j}) + S(r,f). \end{aligned} \tag{21}$$

Lemmas 4 and 6 and the logarithmic derivative lemma imply that, for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$,

$$\begin{aligned} m(r, g_{i,j}) &= m\left(r, \frac{f^{(j)}(z+c_i)}{f(z)}\right) \\ &\le m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) + m\left(r, \frac{f(z+c_i)}{f(z)}\right) \\ &= S(r, f(z+c_i)) + S(r,f) = S(r,f). \end{aligned} \tag{22}$$

Then (15) follows by (21) and (22).

It is easy to find that

$$N(r,P(z,f)) = O\left(N(r,f) + \sum_{i=1}^{n} N(r,f(z+c_i))\right) + S(r,f). \tag{23}$$

Then (16), (23), and Lemma 6 yield that

$$N(r,P(z,f)) = S(r,f). \tag{24}$$

Thus, (17) follows by (15) and (24). □

Lemma 8. *Let $f(z)$ be a transcendental meromorphic function of $\sigma_2(f) < 1$, and let $P(z,f)$ be a differential-difference polynomial of the form* (6)*; then one has*

$$m\left(r, \frac{P(z,f)}{f^{d(P)}}\right) \leq (d(P) - d^*(P))\, m\left(r, \frac{1}{f}\right) + S(r,f). \tag{25}$$

Proof. Similar to (19), we have

$$\begin{aligned}
&\left|\frac{P(z,f)}{f(z)^{d(P)}}\right| \\
&\leq \sum_{\lambda\in I}\left(|a_\lambda(z)| \prod_{i=0}^{n}\prod_{j=0}^{m}\left|g_{i,j}(z)\right|^{\lambda_{i,j}} |g(z)|^{d(P)-d(\lambda)}\right) \\
&\leq \left(\sum_{\lambda\in I} |a_\lambda(z)|\, |g^*(z)|^{d(P)-d(\lambda)}\right) \\
&\quad\times \prod_{i=0}^{n}\prod_{j=0}^{m}\left|g^*_{i,j}(z)\right|^{d(P)} \leq \left(\sum_{\lambda\in I} |a_\lambda(z)|\right) \\
&\quad\times \prod_{i=0}^{n}\prod_{j=0}^{m}\left|g^*_{i,j}(z)\right|^{d(P)} |g^*(z)|^{d(P)-d^*(P)},
\end{aligned} \tag{26}$$

where $g(z) = 1/f(z)$ and

$$g^*(z) = \begin{cases} g(z), & \text{if } |g(z)| > 1, \\ 1, & \text{if } |g(z)| \leq 1. \end{cases} \tag{27}$$

By the definition of $g^*(z)$, we have $m(r,g^*) = m(r,g) = m(r,1/f)$. Thus, (20), (22), and (26) yield that

$$\begin{aligned}
m\left(r, \frac{P(z,f)}{f^{d(P)}}\right) &\leq (d(P) - d^*(P))\, m(r,g^*) \\
&\quad + d(P) \sum_{i=0}^{n}\sum_{j=0}^{m} m\left(r, g^*_{i,j}\right) + S(r,f) \\
&\leq (d(P) - d^*(P))\, m\left(r, \frac{1}{f}\right) + S(r,f),
\end{aligned} \tag{28}$$

that is, (25). □

Now, we can finish the proof of Theorem 1 in the end.

Proof of Theorem 1. We deduce from (8), (24), and Lemma 8 that

$$\begin{aligned}
&d(P)\, T(r,f) \\
&\quad = T\left(r, f^{d(P)}\right) \leq m\left(r, \frac{P(z,f)}{f^{d(P)}}\right) \\
&\qquad + N\left(r, \frac{P(z,f)}{f^{d(P)}}\right) + T(r,P(z,f)) + O(1) \\
&\quad \leq (d(P) - d^*(P))\, m\left(r, \frac{1}{f}\right) + N(r,P(z,f)) \\
&\qquad + d(P)\, N\left(r, \frac{1}{f}\right) + T(r,P(z,f)) + O(1) \\
&\quad \leq (d(P) - d^*(P))\, T(r,f) \\
&\qquad + T(r,P(z,f)) + S(r,f),
\end{aligned} \tag{29}$$

that is,

$$d^*(P)\, T(r,f) + S(r,f) \leq T(r,P(z,f)). \tag{30}$$

Then, (9) follows by (17) and (30). □

3. Deficiencies of Some Differential-Difference Polynomials

In the following, we assume that $\alpha(z)(\not\equiv 0)$ is a meromorphic function of growth $S(r,f)$.

In this section, we will apply Theorem 1 to consider the deficiencies of general homogeneous or nonhomogeneous differential-difference polynomials.

Theorem 9. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and* (8)*, and $P(z,f)$ is a differential-difference polynomial of the form* (6)*.*

(a) *If $P(z,f)$ is a homogeneous differential-difference polynomial, then one has*

$$\overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/(P(z,f) - \alpha))}{T(r,P(z,f))} = 1, \qquad \delta(\alpha, P(z,f)) = 0. \tag{31}$$

(b) *If $P(z,f)$ is a nonhomogeneous differential-difference polynomial with $2d^*(P) > d(P)$, then one has*

$$\begin{aligned}
\overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/(P(z,f) - \alpha))}{T(r,P(z,f))} &\geq \frac{2d^*(P) - d(P)}{d^*(P)}, \\
\delta(\alpha, P(z,f)) &\leq 1 - \frac{2d^*(P) - d(P)}{d^*(P)} < 1.
\end{aligned} \tag{32}$$

Thus, $P(z,f) - \alpha(z)$ has infinitely many zeros, whether $P(z,f)$ is homogeneous or nonhomogeneous.

Furthermore, one considers some differential-difference polynomials of special forms, which are generalizations of both differential cases and difference cases, that is, Theorems C–E.

Theorem 10. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and (16), $P(z,f)$ is a differential-difference polynomial of the form (6), and $F(f) = (f^{\nu} + a_{\nu-1}(z)f^{\nu-1} + \cdots + a_1(z)f + a_0(z))^u$, $u, \nu \in \mathbb{N}$, is a polynomial of $f(z)$ with meromorphic coefficients $a_i(z)$, $i = 0, \ldots, \nu-1$ of growth $S(r,f)$. If $u\nu > d(P)$, $u \neq 1$, then*

$$Q_1(z,f) = F(f)\,P(z,f) \tag{33}$$

satisfies

$$\begin{gathered}\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/(Q_1(z,f)-\alpha))}{T(r,Q_1(z,f))} \geq \frac{(u-1)(u\nu-d(P))}{u(u\nu+d(P))},\\ \delta(\alpha,Q_1(z,f)) \leq 1-\frac{(u-1)(u\nu-d(P))}{u(u\nu+d(P))} < 1.\end{gathered} \tag{34}$$

Thus, $Q_1(z,f)-\alpha(z)$ has infinitely many zeros.

When $F(f)$ is of a special form f^{ν}, we can deduce the following result from Theorem 9.

Theorem 11. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and (16), and $P(z,f)$ is a differential-difference polynomial of the form (6). If $\nu \in \mathbb{N}\setminus\{1\}$ and $\nu + 2d^*(P) > d(P)$, then*

$$Q_2(z,f) = f^{\nu}P(z,f) \tag{35}$$

satisfies $\delta(\beta, Q_2(z,f)) < 1$, where $\beta \in \mathbb{C}\setminus\{0\}$. Thus, $Q_2(z,f)-\beta$ has infinitely many zeros.

Remark 12. On the one hand, we can also apply Theorem 9 to $Q_1(z,f)$ with the assumption "$2(d^*(P)+d^*(F)) > d(P)+u\nu$" and obtain the same result as Theorem 10. But our present assumption "$u\nu > d(P)$" has no concern with $d^*(P)$ and $d^*(F)$, so we think Theorem 10 is better to some extent. On the other hand, we can also apply Theorem 10 to $Q_2(z,f)$ with the assumption "$\nu > d(P)$," which is stronger than "$\nu + 2d^*(P) > d(P)$" in Theorem 11, showing Theorem 11 is better to some extent.

Theorem 13. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and (16), $P(z,f)$ is a differential-difference polynomial of the form (6), and $F(f) = (f^{\nu} + a_{\nu-1}(z)f^{\nu-1} + \cdots + a_1(z)f + a_0(z))^u$, $u, \nu \in \mathbb{N}$, is a polynomial of $f(z)$ with meromorphic coefficients $a_i(z)$, $i = 0, \ldots, \nu-1$ of growth $S(r,f)$. If $(u-1)u\nu/(2u-1) > d(P)$, $u \neq 1$, then*

$$Q_3(z,f) = F(f) + P(z,f) \tag{36}$$

satisfies

$$\begin{gathered}\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/(Q_3(z,f)-\alpha))}{T(r,Q_3(z,f))} \geq 1-\frac{1}{u}-\frac{2u-1}{u^2\nu}d(P),\\ \delta(\alpha,Q_3(z,f)) \leq \frac{1}{u}+\frac{2u-1}{u^2\nu}d(P) < 1.\end{gathered} \tag{37}$$

Thus, $Q_3(z,f)-\alpha(z)$ has infinitely many zeros.

When $u = 1$, one can consider some special cases as follows.

Theorem 14. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and (16), and $P(z,f)$ is a differential-difference polynomial of the form (6).*

(a) *If $\nu > d(P)+2 \geq 3$, then*

$$Q_4(z,f) = f^{\nu} + P(z,f) \tag{38}$$

satisfies

$$\begin{gathered}\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/(Q_4(z,f)-\alpha))}{T(r,Q_4(z,f))} \geq 1-\frac{d(P)+2}{\nu},\\ \delta(\alpha,Q_4(z,f)) \leq \frac{d(P)+2}{\nu} < 1.\end{gathered} \tag{39}$$

(b) *If $(\nu-1)\nu/(2\nu-1) > d(P)$, $\nu \geq 3$, then $Q_4(z,f)$ satisfies*

$$\begin{gathered}\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/(Q_4(z,f)-\alpha))}{T(r,Q_4(z,f))} \geq 1-\frac{1}{\nu}-\frac{2\nu-1}{\nu^2}d(P),\\ \delta(\alpha,Q_4(z,f)) \leq \frac{1}{\nu}+\frac{2\nu-1}{\nu^2}d(P) < 1.\end{gathered} \tag{40}$$

Especially, it holds for $\nu = d(P)+2 = 3$.

(c) *If $\nu \geq d(P)+2 \geq 3$ and f also satisfies $N(r,1/f) = S(r,f)$, then $Q_4(z,f)$ satisfies $\delta(\alpha, Q_4(z,f)) < 1$. Especially, it holds for $\nu = d(P)+2 > 3$.*

Thus, $Q_4(z,f)-\alpha(z)$ has infinitely many zeros.

If we assume that $N(r,1/f) = S(r,f)$ in addition, the following result follows immediately by Theorem 9.

Theorem 15. *Suppose that $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and (8), and $P(z,f)$ is a differential-difference polynomial of the form (6). If $2\min\{d^*(P),\nu\} > \max\{d(P),\nu\}$, then $Q_4(z,f)$ satisfies $\delta(\alpha,Q_4(z,f)) < 1$. Thus, $Q_4(z,f)-\alpha(z)$ has infinitely many zeros.*

Remark 16. Noting that, when $\nu > 3$, $(\nu-1)\nu/(2\nu-1) \leq \nu-2$ hold, we see that the assumption "$\nu > d(P)+2$" in Theorem 14(a) is weaker than the assumption "$(\nu-1)\nu/(2\nu-1) > d(P)$" in Theorem 14(b). And these assumptions in Theorem 14 have no concern with $d^*(P)$); thus they are different from the assumption "$2\min\{d^*(P),\nu\} > \max\{d(P),\nu\}$" in Theorem 15.

Remark 17. From the proofs behind, it is easy to find that

$$\lambda(P(z,f)-\alpha)=\sigma(P(z,f))=\sigma(f),$$
$$\lambda(Q_i(z,f)-\alpha)=\sigma(Q_i(z,f))=\sigma(f),\quad i=1,3,4, \tag{41}$$

hold, respectively, in Theorems 9, 10, 13, 14(a) and (b), and 15.

Now, we give the proofs of Theorems 9–15.

Proof of Theorem 9. It follows by Theorem 1 that

$$S(r,f)=S(r,P(z,f)). \tag{42}$$

We deduce from (8), (24), (25), and (42) that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{P(z,f)}\right)&\\ \le \overline{N}\left(r,\frac{1}{f^{d(P)}}\right)+\overline{N}\left(r,\frac{f^{d(p)}}{P(z,f)}\right)&\\ \le \overline{N}\left(r,\frac{1}{f}\right)+m\left(r,\frac{P(z,f)}{f^{d(P)}}\right)&\\ +N\left(r,\frac{P(z,f)}{f^{d(P)}}\right)+O(1)&\\ \le (d(P)-d^*(P))\,m\left(r,\frac{1}{f}\right)+S(r,f)&\\ \le (d(P)-d^*(P))\,m\left(r,\frac{1}{f}\right)+S(r,P(z,f)).&\end{aligned} \tag{43}$$

Thus, an application of the second main theorem and (24), (42), and (43) imply that

$$\begin{aligned}T(r,P(z,f))&\le \overline{N}(r,P(z,f))+\overline{N}\left(r,\frac{1}{P(z,f)}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)+S(r,P(z,f))\\ &\le (d(P)-d^*(P))\,m\left(r,\frac{1}{f}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)+S(r,P(z,f)).\end{aligned} \tag{44}$$

(a) If $d(P)=d^*(P)$, then it follows by (44) that

$$T(r,P(z,f))\le \overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)+S(r,P(z,f)), \tag{45}$$

by which (31) holds.

(b) If $2d^*(P)>d(P)$, then we deduce from (30) and (44) that

$$\begin{aligned}T(r,P(z,f))&\le (d(P)-d^*(P))\,T(r,f)\\ &\quad+\overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)+S(r,P(z,f))\\ &\le \frac{d(P)-d^*(P)}{d^*(P)}T(r,P(z,f))\\ &\quad+\overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)+S(r,P(z,f));\end{aligned} \tag{46}$$

that is,

$$\begin{aligned}\frac{2d^*(P)-d(P)}{d^*(P)}T(r,P(z,f))&\le \overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right)\\ &\quad+S(r,P(z,f)).\end{aligned} \tag{47}$$

Since $2d^*(P)-d(P)>0$, (32) follows immediately by (47). □

Proof of Theorem 10. We deduce from (16), (17), and (24) that

$$T(r,Q_1(z,f))\le (uv+d(P))\,T(r,f)+S(r,f), \tag{48}$$

$$\begin{aligned}N(r,Q_1(z,f))&=O(N(r,f))+N(r,P(z,f))+S(r,f)\\ &=S(r,f)\end{aligned} \tag{49}$$

hold. Next, we consider $\overline{N}(r,1/Q_1(z,f))$. Let z_0 be a zero of $Q_1(z,f)$ and distinguish three cases.

(i) z_0 is not a zero of $F(f)$; then z_0 must be a zero of $P(z,f)$ and

$$u\le \omega\left(\frac{1}{Q_1(z,f)},z_0\right)+(u-1)\,\omega\left(\frac{1}{P(z,f)},z_0\right), \tag{50}$$

where $\omega(f,z_0)$ denotes the order of multiplicity of z_0 or zero according as z_0 is a pole of $f(z)$ or not.

(ii) z_0 is a zero of $F(f)$ but not a pole of $P(z,f)$. Then

$$u\le \omega\left(\frac{1}{Q_1(z,f)},z_0\right). \tag{51}$$

(iii) z_0 is a zero of $F(f)$ and a pole of $P(z,f)$. Then

$$u\le \omega\left(\frac{1}{F(f)},z_0\right)\le \omega\left(\frac{1}{Q_1(z,f)},z_0\right)+\omega(P(z,f),z_0). \tag{52}$$

(24) and (50)–(52) yield that

$$u\overline{N}\left(r,\frac{1}{Q_1(z,f)}\right)\le N\left(r,\frac{1}{Q_1(z,f)}\right) + (u-1)N\left(r,\frac{1}{P(z,f)}\right)+S(r,f). \quad (53)$$

Then (48), (49), (53), and an application of the second main theorem to $Q_1(z,f)$ imply that

$$\begin{aligned} &T(r,Q_1(z,f)) \\ &\quad\le \overline{N}(r,Q_1(z,f))+\overline{N}\left(r,\frac{1}{Q_1(z,f)}\right) \\ &\quad\quad+\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right)+S(r,Q_1(z,f)) \\ &\quad\le \frac{1}{u}N\left(r,\frac{1}{Q_1(z,f)}\right)+\frac{u-1}{u}N\left(r,\frac{1}{P(z,f)}\right) \\ &\quad\quad+\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right)+S(r,f); \end{aligned} \quad (54)$$

consequently,

$$T(r,Q_1(z,f))\le N\left(r,\frac{1}{P(z,f)}\right) + \frac{u}{u-1}\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right)+S(r,f). \quad (55)$$

Moreover, by $f^{d(P)}F(f) = f^{d(P)}Q_1(z,f)/P(z,f)$, (16), (24), (25), and Theorem A, we have

$$\begin{aligned} &(d(P)+uv)\,m(r,f) \\ &\quad= m\left(r,\frac{f^{d(P)}Q_1(z,f)}{P(z,f)}\right)+S(r,f) \\ &\quad\le m(r,Q_1(z,f))+m\left(r,\frac{P(z,f)}{f^{d(P)}}\right) \\ &\quad\quad+N\left(r,\frac{P(z,f)}{f^{d(P)}}\right)-N\left(r,\frac{f^{d(P)}}{P(z,f)}\right)+S(r,f) \\ &\quad\le m(r,Q_1(z,f))+d(P)\,m\left(r,\frac{1}{f}\right) \\ &\quad\quad+d(P)\left(N\left(r,\frac{1}{f}\right)-N(r,f)\right) \\ &\quad\quad+N(r,P(z,f))-N\left(r,\frac{1}{P(z,f)}\right)+S(r,f) \\ &\quad= m(r,Q_1(z,f))+d(P)\,m(r,f) \\ &\quad\quad-N\left(r,\frac{1}{P(z,f)}\right)+S(r,f); \end{aligned} \quad (56)$$

consequently,

$$uvm(r,f)\le m(r,Q_1(z,f))-N\left(r,\frac{1}{P(z,f)}\right)+S(r,f). \quad (57)$$

On the other hand, the evident relation $uv\omega(f,z_0) \le \omega(F(f),z_0) + uv\sum_{j=0}^{v-1}\omega(a_j,z_0)$, where the definition of $\omega(f,z_0)$ is given after (50), results in

$$\begin{aligned} uvN(r,f) &\le N(r,F(f))+S(r,f) \\ &\le N(r,Q_1(z,f))+N\left(r,\frac{1}{P(z,f)}\right)+S(r,f). \end{aligned} \quad (58)$$

We deduce from (57) and (58) that

$$uvT(r,f)\le T(r,Q_1(z,f))+S(r,f). \quad (59)$$

Then (17), (55), and (59) yield that

$$\begin{aligned} &uvT(r,f) \\ &\le N\left(r,\frac{1}{P(z,f)}\right)+\frac{u}{u-1}\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right)+S(r,f) \\ &\le d(P)\,T(r,f)+\frac{u}{u-1}\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right)+S(r,f); \end{aligned} \quad (60)$$

that is,

$$\frac{(u-1)(uv-d(P))}{u}T(r,f)\le\overline{N}\left(r,\frac{1}{Q_1(z,f)-\alpha}\right) +S(r,f). \quad (61)$$

From (48) and (61), we deduce that

$$\begin{aligned} &\underline{\lim}_{r\to\infty}\frac{\overline{N}(r,1/(Q_1(z,f)-\alpha))}{T(r,Q_1(z,f))}\ge\frac{(u-1)(uv-d(P))}{u(uv+d(P))}, \\ &\delta(\alpha,Q_1(z,f))\le 1-\frac{(u-1)(uv-d(P))}{u(uv+d(P))}<1. \end{aligned} \quad (62)$$

□

Proof of Theorem 11. Assume to the contrary that $\delta(\beta,Q_2(z,f))=1$. Denoting

$$Q_2(z,f)-\beta=f^{v}(z)P(z,f)-\beta=G(z), \quad (63)$$

we deduce from (16) and (17) that

$$\begin{aligned} N\left(r,\frac{1}{G}\right)&=N\left(r,\frac{1}{Q_2(z,f)-\beta}\right) \\ &=S(r,Q_2(z,f))=S(r,f). \end{aligned} \quad (64)$$

On the other hand, (16) and (24) yield that

$$N(r,G) = N(r,Q_2(z,f)-\beta) = S(r,f). \quad (65)$$

Differentiating both sides of (63), we obtain

$$f^{\nu-1}(z)R(z,f) = G'(z), \quad (66)$$

where $R(z,f) = \nu f'(z)P(z,f) + f(z)P'(z,f)$. Clearly, we deduce from (16) and (24) that

$$N(r,R(z,f)) = S(r,f). \quad (67)$$

Moreover, (64) and (65) yield that

$$\begin{aligned} N\left(r,\frac{1}{G'}\right) &\le N\left(r,\frac{G}{G'}\right) + N\left(r,\frac{1}{G}\right) \\ &\le T\left(r,\frac{G'}{G}\right) + N\left(r,\frac{1}{G}\right) + O(1) \\ &\le m\left(r,\frac{G'}{G}\right) + \overline{N}(r,G) + 2N\left(r,\frac{1}{G}\right) + O(1) \\ &= S(r,G) + S(r,f) = S(r,f). \end{aligned} \quad (68)$$

It follows by (66)–(68) that

$$N\left(r,\frac{1}{f}\right) = \frac{1}{\nu-1}N\left(r,\frac{R(z,f)}{G'}\right) = S(r,f). \quad (69)$$

Then (16), (69), and the fact $2(d^*(P)+\nu) > d(P)+\nu$ imply that the assumptions of Theorem 9(b) are satisfied. Thus, Theorem 9(b) yields that $\delta(\beta,Q_2(z,f)) < 1$, a contradiction. Therefore, we have $\delta(\beta,Q_2(z,f)) < 1$. □

Proof of Theorem 13. We deduce from (16), (17), and (24) that

$$\begin{aligned} T(r,Q_3(z,f)) &\le \max\{u\nu, d(P)\}T(r,f) + S(r,f) \\ &= u\nu T(r,f) + S(r,f). \end{aligned} \quad (70)$$

Denote

$$H(z) = \frac{-P(z,f)+\alpha(z)}{F(f)}. \quad (71)$$

Now, we estimate the poles, the zeros, and 1-points of $H(z)$ accurately. On the one hand, we see by (71) that the poles of $H(z)$ occur at zeros of $F(f)$ and poles of $-P(z,f)+\alpha(z)$ which are not simultaneously 1-points of $H(z)$, and those poles of $H(z)$ which are zeros of $F(f)$ but not simultaneously zeros of $-P(z,f)+\alpha(z)$ also have multiplicities at least u. On the other hand, we also see by (71) that the zeros of $H(z)$ occur at zeros of $-P(z,f)+\alpha(z)$ and poles of $F(f)$ which are not simultaneously 1-points of $H(z)$. Moreover, 1-points of $H(z)$ occur at zeros of $Q_3(z,f)-\alpha(z)$ and occur at the common poles, zeros of $F(f)$ and $-P(z,f)+\alpha(z)$ with the same multiplicities. Thus, it follows by (16) and (24) that

$$\begin{aligned} &\overline{N}(r,H) + \overline{N}\left(r,\frac{1}{H}\right) + \overline{N}\left(r,\frac{1}{H-1}\right) \\ &\quad\le \frac{1}{u}N(r,H) + \overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right) \\ &\quad\quad + \overline{N}\left(r,\frac{1}{Q_3(z,f)-\alpha}\right) + S(r,f). \end{aligned} \quad (72)$$

Then (17), (72), and the second main theorem result in

$$\begin{aligned} T(r,H) &\le \overline{N}(r,H) + \overline{N}\left(r,\frac{1}{H}\right) + \overline{N}\left(r,\frac{1}{H-1}\right) + S(r,H) \\ &\le \frac{1}{u}T(r,H) + \overline{N}\left(r,\frac{1}{P(z,f)-\alpha}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{Q_3(z,f)-\alpha}\right) + S(r,f) \\ &\le \frac{1}{u}T(r,H) + d(P)T(r,f) \\ &\quad + \overline{N}\left(r,\frac{1}{Q_3(z,f)-\alpha}\right) + S(r,f); \end{aligned} \quad (73)$$

that is,

$$\begin{aligned} \left(1-\frac{1}{u}\right)T(r,H) &\le d(P)T(r,f) \\ &\quad + \overline{N}\left(r,\frac{1}{Q_3(z,f)-\alpha}\right) + S(r,f). \end{aligned} \quad (74)$$

Moreover, Theorem A and (17) imply that

$$\begin{aligned} u\nu T(r,f) + S(r,f) &= T(r,F(f)) = T\left(r,\frac{-P(z,f)+\alpha}{H}\right) \\ &\le d(P)T(r,f) + T(r,H) + S(r,f); \end{aligned} \quad (75)$$

that is,

$$(u\nu - d(P))T(r,f) \le T(r,H) + S(r,f). \quad (76)$$

Then (74) and (76) yield that

$$\begin{aligned} \left((u-1)\nu - \frac{2u-1}{u}d(P)\right)T(r,f) &\le \overline{N}\left(r,\frac{1}{Q_3(z,f)-\alpha}\right) \\ &\quad + S(r,f). \end{aligned} \quad (77)$$

From (70) and (77), we deduce that

$$\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,1/\left(Q_3\left(z,f\right)-\alpha\right)\right)}{T\left(r,Q_3\left(z,f\right)\right)}\geq 1-\frac{1}{u}-\frac{2u-1}{u^2v}d\left(P\right),$$
$$\delta\left(\alpha,Q_3\left(z,f\right)\right)\leq\frac{1}{u}+\frac{2u-1}{u^2v}d\left(P\right)<1. \quad (78)$$

□

To prove Theorem 14(c), we also need the following lemma of one of Tumura-Clunie type theorems.

Lemma 18 (see [24]). *Let $f(z)$ be a meromorphic function, and suppose that $\Psi = a_n f^n + \cdots + a_0$ has small meromorphic coefficients $a_j(z)$, $a_n(z) \not\equiv 0$, in the sense of $T(r,a_j) = S(r,f)$. Moreover, assume that $\overline{N}(r,1/\Psi) + \overline{N}(r,f) = S(r,f)$. Then $\Psi = a_n(f + (a_{n-1}/na_n))^n$.*

Proof of Theorem 14. (a) We deduce from (16), (17), and (24) that

$$T\left(r,Q_4\left(z,f\right)\right)\leq vT\left(r,f\right)+S\left(r,f\right). \quad (79)$$

Denote

$$K\left(z\right)=Q_4\left(z,f\right)-\alpha\left(z\right)=f^{v}\left(z\right)+P\left(z,f\right)-\alpha\left(z\right). \quad (80)$$

Differentiating both sides of (80), we obtain

$$\begin{aligned} &vf^{v-1}\left(z\right)f'\left(z\right)+P'\left(z,f\right)-\alpha'\left(z\right)\\ &\quad=K'\left(z\right)=\left(f^{v}\left(z\right)+P\left(z,f\right)-\alpha\left(z\right)\right)\frac{K'\left(z\right)}{K\left(z\right)}; \end{aligned} \quad (81)$$

that is,

$$\begin{aligned} &f^{v-1}\left(z\right)\left(\left(v\frac{f'\left(z\right)}{f\left(z\right)}-\frac{K'\left(z\right)}{K\left(z\right)}\right)f\left(z\right)\right)\\ &\quad=\left(P\left(z,f\right)-\alpha\left(z\right)\right)\frac{K'\left(z\right)}{K\left(z\right)}-\left(P'\left(z,f\right)-\alpha'\left(z\right)\right)\\ &\quad=\left(P\left(z,f\right)-\alpha\left(z\right)\right)\left(\frac{K'\left(z\right)}{K\left(z\right)}-\frac{P'\left(z,f\right)-\alpha'\left(z\right)}{P\left(z,f\right)-\alpha\left(z\right)}\right). \end{aligned} \quad (82)$$

It follows by (15)–(17), (24), (79), and (82) that

$$\begin{aligned} &m\left(r,f^{v-1}\right)\\ &\leq m\left(r,P\left(z,f\right)-\alpha\right)+m\left(r,\frac{K'}{K}\right)\\ &\quad+m\left(r,\frac{P'\left(z,f\right)-\alpha'}{P\left(z,f\right)-\alpha}\right)+m\left(r,\frac{1}{\left(v\left(f'/f\right)-\left(K'/K\right)\right)f}\right)\\ &\leq d\left(P\right)m\left(r,f\right)+m\left(r,\left(v\frac{f'}{f}-\frac{K'}{K}\right)f\right)\\ &\quad+N\left(r,\left(v\frac{f'}{f}-\frac{K'}{K}\right)f\right)+S\left(r,K\right)+S\left(r,f\right)\\ &\leq\left(d\left(P\right)+1\right)m\left(r,f\right)+N\left(r,\frac{K'}{K}\right)+S\left(r,f\right)\\ &\leq\left(d\left(P\right)+1\right)m\left(r,f\right)+\overline{N}\left(r,\frac{1}{K}\right)+S\left(r,f\right); \end{aligned} \quad (83)$$

that is,

$$\left(v-d\left(P\right)-2\right)T\left(r,f\right)\leq\overline{N}\left(r,\frac{1}{Q_4\left(z,f\right)-\alpha}\right)+S\left(r,f\right). \quad (84)$$

From (79) and (84), we deduce that

$$\begin{aligned} &\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,1/\left(Q_4\left(z,f\right)-\alpha\right)\right)}{T\left(r,Q_4\left(z,f\right)\right)}\geq 1-\frac{d\left(P\right)+2}{v},\\ &\delta\left(\alpha,Q_4\left(z,f\right)\right)\leq\frac{d\left(P\right)+2}{v}<1. \end{aligned} \quad (85)$$

(b) It suffices to note that we may see f^v as $(f^1)^v$; then Theorem 14(b) follows immediately by Theorem 13.

(c) By using a similar reasoning as [13, Theorem 1], we can rearrange the expression for the differential-difference polynomial $P(z,f)$ by collecting together all terms having the same total degree and then writing $P(z,f)$ in the form $P(z,f) = \sum_{k=0}^{d(P)} b_k(z)f^k(z)$. Now each of the coefficients $b_k(z)$ is a finite sum of products of functions of the form $(f^{(j)}(z+c_i)/f(z))^{\lambda_{i,j}} = (f^{(j)}(z+c_j)/f(z+c_i))^{\lambda_{i,j}}(f(z+c_i)/f(z))^{\lambda_{i,j}}$, with each such product being multiplied by one of the original coefficients $a_\lambda(z)$. We deduce from the logarithmic derivative lemma and Lemmas 4 and 6 that $m(r,b_k) = S(r,f)$. Clearly, $N(r,b_k) = S(r,f)$ holds by (8) and Lemma 6. Thus, $T(r,b_k) = S(r,f)$. Denote

$$L\left(z\right)=Q_4\left(z,f\right)-\alpha\left(z\right)=f^{v}\left(z\right)+\sum_{k=0}^{d(P)}b_k\left(z\right)f^{k}\left(z\right)-\alpha\left(z\right). \quad (86)$$

Assume to the contrary that $\delta(\alpha,Q_4(z,f)) = 1$. Thus, Theorem A yields that

$$\begin{aligned} N\left(r,\frac{1}{L}\right)&=N\left(r,\frac{1}{Q_4\left(z,f\right)-\alpha}\right)\\ &=S\left(r,Q_4\left(z,f\right)\right)=S\left(r,f\right). \end{aligned} \quad (87)$$

Then (8), (86), (87), Lemma 18, and the assumption that $v \geq d(P)+2$ imply that $L(z) \equiv f(z)^v$; that is,

$$P\left(z,f\right)=\sum_{k=0}^{d(P)}b_k\left(z\right)f^{k}\left(z\right)\equiv\alpha\left(z\right). \quad (88)$$

Noting the fact that $T(r,b_k) = S(r,f)$ and $T(r,\alpha) = S(r,f)$, we deduce from Theorem A that (88) is a contradiction. Therefore, we have $\delta(\alpha,Q_4(z,f)) < 1$. □

4. Examples

Example 1. We consider nonhomogeneous differential-difference polynomials

$$\begin{aligned} P_1(z,f) &= f(z)f^2(z+\log 4) - 4f''(z)f(z+\log 2) \\ &\quad\times f'(z+\log 2) + f''^2(z+\log 3), \\ P_2(z,f) &= 3f^3(z)f'^2(z+\log 4) \\ &\quad - 2f'(z)f(z+\log 3)f''^3(z+\log 2) \\ &\quad + f^4(z) - f''^3(z), \\ P_3(z,f) &= f(z)f'(z+\log 2)f''(z+\log 3) - 6f''^2(z) \end{aligned} \tag{89}$$

and a homogeneous differential-difference polynomial

$$\begin{aligned} P_4(z,f) &= f''^3(z+\log 2) - f'(z)f(z+\log 2) \\ &\quad\times f'(z+\log 3) - f(z)f'(z)f''(z), \end{aligned} \tag{90}$$

where $d(P_1) = 3 > 2 = d^*(P_1)$, $d(P_2) = 5 > 3 = d^*(P_2)$, $d(P_3) = 3 > 2 = d^*(P_3)$, and $d(P_4) = 3 = d^*(P_4)$. Clearly, the function $f(z) = e^z$ satisfies (8) and $\sigma_2(f) = 0 < 1$. Then we have

$$\begin{aligned} d^*(P_1)T(r,e^z) + O(1) &= T(r,P_1(z,e^z)) = \frac{2r}{\pi} + O(1) \\ &< d(P_1)T(r,e^z) + O(1), \\ d^*(P_2)T(r,e^z) + O(1) &< T(r,P_2(z,e^z)) = \frac{4r}{\pi} + O(1) \\ &< d(P_2)T(r,e^z) + O(1), \\ d^*(P_3)T(r,e^z) + O(1) &< T(r,P_3(z,e^z)) = \frac{3r}{\pi} + O(1) \\ &= d(P_3)T(r,e^z) + O(1), \\ d^*(P_4)T(r,e^z) + O(1) &= T(r,P_4(z,e^z)) = \frac{3r}{\pi} + O(1) \\ &= d(P_4)T(r,e^z) + O(1). \end{aligned} \tag{91}$$

This example shows that (9) is best possible.

Example 2. Consider $f(z) = e^z$ again. Then the homogeneous case $P_4(z,f)$ in Example 1 also illustrates Theorem 9(a). And the nonhomogeneous differential-difference polynomials $P_i(z,f)$, $i = 1,2,3$, in Example 1 also illustrate Theorem 9(b), where $\delta(\alpha,P_1(z,f)) = 0$, $\delta(\alpha,P_2(z,f)) \le 1/4 < 2/3 = 1-((2d^*(P_2)-d(P_2))/d^*(P_2))$, and $\delta(\alpha,P_3(z,f)) \le 1/3 < 1/2 = 1-((2d^*(P_3)-d(P_3))/d^*(P_3))$. Next, we consider the nonhomogeneous differential-difference polynomial

$$\begin{aligned} P_5(z,f) &= f'(z)f(z+\log 2) - f^2(z) \\ &\quad + f'(z+\log 3) - 3f(z) + 1, \end{aligned} \tag{92}$$

where $d(P_5) = 2, d^*(P_5) = 0$. Clearly, $\delta(1,P_5(z,f)) = \delta(1,e^{2z}+1) = 1$. Note that $2d^*(P_5) > d(P_5)$ fails; then this example shows that the assumption "$2d^*(P) > d(P)$" cannot be omitted in Theorem 9(b).

Example 3. We consider the differential-difference polynomials

$$\begin{aligned} Q_1(z,f) &= (f^2)^2 P_6(z,f) \\ &= f^4(z)\left(f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi)\right. \\ &\qquad \left. + f^2(z+\pi)\right), \\ Q_2(z,f) &= f^2 P_6(z,f) \\ &= f^2(z)\left(f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi)\right. \\ &\qquad \left. + f^2(z+\pi)\right), \end{aligned} \tag{93}$$

and the function $f(z) = \sin z$. On the one hand, $N(r,f) = S(r,f)$, $\sigma_2(f) = 0 < 1$, and $uv_{Q_1} > d(P_6)$ and $v_{Q_2} + 2d^*(P_6) > d(P_6)$ hold, where $v_{Q_1} = v_{Q_2} = u = 2$ and $d(P_6) = 3 > 2 = d^*(P_6)$. On the other hand, $\delta(\alpha,Q_1(z,f)) \le 1-(11/14) < 1-(1/14) = 1-(u-1)(uv_{Q_1}-d(P))/u(uv_{Q_1}+d(P)) < 1$ and $\delta(\alpha,Q_2(z,f)) < 1$ hold. This example shows that Theorems 10 and 11 may hold.

Example 4. We consider the differential-difference polynomials

$$\begin{aligned} Q_4^{(1)}(z,f) &= (f^2)^4 + P_7(z,f) = f^8 + P_7(z,f) \\ &= f^8(z) + f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi), \\ Q_4^{(2)}(z,f) &= f^2 + P_7(z,f) \\ &= f^2(z) + f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi), \\ Q_4^{(3)}(z,f) &= 2f^3 + P_7(z,f) \\ &= 2f^3(z) + f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi), \\ Q_4^{(4)}(z,f) &= f^4 + P_7(z,f) \\ &= f^4(z) + f'\left(z+\frac{\pi}{2}\right)f(z+\pi)f''(z+2\pi), \end{aligned} \tag{94}$$

and the function $f(z) = \sin z$ again. On the one hand, $Q_4^{(1)}(z,f)$ satisfies $(u-1)uv_{Q_4^{(11)}}/(2u-1) > d(P_7)$ and $v_{Q_4^{(12)}} - 2 > (v_{Q_4^{(12)}}-1)v_{Q_4^{(12)}}/(2v_{Q_4^{(12)}}-1) > d(P_7)$, respectively, where $u = 4$, $v_{Q_4^{(11)}} = 2$, $v_{Q_4^{(12)}} = 8$, $d(P_7) = d^*(P_7) = 3$, and, for $i = 2,3,4$, $Q_4^{(i)}(z,f)$ satisfies $2\min\{d^*(P_7), v_{Q_4^{(i)}}\} > \max\{d(P_7), v_{Q_4^{(i)}}\}$, where $v_{Q_4^{(i)}} = i$. On the other hand, $\delta(\alpha,Q_4^{(i)}(z,f)) < 1$, $i = 1,2,3,4$, hold. This example shows that Theorems 13–15 may hold. Moreover, this example also shows the assumption "$N(r, 1/f) = S(r, f)$" is not necessary to Theorems 14(c) and 15, but it is regrettable for us not removing it in our proofs.

Example 5. We consider the differential-difference polynomials

$$\begin{aligned} R_1(z,f) &= f^2 P_8(z,f) \\ &= f^2(z)\left(f^2(z+\pi) + \frac{1}{\sin^2 2z} f'^2\left(z+\frac{\pi}{2}\right)\right), \\ R_2(z,f) &= f^7 + P_9(z,f) \\ &= f^7(z) + \sin 2z f'\left(z+\frac{\pi}{2}\right) f^2\left(z+\frac{\pi}{2}\right) \\ &\quad + f'^2\left(z+\frac{\pi}{2}\right) f\left(z+\frac{3\pi}{2}\right) \\ &\quad + zf(z) f\left(z+\frac{\pi}{2}\right), \end{aligned} \tag{95}$$

and the function $f(z) = e^{\sin^2 z}$. On the one hand, $R_1(z,f)$ satisfies $\nu_{R_1} + 2d^*(P_8) > d(P_8)$, and $R_2(z,f)$ satisfies $\nu_{R_2} - 2 > (\nu_{R_2} - 1)\nu_{R_2}/(2\nu_{R_2} - 1) > d(P_9)$, respectively, where $\nu_{R_1} = 2$ and $d(P_8) = d^*(P_8) = 2$, and $\nu_{R_2} = 7$ and $d(P_9) = 3$. On the other hand, $\delta(e^2, R_1(z,f)) = \delta(ez, R_2(z,f)) = 1$ hold, showing that Theorems 11 and 14 fail. Noting that the function $f(z) = e^{\sin^2 z}$ satisfies $\sigma_2(f) = 1$, we know that the assumption "$\sigma_2(f) < 1$" is essential for Theorems 11 and 14. In fact, it is also essential for our other results in the whole paper, but it is unnecessary to give examples one by one.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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