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Research Article

Stability Conditions of Second Order Integrodifferential Equations with Variable Delay

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We investigate integrodifferential functional differential equations $\ddot{x} + f(t,x,\dot{x})\dot{x} + \int_{t-r(t)}^{t} a(t,s)g(x(s))ds = 0$ with variable delay. By using the fixed point theory, we obtain conditions which ensure that the zero solution of this equation is stable under an exponentially weighted metric. Then we establish necessary and sufficient conditions ensuring that the zero solution is asymptotically stable. We will give an example to apply our results.

1. Introduction

Functional differential equations have many applications in control theory, biology, and so on. The stability of the solution of functional differential equations has been a hot issue for researchers for many years. It is well known that Lyapunov's direct method has been widely applied to study the stability problems for a long time; see, for example, [1, 2]. Recently, many authors have applied the fixed points theory to study the stability of solution of integral equations and several functional differential equations with variable delays; see, for example, [3–5] and the references therein.

In [6], Levin and Nohel investigated the behavior of solution of a nonlinear equation

$$\dot{x}(t) = -\frac{1}{L}\int_{t-L}^{t} (L-(t-\tau))\, g(x(\tau(t)))\, d\tau, \quad (0 \le t < \infty), \tag{1}$$

where L is a constant. This equation was equivalent to

$$\ddot{x}(t) + g(x(t)) = \frac{1}{L}\int_{t-L}^{t} g(x(\tau))\, d\tau, \quad (0 \le t < \infty). \tag{2}$$

Burton studied stability of a nonconvolution equation

$$\dot{x} = -\int_{t-r}^{t} a(t,s)\, g(x(s))\, ds, \tag{3}$$

where r was a positive constant. He gave conditions on functions a and g to ensure that the zero solution was asymptotically stable by applying fixed point theorem; see [7].

Becker and Burton studied the following differential equation:

$$\dot{x} = -\int_{t-r(t)}^{t} a(t,s)\, g(x(s))\, ds \tag{4}$$

and equation

$$\dot{x} = -a(t)\, g(x(t-r(t))), \tag{5}$$

for $t \ge 0$, where $r(t) : [0,\infty) \to [0,\infty)$, $a(t,s) : [0,\infty) \times [-r(0),\infty) \to R$, $g : R \to R$, $a(t) : [0,+\infty) \to [0,+\infty)$ are continuous. In addition, they assumed that

(A_1) $r(t)$ is differentiable;

(A_2) the function $t - r(t) : [0,\infty) \to [-r(0),\infty)$ is strictly increasing;

(A_3) $t - r(t) \to \infty$ as $t \to \infty$.

They obtained sufficient conditions ensuring that the zero solution was asymptotically stable by changing the supremum metric to an exponentially weighted metric. Moreover, they hoped to relax condition (A_2); see, for example, [8].

Jin and Luo succeeded in eliminating condition (A_2) in their work; they did not need the condition that $t - r(r)$

was invertible. Moreover, they established necessary and sufficient conditions that could ensure that the zero solution of this equation was asymptotically stable; see, for example, [9]. Dung [10] studied linear case of this equation and gave new stability results by using a new expression of the solution. Other results on fixed points and stability properties in equations with variable delays can be found in [3, 11] and the references therein.

Levin and Nohel [12] studied the global asymptotic stability of a class of nonlinear systems

$$\ddot{x}(t) + h(t, x, \dot{x})\dot{x} + f(x) = e(t). \tag{6}$$

Burton [13] used the fixed points theory to study the stability problems of some second order functional differential equations. He considered the equation

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t) g(x(t-L)) = 0, \tag{7}$$

where L is a positive constant. He obtained sufficient conditions under which each solution $x(t)$ satisfied $(x(t), \dot{x}(t)) \to 0$ via the fixed point theorem.

We generalized the above equation to an equation with a variable delay [11]

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t) g(x(t-\tau(t))) = 0 \tag{8}$$

and obtained some results on asymptotic stability of the zero solution. Before we introduce our new results we recall the main results in [11]. There are *basic assumptions* on the delay function $\tau(t)$.

($\mathscr{A}$) $t-\tau(t)$ is strictly increasing and $\lim_{t\to\infty} t-\tau(t) = \infty$. The inverse of $t-\tau(t)$ exists and denotes it by $p(t)$. Moreover, $0 \le b(t) \le M$ for some constant $M > 0$.

The main results in [11] can be stated as follows.

Theorem 1. *Suppose* ($\mathscr{A}$) *and the following conditions.*

(i) *There exists a constant* $l > 0$ *such that* $g(x)$ *satisfies the Lipschitz condition on* $[-l, l]$. *The function* $g(x)$ *is odd and is strictly increasing on* $[-l, l]$, *and* $x - g(x)$ *is nondecreasing on* $[0, l]$.

(ii) *There exist an* $\alpha \in (0,1)$ *and a continuous function* $a(t) : [0,\infty) \to [0,\infty)$ *such that* $f(t, x, y) \ge a(t)$ *for* $t \ge 0, x \in \mathbb{R}, y \in \mathbb{R}$, $\int_0^\infty a(t)dt = \infty$, *and*

$$\begin{aligned} &2\sup_{t\ge0}\int_t^{p(t)}\int_0^\infty e^{-\int_s^{w+s}a(v)dv}b(s)\,dw\,ds \\ &\quad + 2\sup_{t\ge0}\int_0^t\int_{t-s}^\infty e^{-\int_s^{w+s}a(v)dv}b(s)\,dw\,ds \le \alpha. \end{aligned} \tag{9}$$

(iii) *There exist constants* $a_0 > 0$ *and* $Q > 0$ *such that, for each* $t \ge 0$, *if* $J \ge Q$, *then*

$$\int_t^{t+J} a(v)\,dv \ge a_0 J. \tag{10}$$

(iv) *There exist continuous functions* $F : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ *and* $c(t) : [0,\infty) \to [0,\infty)$ *such that* $\forall t \ge 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $f(t, x, y) \le F(x, y)c(t)$. *The function* $g'(x)$ *is continuous on* $[-l, l]$, $g'(0) \ne 0$.

The following statements hold.

(a) *If, for each* $\gamma > 0$,

$$\iint_0^\infty e^{-\int_s^{w+s}\gamma c(v)}b(s)\,dw\,ds = \infty, \tag{11}$$

then the zero solution of (8) *is asymptotically stable.*

(b) *If the zero solution of* (8) *is asymptotically stable, then*

$$\iint_0^\infty e^{-\int_s^{w+s}a(v)dv}b(s)\,dw\,ds = \infty. \tag{12}$$

This theorem failed to offer a necessary and sufficient condition which ensures that the zero solution was asymptotically stable. In this paper, we will establish a necessary and sufficient condition which ensures that the zero solution of related equation is asymptotically stable.

In this paper we consider equation

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + \int_{t-r(t)}^t a(t,s) g(x(s))\,ds = 0 \tag{13}$$

for $t \ge 0$, where $r(t) : [0,\infty) \to [0,\infty)$. $a(t,s) : [0,\infty) \times [-r(0),\infty) \to R, g : R \to R, a(t) : R^+ \to R, f : R\times R\times R \to R^+$ are all continuous, where $R^+ = [0, +\infty)$. We assume that $t - r(t) \to \infty$ as $t \to \infty$.

For each $t_0 \ge 0$, define $m(t_0) = \inf\{s - r(s) : s \ge t_0\}$. Set $C(t_0) = C([m(t_0), t_0], R)$ with the continuous function norm $\|\cdot\|$, where $\|\psi\| = \sup\{|\psi(s)| : m(t_0) \le s \le t_0\}$. It will cause no confusion even though we use $\|\phi\|$ to express the supremum on $[m(t_0), \infty)$ later. It is well known that in [2], for a given continuous function ϕ, there exists a solution of (13) on an interval $[t_0, T)$; if the solution remains bounded, then $T = \infty$. We denote by $(x(t), y(t))$ the solution $(x(t, t_0, \phi), y(t, t_0, \phi))$.

We will give a necessary and sufficient condition ensuring that the zero solution of this equation is asymptotically stable. To our knowledge, there are few results about its stability. From the solution $(x(t), y(t))$, we denote $A(t) := f(t, x(t), y(t))$. We can write (13) as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -A(t)y - \int_{t-r(t)}^t a(t,s) g(x(s))\,ds. \end{aligned} \tag{14}$$

For each $t_0 \ge 0$, let $m(t_0) = \inf\{s - r(s) : s \ge t_0\}, C(t_0) = C([m(t_0), t_0], R)$ with the continuous function norm $\|\cdot\|$, where $\|\psi\| = \sup\{|\psi(s)| : m(t_0) \le s \le t_0\}$.

This paper is organized as follows. In the next section we will state our main results. Their proofs will be given in Sections 3 and 4. We will give an example to apply our results in Section 5.

2. Statement of Main Results

We make the following *basic assumptions* on the delay function $r(t)$ of (13).

($\mathscr{A}_1$) $\lim_{t\to\infty} t - r(t) = \infty$. $p(t)$ is the inverse of $t - r(t)$. $G(t,t) = \int_t^{p(t)} a(u,t)du$ and $G(t,s) = \int_t^{p(s)} a(u,s)du$. There exists a constant $M > 0$ such that $|G(t,t)| \le M$. The following are our main results.

Theorem 2. *Assume that ($\mathscr{A}_1$) holds and the following conditions hold.*

(i) *There exists a constant $l > 0$ such that $g(x)$ satisfies Lipschitz condition on $[-l,l]$. $g(x)$ is odd and it is strictly increasing on $[-l,l]$, and $x - g(x)$ is nondecreasing on $[0,l]$.*

(ii) *There exist an $\alpha \in (0,1)$ and a continuous function $a(t) : [0,\infty) \to [0,\infty)$ such that $f(t,x,y) \ge a(t)$ for $t \ge 0$. For $\forall t \ge 0$, $\int_{t-r(t)}^t |G(t,v)|dv$ is increasing with respect to t, and $\int_{t-r(t)}^t |a(t,v)|dv$ is bounded and for $t \ge 0$ and for $t \ge 0$, $x \in R$, $y \in R$,*

$$2\int_t^{p(t)}\int_0^\infty e^{-\int_s^{w+s} a(v)dv} G(s,s)\,dw\,ds + 2\int_0^t\int_{t-s}^\infty e^{-\int_s^{w+s} a(v)dv} G(s,s)\,dw\,ds + 2\int_0^t e^{-\int_s^t a(v)dv}\int_{s-r(s)}^s |G(s,v)|\,dv\,ds \le \alpha. \tag{15}$$

(iii) *There exist constants $a_0 > 0$ and $Q > 0$ such that, for each $t \ge 0$, if $J \ge Q$, then*

$$\int_t^{t+J} a(s)\,ds \ge a_0 J. \tag{16}$$

Then the zero solution of (13) *is stable.*

In addition, we have the following.

Theorem 3. *Assume that $\lim_{t\to\infty} t - r(t) = \infty$ and there exists a function $h(t) \in C(R^+, R^+)$ such that for $t \ge 0$ the following conditions hold.*

(i) *There exists constant $l > 0$ such that $g(x)$ satisfies Lipschitz condition on $[-l,l]$. L is the Lipschitz constant. $g(x)$ is odd and it is strictly increasing on $[-l,l]$, and $x - g(x)$ is nondecreasing on $[0,l]$.*

(ii) *There exist a constant $\alpha \in (0,1)$ and a continuous function $a(t) : R^+ \to R^+$ such that $f(t,x,y) \ge a(t)$. For $\forall t \ge 0$, $\int_{t-r(t)}^t |a(t,v)|dv$ is bounded and for $t \ge 0$, $x \in R$, $y \in R$,*

$$2L\int_t^{p(t)}\int_0^\infty e^{-\int_s^{w+s} a(v)dv} G(s,s)\,dw\,ds + 2L\int_0^t\int_{t-s}^\infty e^{-\int_s^{w+s} a(v)dv} G(s,s)\,dw\,ds + 2L\int_0^t e^{-\int_u^t h(v)dv}\left(\int_{u-r(u)}^u |a(u,v)|\,dv\right)du + L\int_0^{p(t)} e^{-\int_{w-r(w)}^t h(v)dv}\int_0^\infty e^{-\int_w^{w+u} a(s)ds}|G(u,u)|\,du\,dw + \int_0^t e^{-\int_u^t h(v)dv}|h(u) - H(u)|\,du \le \alpha. \tag{17}$$

(iii) *There exist constants $a_0 > 0$ and $Q > 0$ such that, for each $t \ge 0$, if $J \ge Q$, then*

$$\int_t^{t+J} a(s)\,ds \ge a_0 J. \tag{18}$$

Then the zero solution of (13) *is asymptotically stable if and only if*

(iv)

$$\int_0^t h(s)\,ds \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{19}$$

Remark 4. We give some new notations:

$$\begin{gathered} e^{-\int_u^t A(s)ds} G(u,u) \triangleq C(t,u), \\ \int_{t_0}^\infty C(u + t - t_0, t)\,du \triangleq D(t) \ge 0, \\ \dot{x}(t_0) e^{-\int_{t_0}^t A(s)ds} \triangleq B(t), \qquad \frac{D(t)}{1 - \dot{r}(t)} \triangleq \widetilde{D}(t), \\ \widetilde{D}(p(t)) \triangleq H(t), \\ \int_{t_0+t-s}^\infty C(u + s - t_0, s)\,du \triangleq E(t,s) \ge 0. \end{gathered} \tag{20}$$

3. Proof of Theorem 2

In this section, we will prove Theorem 2 by applying the fixed point theory. We will give the expression of the solution of the related equation. The following result can be found in [8].

Lemma 5. *Let the function $p : [-r(0),\infty) \to [0,\infty)$ denote the inverse of $t - r(t)$. Then*

$$\dot{x}(t) = -\int_{t-r(t)}^t a(t,s)\,g(x(s))\,ds \tag{21}$$

is equivalent to

$$\dot{x}(t) = -G(t,t)\,g(x(t)) + \frac{d}{dt}\int_{t-r(t)}^{t} G(t,s)\,g(x(s))\,ds. \tag{22}$$

Lemma 6. *Let $\psi : [m(t_0), t_0] \to R$ be a given continuous function; if $(x(t), y(t))$ is the solution of (13) on $[t_0, T_1)$ satisfying $x(t) = \psi(t)$, $t \in [m(t_0), t_0]$, and $y(t_0) = \dot{x}(t_0)$, then $x(t)$ is the solution of the following integral equation:*

$$\begin{aligned}
x(t) &= \psi(t_0)\,e^{-\int_{t_0}^{t} H(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds} B(u)\,du \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds} H(u)\left[x(u) - g(x(u))\right]du \\
&+ \int_{t_0}^{t} E(t,s)\,g(x(s))\,ds + \int_{t-r(t)}^{t} H(s)\,g(x(s))\,ds \\
&- e^{-\int_{t_0}^{t} H(s)ds} \cdot \int_{t_0-r(t_0)}^{t_0} H(s)\,g(\psi(s))\,ds \\
&- \int_{t_0}^{t}\left[\int_{u-r(u)}^{u} H(s)\,g(x(s))\,ds\right] e^{-\int_{u}^{t} H(s)ds} H(u)\,du \\
&- \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)\,g(x(s-r(s)))\,ds\right] e^{-\int_{u}^{t} H(s)ds} H(u)\,du \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds}\left(\int_{u-r(u)}^{u} G(u,v)\,g(x(v))\,dv\right)du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds} e^{-\int_{t_0}^{u} A(s)ds} \\
&\quad \times \left(\int_{t_0-r(t_0)}^{t_0} G(t_0,v)\,g(\psi(v))\,dv\right)du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} H(s)ds}\left[\int_{t_0}^{u} e^{-\int_{s}^{u} A(v)dv} A(s)\right. \\
&\quad \left. \times \left(\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds\right]du.
\end{aligned} \tag{23}$$

Proof. We apply the variation of parameters formula to the second equation of (14); then we obtain

$$\begin{aligned}
\dot{x}(t) &= \dot{x}(t_0)\,e^{-\int_{t_0}^{t} A(s)ds} \\
&- \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv}\left(\int_{s-r(s)}^{s} a(s,v)\,g(x(v))\,dv\right)ds.
\end{aligned} \tag{24}$$

Equation (24) can be written as

$$\begin{aligned}
\dot{x}(t) &= B(t) \\
&- \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv}\left(G(s,s)\,g(x(s))\right. \\
&\quad \left. - \frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds.
\end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned}
\dot{x}(t) &= B(t) - \int_{t_0}^{t} C(t,s)\,g(x(s))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv}\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds.
\end{aligned} \tag{26}$$

Since $|G(t,t)| \le M$, we have

$$\begin{aligned}
\left|\int_{t-s+t_0}^{\infty} C(u+s-t_0,s)\,du\right| &\le \int_{t-s}^{\infty} e^{-\int_{s}^{w+s} A(v)dv}\,|G(s,s)|\,dw \\
&= \int_{t-s}^{Q} e^{-\int_{s}^{w+s} A(v)dv}\,|G(s,s)|\,dw \\
&\quad + \int_{Q}^{\infty} e^{-\int_{s}^{w+s} A(v)dv}\,|G(s,s)|\,dw, \\
\int_{Q}^{\infty} e^{-\int_{s}^{w+s} A(v)dv}\,|G(s,s)|\,dw &\le M\frac{e^{-a_0 Q}}{a_0}.
\end{aligned} \tag{27}$$

This implies that the integral $\int_{t-s+t_0}^{\infty} C(u+s-t_0,s)du$ is convergent. Hence, we have

$$\begin{aligned}
\dot{x}(t) &= B(t) - g(x(t-r(t)))\,D(t) \\
&+ \frac{d}{dt}\int_{t_0}^{t} E(t,s)\,g(x(s-r(s)))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv}\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds.
\end{aligned} \tag{28}$$

Then we have

$$\begin{aligned}
\dot{x}(t) &= B(t) - \widetilde{D}(p(t))\,g(x(t)) \\
&+ \frac{d}{dt}\int_{t-r(t)}^{t} \widetilde{D}(p(s))\,g(x(s))\,ds \\
&+ \frac{d}{dt}\int_{t_0}^{t} E(t,s)\,g(x(s-r(s)))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv}\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds,
\end{aligned} \tag{29}$$

$$\begin{aligned}\dot{x}(t) &= B(t) - H(t)x(t) + H(t)[x(t) - g(x(t))] \\ &\quad + \frac{d}{dt}\int_{t-r(t)}^{t} H(s)\,g(x(s))\,ds \\ &\quad + \frac{d}{dt}\int_{t_0}^{t} E(t,s)\,g(x(s-r(s)))\,ds \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t A(v)dv}\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds.\end{aligned} \tag{30}$$

For $\forall t \in [t_0, T_1]$, by the variation of parameters formula, we obtain that

$$\begin{aligned} x(t) &= \psi(t_0)\,e^{-\int_{t_0}^t H(s)ds} + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} B(u)\,du \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds}\left[\frac{d}{du}\int_{u-r(u)}^{u} H(s)\,g(x(s))\,ds\right]du \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} H(u)\,[x(u) - g(x(u))]\,du \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds}\left[\frac{d}{du}\int_{t_0}^{u} E(u,s)\,g(x(s-r(s)))\,ds\right]du. \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds}\left[\int_{t_0}^{u} e^{-\int_s^u A(v)dv}\right. \\ &\qquad\qquad \left.\times\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)\,g(x(v))\,dv\right)ds\right]du. \end{aligned} \tag{31}$$

If we integrate the last several terms by parts, we have (23). This ends the proof of this lemma. □

Let $(C, \|\cdot\|)$ be the Banach space of bounded continuous functions on $[m(t_0), \infty)$ with the supremum norm. For a given continuous initial function $\psi : [m(t_0), t_0] \to R$, define the set $C_\psi \subset C$ by

$$\begin{aligned} C_\psi &= \{\phi : [m(t_0), \infty) \longrightarrow R \mid \phi \in C,\ \phi(t) = \psi(t), \\ &\qquad t \in [m(t_0), t_0]\}, \\ C_\psi^l &:= \{\phi : [m(t_0), \infty) \longrightarrow R \mid \phi \in C,\ \phi(t) = \psi(t), \\ &\qquad t \in [m(t_0), t_0],\ |\phi(t)| \le l,\ t \ge m(t_0)\}, \end{aligned} \tag{32}$$

where $\psi : [m(t_0), t_0] \to [-l, l]$ is a given initial function and l is a positive constant. We will also use $\|\cdot\|$ to denote the supremum norm of an initial function. Let P_1 be a mapping defined on C_ψ^l as follows: for $\phi \in C_\psi^l$, if $t \in [m(t_0), t_0]$, $(P_1\phi)(t) = \psi(t)$. If $t > t_0$,

$$\begin{aligned} (P_1\phi)(t) &= \psi(t_0)\,e^{-\int_{t_0}^t H(s)ds} + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} B(u)\,du \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} H(u)\,[\phi(u) - g(\phi(u))]\,du \\ &\quad + \int_{t_0}^{t} E(t,s)\,g(\phi(s))\,ds + \int_{t-\tau(t)}^{t} H(s)\,g(\phi(s))\,ds \\ &\quad - e^{-\int_{t_0}^t H(s)ds}\cdot\int_{t_0-r(t_0)}^{t_0} H(s)\,g(\psi(s))\,ds \\ &\quad - \int_{t_0}^{t}\left[\int_{u-r(u)}^{u} H(s)\,g(\phi(s))\,ds\right] e^{-\int_u^t H(s)ds} H(u)\,du \\ &\quad - \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)\,g(\phi(s-r(s)))\,ds\right] e^{-\int_u^t H(s)ds} H(u)\,du \\ &\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds}\left(\int_{u-r(u)}^{u} G(u,v)\,g(\phi(v))\,dv\right)du \\ &\quad - \int_{t_0}^{t} e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^u A(s)ds} \\ &\qquad\qquad \times\left(\int_{t_0-r(t_0)}^{t_0} G(t_0,v)\,g(\psi(v))\,dv\right)du \\ &\quad - \int_{t_0}^{t} e^{-\int_u^t H(s)ds}\left[\int_{t_0}^{u} e^{-\int_s^u A(v)dv} A(s)\right. \\ &\qquad\qquad \left.\times\left(\int_{s-r(s)}^{s} G(s,v)\,g(\phi(v))\,dv\right)ds\right]du. \end{aligned} \tag{33}$$

Note that P_1 may not be a contraction mapping. We solve this problem in Lemma 7 by introducing an exponentially weighted metric.

Lemma 7. *Suppose that there exists a constant $l > 0$ such that $g(x)$ satisfies Lipschitz condition on $[-l, l]$. Then there exists a metric d on C_ψ^l such that*

(i) *the metric space (C_ψ^l, d) is complete;*

(ii) *P_1 is a contraction mapping on (C_ψ^l, d) if P_1 maps C_ψ^l into itself.*

Proof. (i) We change the supremum norm to an exponentially weighted norm $|\phi|_h$, which is defined on C_ψ^l. Let S be the space of all continuous functions $\phi : [m(t_0), \infty) \to R$ such that

$$|\phi|_h := \sup\left\{|\phi(t)|\,e^{-h(t)} : t \in [m(t_0), \infty)\right\} < \infty, \tag{34}$$

where $h(t) = kL\int_{t_0}^{t}[H(s) + D(s) + \int_{s-r(s)}^{s}|G(s,v)|dv]ds$, k is a constant and $k > 7$, and L is the common Lipschitz constant for $x - g(x)$ and $g(x)$. Then $(S, |\cdot|_h)$ is a Banach space. Thus, (S, d) is a complete metric space, where d denotes the induced metric: $d(\phi,\eta) = |\phi - \eta|_h$, where $\phi,\eta \in S$. Under this metric, the space C_{ψ}^{l} is a closed subset of S. Therefore, the metric space (C_{ψ}^{l}, d) is complete.

(ii) Suppose that $P_1 : C_{\psi}^{l} \rightarrow C_{\psi}^{l}$. For $\phi,\eta \in C_{\psi}^{l}$, since $H(t) \geq 0$ and $E(t,s) \geq 0$, then

$$\begin{aligned}
&\left|(P_1\phi)(t) - (P_1\eta)(t)\right| e^{-h(t)} \\
&\leq \int_{t_0}^{t} e^{-\int_u^t H(s)ds} H(u) \\
&\quad \times \left|[\phi(u) - g(\phi(u))] - [\eta(u) - g(\eta(u))]\right| e^{-h(t)} du \\
&\quad + \int_{t_0}^{t} E(t,s)\left|g(\phi(s)) - g(\eta(s))\right| e^{-h(t)} ds \\
&\quad + \int_{t-r(t)}^{t} H(s)\left|g(\phi(s)) - g(\eta(s))\right| e^{-h(t)} ds \\
&\quad + \int_{t_0}^{t}\left[\int_{u-r(u)}^{u} H(s)\left|g(\phi(s)) - g(\eta(s))\right| e^{-h(t)} ds\right] \\
&\quad \times e^{-\int_u^t H(s)ds} H(u)\, du \\
&\quad + \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)\left|g(\phi(s-r(s))) - g(\eta(s-r(s)))\right| \right.\\
&\quad \left.\times e^{-h(t)} ds\right] e^{-\int_u^t H(s)ds} H(u)\, du \\
&\quad + \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)\, g(\phi(s-r(s)))\, ds\right] \\
&\quad \times e^{-h(t)} e^{-\int_u^t H(s)ds} H(u)\, du \\
&\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} e^{-h(t)} x \\
&\quad \times \left(\int_{u-r(u)}^{u} |G(u,v)|\left|g(\phi(v)) - g(\eta(v))\right| dv\right) du \\
&\quad + \int_{t_0}^{t} e^{-\int_u^t H(s)ds} e^{-h(t)} \left(\int_{t_0}^{u} e^{-\int_s^u A(v)dv} A(s)\right. \\
&\quad \times \left(\int_{s-r(s)}^{s} |G(s,v)|\right. \\
&\quad \times \left|g(\phi(v))\right. \\
&\quad \left.\left.\left. - g(\eta(v))\right| dv\right) ds\right) du.
\end{aligned} \tag{35}$$

For $v \leq t$, since $D(t), H(t) \geq 0$, we have

$$\begin{aligned}
h(v) - h(t) &= -kL\int_{v}^{t}\left[H(s) + D(s) + \int_{s-r(s)}^{s}|G(s,v)|\, dv\right] ds \\
&\leq -kL\int_{v}^{t}\int_{s-r(s)}^{s}|G(s,v)|\, dv\, ds.
\end{aligned} \tag{36}$$

For $u \leq t$, since $D(t) \geq 0$, we have

$$h(u) - h(t) \leq -kL\int_{u}^{t} H(s)\, ds. \tag{37}$$

For $s \leq t$,

$$h(s - r(s)) - h(t) \leq -kL\int_{s}^{t} D(u)\, du. \tag{38}$$

Since $E(t,s) \geq 0$, then

$$\begin{aligned}
E(t,s) &= \int_{t-s+t_0}^{\infty} C(u + s - t_0, s)\, du \\
&\leq \int_{t_0}^{\infty} C(u + s - t_0, s)\, du = D(s).
\end{aligned} \tag{39}$$

For $s \leq u$,

$$\int_{s-r(s)}^{s}|G(s,v)|\, dv \leq \int_{u-r(u)}^{u}|G(u,v)|\, dv. \tag{40}$$

Easy calculation shows that

$$\begin{aligned}
&\left|(P_1\phi)(t) - (P_1\eta)(t)\right| e^{-h(t)} \\
&\quad \leq \left\{\frac{1}{kL+1} + \frac{2}{kL} + \int_{t_0}^{t}\frac{1}{kL}e^{-\int_u^t H(s)ds}H(u)\, du \right.\\
&\qquad \left. + \int_{t_0}^{t}\frac{1}{kL}e^{-\int_u^t H(s)ds}H(u)\, du + \frac{2}{kL}\right\} L|\phi - \eta|_h \\
&\quad \leq \frac{7}{k}|\phi - \eta|_h, \quad t > t_0.
\end{aligned} \tag{41}$$

For $t \in [m(t_0), t_0]$, $(P_1\phi)(t) = (P_1\eta)(t) = \psi(t)$. Hence, $d(P_1\phi - P_1\eta) \leq (7/k)d(\phi - \eta)$. Note that $k > 7$; thus P_1 is a contraction mapping on (C_{ψ}^{l}, d). □

We continue to prove Theorem 2. Choose $\psi : [m(t_0), t_0] \rightarrow R$ and $\dot{x}(t_0)$ satisfying $\|\psi\| + |\dot{x}(t_0)| \leq \delta$ such that

$$\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\delta + \delta + g(\delta)\int_{t_0 - r(t_0)}^{t_0} H(s)\, ds \leq (1-\alpha)\, g(l). \tag{42}$$

Since (i) implies that $g(0) = 0$, thus $g(l) \leq l$. Since $g(x)$ satisfies the Lipschitz condition on $[-l, l]$, thus $g(x)$ is continuous on $[-l, l]$, so such a δ exists and $\delta < l$.

By the expression of $(P_1\phi)(t)$, and condition (ii), we have

$$\begin{aligned}\left|(P_1\phi)(t)\right| &\le \delta + g(\delta)\int_{t_0-r(t_0)}^{t_0} H(s)\,ds + [l-g(l)] + \alpha g(l)\\&\quad+\int_{t_0}^{t} e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^{u} A(s)ds}du\cdot\delta\\&\le \delta + g(\delta)\int_{t_0-r(t_0)}^{t_0} H(s)\,ds + \int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds}du\cdot\delta\\&\quad+[l-g(l)]+\alpha g(l).\end{aligned}\tag{43}$$

By condition (iii), some easy computation shows that

$$\begin{aligned}\int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds}du &= \int_{t_0}^{t_0+Q} e^{-\int_{t_0}^{u} A(s)ds}du + \int_{t_0+Q}^{t} e^{-\int_{t_0}^{u} A(s)ds}du\\&\le \int_{t_0}^{t_0+Q} du + \int_{t_0+Q}^{t} e^{-a_0(u-t_0)}du\\&\le Q+\frac{e^{-a_0Q}}{a_0}.\end{aligned}\tag{44}$$

Hence,

$$\left|(P_1\phi)(t)\right| \le (1-\alpha)g(l) + (\alpha-1)g(l) + l = l.\tag{45}$$

Observe that if $t\in[m(t_0),t_0]$, then $(P_1\phi)(t)=\psi(t)$. We obtain that $|(P_1\phi)(t)|\le l$, $t\in[m(t_0),\infty)$. Thus, $P_1\phi : C_\psi^l \to C_\psi^l$. Since we have proved that P_1 is a contraction mapping, hence P_1 has a unique fixed point $x(t)$ and $|x(t)|\le l$.

Recall (24); we have

$$\begin{aligned}\left|y(t)\right| &\le \left|\dot{x}(t_0)\right| e^{-\int_{t_0}^{t} A(s)ds}\\&\quad+\int_{t_0}^{t} e^{-\int_s^t A(v)dv}\int_{s-r(s)}^{s}|a(s,v)|\left|g(x(v))\right|dv\,ds.\end{aligned}\tag{46}$$

Since $\int_{t-r(t)}^{t}|a(t,v)|dv$ is bounded, $\exists$ a constant $N>0$ such that $\int_{t-r(t)}^{t}|a(t,v)|dv\le N$, then

$$\begin{aligned}\left|y(t)\right| &\le \left|x'(t_0)\right| + NLl\cdot\int_{t_0}^{t} e^{-\int_u^t A(s)ds}du\\&< l\left(1+NL\left(Q+\frac{e^{-a_0Q}}{a_0}\right)\right).\end{aligned}\tag{47}$$

It follows that

$$|x(t)|+\left|y(t)\right| < l\left(2+NL\left(Q+\frac{e^{-a_0Q}}{a_0}\right)\right).\tag{48}$$

To show the stability of zero solution, let $\forall\epsilon>0$ be given; we only need to replace ϵ by l. This completes the proof of Theorem 2.

4. Proof of Theorem 3

In this section, we will prove Theorem 3. First of all, we will obtain a new expression of the solution of (13). We multiply $e^{\int_{t_0}^{t} h(s)ds}$ by both sides of (30); then

$$\begin{aligned}x(t) &= \psi(t_0)e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_u^t h(s)ds}B(u)\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}h(u)x(u)\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\frac{d}{du}\int_{u-r(u)}^{u} H(s)g(x(s))\,ds\right]du\\&\quad-\int_{t_0}^{t} e^{-\int_u^t h(s)ds}H(u)g(x(u))\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\frac{d}{du}\int_{t_0}^{u} E(u,s)g(x(s-r(s)))\,ds\right]du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\int_{t_0}^{u} e^{-\int_s^u A(v)dv}\right.\\&\qquad\times\left.\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))\,dv\right)ds\right]du.\end{aligned}\tag{49}$$

We have

$$\begin{aligned}x(t) &= \psi(t_0)e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_u^t h(s)ds}B(u)\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}[h(u)-H(u)]x(u)\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\frac{d}{du}\int_{u-r(u)}^{u} H(s)g(x(s))\,ds\right]du\\&\quad-\int_{t_0}^{t} e^{-\int_u^t h(s)ds}H(u)[g(x(u))-x(u)]\,du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\frac{d}{du}\int_{t_0}^{u} E(u,s)g(x(s-r(s)))\,ds\right]du\\&\quad+\int_{t_0}^{t} e^{-\int_u^t h(s)ds}\left[\int_{t_0}^{u} e^{-\int_s^u A(v)dv}\right.\\&\qquad\times\left.\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))\,dv\right)ds\right]du.\end{aligned}\tag{50}$$

Performing an integration by parts, then we have

$$\begin{aligned}
x(t) &= \psi(t_0) e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} h(u)du} B(s)\, ds \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} [h(u) - H(u)]\, x(u)\, du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} H(u) [g(x(u)) - x(u)]\, du \\
&+ \int_{t_0}^{t} E(t,s)\, g(x(s))\, ds + \int_{t-r(t)}^{t} H(s)\, g(x(s))\, ds - e^{-\int_{t_0}^{t} h(s)ds} \\
&\cdot \int_{t_0-r(t_0)}^{t_0} H(s)\, g(x(s))\, ds - \int_{t_0}^{t} \left[\int_{u-r(u)}^{u} H(s)\, g(x(s))\, ds\right] \\
&\qquad \times e^{-\int_{u}^{t} h(s)ds} h(u)\, du \\
&- \int_{t_0}^{t} \left[\int_{t_0}^{u} E(u,s)\, g(x(s-r(s)))\, ds\right] e^{-\int_{u}^{t} h(s)ds} h(u)\, du \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} \left(\int_{u-r(u)}^{u} G(u,v)\, g(x(v))\, dv\right) du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} \left(\int_{t_0-r(t_0)}^{t_0} G(t_0,v)\, g(\psi(v))\, dv\right) du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} \left[\int_{t_0}^{u} e^{-\int_{s}^{u} A(v)dv} A(s) \right. \\
&\qquad \left. \times \left(\int_{s-r(s)}^{s} G(s,v)\, g(x(v))\, dv\right) ds\right] du.
\end{aligned} \tag{51}$$

We define

$$\begin{aligned}
C_{\psi}^{0} = \{\phi : [m(t_0), \infty) \longrightarrow R \mid \phi \in C,\ \phi(t) = \psi(t), \\
t \in [m(t_0), t_0],\ \phi(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty.\}
\end{aligned} \tag{52}$$

Let P_2 be a mapping defined on C_ψ as follows: for $\phi \in C_\psi$, if $t \in [m(t_0), t_0]$, $(P_2\phi)(t) = \psi(t)$. If $t > t_0$, we define

$$\begin{aligned}
(P_2\phi)(t) &= \psi(t_0) e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} B(u)\, du \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} [h(u) - H(u)]\, \phi(u)\, du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} H(u) [g(\phi(u)) - \phi(u)]\, du \\
&+ \int_{t_0}^{t} E(t,s)\, g(\phi(s))\, ds + \int_{t-r(t)}^{t} H(s)\, g(\phi(s))\, ds - e^{-\int_{t_0}^{t} h(s)ds} \\
&\cdot \int_{t_0-r(t_0)}^{t_0} H(s)\, g(\psi(s))\, ds - \int_{t_0}^{t} \left[\int_{u-r(u)}^{u} H(s)\, g(\phi(s))\, ds\right] \\
&\qquad \times e^{-\int_{u}^{t} h(s)ds} h(u)\, du \\
&- \int_{t_0}^{t} \left[\int_{t_0}^{u} E(u,s)\, g(\phi(s-r(s)))\, ds\right] e^{-\int_{u}^{t} h(s)ds} h(u)\, du \\
&+ \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} \left(\int_{u-r(u)}^{u} G(u,v)\, g(\phi(v))\, dv\right) du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} \left(\int_{t_0-r(t_0)}^{t_0} G(t_0,v)\, g(\psi(v))\, dv\right) du \\
&- \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} \left[\int_{t_0}^{u} e^{-\int_{s}^{u} A(v)dv} A(s) \right. \\
&\qquad \left. \times \left(\int_{s-r(s)}^{s} G(s,v)\, g(\phi(v))\, dv\right) ds\right] du.
\end{aligned} \tag{53}$$

If $\phi \in C_{\psi}^{0}$, since $\int_0^t h(s)ds \to \infty$ as $t \to \infty$, the first term and fourth term of $(P_2\phi)(t) \to 0$, as $t \to \infty$.

Note that

$$\int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} B(u)\, du = \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du. \tag{54}$$

Since $A(t) = f(t, x(t), y(t)) \geq a(t) \geq 0$, then

$$\int_{t_0}^{t} e^{-\int_{u}^{t} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du \leq \int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds} du. \tag{55}$$

For a given $\epsilon > 0$, there exists $T_1 > Q + t_0$ such that $e^{a_0 t_0 - a_0 T} < a_0 \epsilon$. For $T_1 < t$ and $t_0 < Q < T_1 \leq u$, we have

$$\begin{aligned}
\int_{T_1}^{t} e^{-\int_{t_0}^{u} A(s)ds} du &\leq \int_{T_1}^{t} e^{-a_0(u-t_0)} du = \frac{e^{a_0 t_0}}{a_0} \left(e^{-a_0 T_1} - e^{-a_0 t}\right) \\
&< \frac{e^{a_0 t_0 - a_0 T_1}}{a_0} < \epsilon.
\end{aligned} \tag{56}$$

For $t > T_1$, we have

$$\begin{aligned}
\int_{t_0}^{T_1} e^{-\int_{u}^{t} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du &= e^{-\int_{T_1}^{t} h(s)ds} \\
&\cdot \int_{t_0}^{T_1} e^{-\int_{u}^{T_1} h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du.
\end{aligned} \tag{57}$$

Then this term of $(P_2\phi)(t) \to 0$, as $t \to \infty$. Analogously, we can prove other terms of $(P_2\phi)(t) \to 0$, as $t \to \infty$. Then we can easily check that P_2 is a contraction mapping on C_{ψ}^{0} by using condition (ii). By the contraction mapping principle, P_2 has a unique fixed point $x(t)$ in C^0_ψ. Thus, $x(t) \to 0$ as $t \to +\infty$.

Remark.8 By using the new expression (53), we do not need to change the supremum metric to an exponentially weighted metric. We can easily check that P_2 is a contraction mapping.

In order to obtain the asymptotic stability, we now need to show that the zero solution is stable. Let $\epsilon > 0$ be given; we choose $\psi : [m(t_0), t_0] \to R$ and $\dot{x}(t_0)$ satisfying $\|\psi\| + |\dot{x}(t_0)| \leq \delta$ such that

$$\delta + \left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\delta + \int_{t_0 - r(t_0)}^{t_0} H(s)\,ds \cdot L\delta \leq (1-\alpha)\epsilon. \tag{58}$$

By (51), we have

$$|x(t)| \leq \delta + (|\dot{x}(t_0)| + \|\psi\|) \int_{t_0}^{t} e^{-\int_u^t h(s)ds} e^{-\int_{t_0}^{u} A(s)ds} du + \int_{t_0 - r(t_0)}^{t_0} H(s)\,ds \cdot L\delta + \alpha\epsilon. \tag{59}$$

We have obtained that

$$\int_{t_0}^{t} e^{-\int_{t_0}^{u} A(s)ds} du \leq \int_{t_0}^{t_0+Q} du + \int_{t_0+Q}^{t} e^{-a_0(u-t_0)} du \leq Q + \frac{e^{-a_0 Q}}{a_0}. \tag{60}$$

Hence,

$$|x(t)| \leq \epsilon. \tag{61}$$

Recall (24); we have

$$|\dot{x}(t)| \leq |\dot{x}(t_0)| + \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \left(\int_{s-r(s)}^{s} |a(s,v)\,g(x(v))|\,dv\right) ds. \tag{62}$$

It follows that

$$|x(t)| + |y(t)| < \epsilon\left(2 + NL\left(Q + \frac{e^{-a_0 Q}}{a_0}\right)\right). \tag{63}$$

Therefore, the zero solution is stable; since we have obtained that $x(t) \to 0$ as $t \to \infty$, it follows that the zero solution is asymptotically stable.

The necessary condition is as follows: for each $t_0 \geq 0$, we denote

$$K = \sup_{t\geq 0}\left\{e^{-\int_0^t h(s)ds}\right\}. \tag{64}$$

We will prove that

$$\int_0^\infty h(s)\,ds = \infty \tag{65}$$

by way of contradiction. If

$$\int_0^\infty h(s)\,ds < \infty, \tag{66}$$

since $h(t) \geq 0$, $\exists$ a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} h(s)ds = \sigma$, for a certain finite number $\sigma \in R$. Choose η such that $-\eta \leq \int_0^{t_n} h(s)ds \leq \eta$ holds, for $\forall n \geq 1$.

Denote

$$\omega(u) \triangleq h(u)\int_{u-r(u)}^{u} LH(s)\,ds + LH(u) + |h(u) - H(u)| + 2L\int_{u-r(u)}^{u} |G(u,v)|\,dv + L\int_{t_0}^{u} E(u,s)\,ds. \tag{67}$$

By conditions of Theorem 3, we have

$$0 \leq \int_0^{t_n} \omega(u)\, e^{-\int_u^{t_n} h(s)ds} du \leq \alpha. \tag{68}$$

Then

$$\int_0^{t_n} \omega(u)\, e^{\int_0^u h(s)ds} du \leq \alpha e^{\int_0^{t_n} h(s)ds} \leq \alpha e^{\eta}. \tag{69}$$

Thus, the sequence $\{\int_0^{t_n} \omega(u) e^{\int_0^u h(s)ds} du\}$ is bounded; there exists a convergent subsequence; we assume that $\lim_{n\to\infty} \int_0^{t_n} \omega(u) e^{\int_0^u h(s)ds} du = I$, $I \geq 0$. We can choose a positive integer $\overline{k}$ large enough such that

$$\int_{t_{\overline{k}}}^{t_n} \omega(u)\, e^{\int_0^u h(s)ds} du < \frac{\delta_0}{8K} \tag{70}$$

for $\forall n \geq \overline{k}$, where $0 < \delta_0 (\delta_0 < \epsilon)$ satisfying

$$\delta_0 + 2\max\left\{Q + \frac{e^{-a_0 Q}}{a_0}, 1\right\}\delta_0 + \delta_0 \int_{t_{\overline{k}} - r(t_{\overline{k}})}^{t_{\overline{k}}} H(s)\,ds \leq (1-\alpha). \tag{71}$$

Now we consider the solution $x(t) = x(t, \psi, \dot{x}(t_{\overline{k}}))$ of (8) which satisfies

$$\psi(t_{\overline{k}}) = \frac{3\delta_0}{4}, \qquad \dot{x}(t_{\overline{k}}) = \frac{\delta_0}{4}, \qquad |\psi(s)| + |\dot{x}(s)| \leq \delta_0, \quad s \leq t_{\overline{k}}. \tag{72}$$

We can obtain that $|x(t)| \leq 1$ by a similar argument with (61) if we replace ϵ by 1. Then

$$\begin{aligned} &\psi(t_{\bar{k}}) - \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} h(s)ds} e^{-\int_{t_{\bar{k}}}^u A(s)ds} \\ &\quad \times \left(\int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} G(t_{\bar{k}}, v) g(\psi(v)) dv \right) du \\ &\quad - \int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} H(s)\psi(s) ds \geq \psi(t_{\bar{k}}) - \int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} H(s)|\psi(s)| ds \\ &\quad - \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} h(s)ds} e^{-\int_{t_{\bar{k}}}^u A(s)ds} \\ &\quad \times \left(\int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} G(t_{\bar{k}}, v) |g(\psi(v))| dv \right) du \\ &\quad \geq \frac{3\delta_0}{4} - \frac{\delta_0}{2} = \frac{\delta_0}{4}. \end{aligned} \tag{73}$$

By (23) and $|x(t)| \leq 1$, if $t_n \geq t_{\bar{k}}$, we have

$$\begin{aligned} &\left| x(t_n) - \int_{t_{\bar{k}}}^{t_n} E(t_n, s) g(x(s)) ds - \int_{t_n - r(t_n)}^{t_n} H(s) g(x(s)) ds \right| \\ &\quad \geq \left| \psi(t_{\bar{k}}) e^{-\int_{t_{\bar{k}}}^{t_n} h(s)ds} + \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} h(s)ds} B(u) du \right. \\ &\qquad - \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} h(s)ds} e^{-\int_{t_{\bar{k}}}^u A(s)ds} \\ &\qquad \times \left(\int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} G(t_{\bar{k}}, v) g(\psi(v)) dv \right) du \\ &\qquad \left. - \int_{t_{\bar{k}}-r(t_{\bar{k}})}^{t_{\bar{k}}} H(s)\psi(s) ds \cdot e^{-\int_{t_{\bar{k}}}^{t_n} h(u)du} \right| \\ &\qquad - \left| \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} h(s)ds} \omega(u) du \right|. \end{aligned} \tag{74}$$

This implies that

$$\begin{aligned} &\left| x(t_n) - \int_{t_{\bar{k}}}^{t_n} E(t_n, s) g(x(s)) ds - \int_{t_n - r(t_n)}^{t_n} H(s) g(x(s)) ds \right| \\ &\quad \geq e^{-\int_{t_{\bar{k}}}^{t_n} h(s)ds} \left[\left| \frac{\delta_0}{4} + \int_{t_{\bar{k}}}^{t_n} e^{\int_{t_{\bar{k}}}^u h(s)ds} \dot{x}(t_{\bar{k}}) e^{\int_{t_{\bar{k}}}^u A(s)ds} du \right| \right. \\ &\qquad \left. - \int_{t_{\bar{k}}}^{t_n} \omega(u) e^{\int_{t_{\bar{k}}}^u h(s)ds} du \right]. \end{aligned} \tag{75}$$

Note that $\dot{x}(t_{\bar{k}}) = \delta_0/4 > 0$; it follows that

$$\begin{aligned} &\left| x(t_n) - \int_{t_{\bar{k}}}^{t_n} E(t_n, s) g(x(s)) ds - \int_{t_n - r(t_n)}^{t_n} H(s) g(x(s)) ds \right| \\ &\quad \geq e^{-\int_{t_{\bar{k}}}^{t_n} h(s)ds} \left[\frac{\delta_0}{4} - e^{-\int_0^{t_{\bar{k}}} h(s)ds} \cdot \int_{t_{\bar{k}}}^{t_n} \omega(u) e^{\int_0^u h(s)ds} du \right] \\ &\quad \geq \frac{\delta_0}{8} e^{-\int_{t_{\bar{k}}}^{t_n} h(s)ds} \geq \frac{\delta_0}{8} e^{-2\eta} > 0. \end{aligned} \tag{76}$$

If the zero solution $x(t)$ is asymptotically stable, then $x(t) \to 0$ as $t \to \infty$. By the mean value theorem, easy computation shows that

$$\begin{aligned} &\left| \int_{t_n - r(t_n)}^{t_n} H(s) g(x(s)) ds \right| \\ &\quad = \left| g(x(\varpi)) \int_{t_n - r(t_n)}^{t_n} H(s) ds \right| \leq |x(\varpi)|. \end{aligned} \tag{77}$$

So this term tends to 0 as $t_n \to \infty$.

Easy computation shows that

$$x(t_n) - \int_{t_{\bar{k}}}^{t_n} E(t_n, s) g(x(s)) ds - \int_{t_n - r(t_n)}^{t_n} H(s) g(x(s)) ds \tag{78}$$

tends to 0 as t_n tends to ∞. This is a contradiction to (76).

Hence,

$$\int_0^\infty h(s) ds = \infty. \tag{79}$$

This completes the proof of Theorem 3.

5. An Example

In this section, we will give an example to apply our results. Let

$$\begin{aligned} &r(t) = \frac{t}{2}, \qquad p(t) = 2t, \qquad a(t,s) = a^* e^{-(t-s)}, \\ &\int_{t-r(t)}^t |a(t,v)| dv = \int_{t/2}^t a^* e^{-(t-v)} dv = a^* \left(1 - e^{-3t/2}\right) \leq a^*, \\ &\qquad \text{for } t \geq 0, \end{aligned} \tag{80}$$

where a^* is a very small positive constant. Consider

$$\begin{aligned} G(t,v) &= \int_t^{p(v)} a(u,v) du \\ &= \int_t^{2v} a^* e^{-(u-v)} du = a^* \left(e^{v-t} - e^{-v}\right), \end{aligned} \tag{81}$$

$$\begin{aligned} \int_{t-r(t)}^t |G(t,v)| dv &= \int_{t/2}^t a^* \left| e^{v-t} - e^{-v} \right| dv \\ &= a^* \left(1 + e^{-t} - 2e^{-t/2}\right). \end{aligned}$$

This implies that $\int_{t-r(t)}^{t} |G(t,v)|dv$ is increasing with respect to t for $t \geq 0$. Consider

$$G(s,s) = \int_{s}^{p(s)} a(u,s)\,du = \int_{s}^{2s} a^{*} e^{-(u-s)} du = a^{*}\left(1 - e^{-s}\right). \tag{82}$$

Set $a(t) = 2t$; we have $\int_{t}^{t+J} 2s\,ds = J^2 + 2tJ \geq J^2$ for $J \geq 0$, $t \geq 0$. Consider

$$\begin{aligned} &\int_{t}^{2t}\int_{0}^{\infty} e^{-\int_{s}^{s+w} 2v dv} G(s,s)\,dw\,ds \\ &\quad = \int_{t}^{2t}\int_{0}^{\infty} e^{-w^2-2ws} G(s,s)\,dw\,ds \\ &\quad \leq a^{*}\int_{t}^{2t}\int_{0}^{\infty} e^{-2ws}\,dw\,ds \\ &\quad = a^{*}\int_{t}^{2t}\frac{1}{2s}\,ds = \frac{1}{16}\ln 2, \quad \text{for } t > 0, \\ &\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{s+w} 2v dv} G(s,s)\,dw\,ds \\ &\quad = \int_{0}^{t}\int_{t-s}^{\infty} e^{-w^2-2ws} G(s,s)\,dw\,ds \\ &\quad \leq a^{*}\int_{0}^{t}\int_{t-s}^{\infty} e^{-w^2} e^{-2(t-s)s}\,dw\,ds. \end{aligned} \tag{83}$$

Since $0 \leq s \leq t$, then

$$\int_{t-s}^{\infty} e^{-w^2} dw \leq \int_{0}^{\infty} e^{-w^2} dw = \frac{\sqrt{\pi}}{2}. \tag{84}$$

Thus, we have

$$\begin{aligned} &\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{s+w} 2v dv} G(s,s)\,dw\,ds \\ &\quad \leq a^{*}\frac{\sqrt{\pi}}{2}\int_{0}^{t} e^{-2(t-s)s} ds \\ &\quad = a^{*}\frac{\sqrt{\pi}}{2}\int_{0}^{t} e^{2(s-t/2)^2 - t^2/2} ds \triangleq a^{*}\frac{\sqrt{\pi}}{2} M(t). \end{aligned} \tag{85}$$

Easy computation shows that $\lim_{t\to\infty} M(t) = 0$; thus, $M(t)$ is bounded. Consider

$$\begin{aligned} &\int_{0}^{t} e^{-\int_{s}^{t} 2v dv}\int_{s/2}^{s} |G(s,v)|\,dv\,ds \\ &\quad = \frac{\int_{0}^{t} a^{*} e^{s^2}\left(1 + e^{-s} - 2e^{-s/2}\right) ds}{e^{t^2}} \triangleq a^{*} N(t). \end{aligned} \tag{86}$$

Since $\lim_{t\to\infty} N(t) = 0$, thus $N(t)$ is bounded. Hence, conditions (ii) and (iii) in Theorem 2 hold. Choose $g(x) = (1/3)x^3$. Let $l = 1$; $g(x)$ satisfies the Lipschitz condition on $[-l, l]$. $x - g(x) = x - (1/3)x^3$, $(x - g(x))' = 1 - x^2 \geq 0$ on $[0,1]$. Thus, $x - g(x)$ is nondecreasing on $[0,1]$. Then by Theorem 2 the zero solution of (13) is stable.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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