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Research Article

Global Existence of Solutions to the 2D Incompressible Generalized Liquid Crystal Flow

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We consider the global existence of solutions to the 2D incompressible generalized liquid crystal flow. It is proved that the local solution exists globally with $\beta = 0$, $\alpha \geq 2$.

1. Introduction

In this paper, we consider the following 2D liquid crystal flow:

$$u_t + u \cdot \nabla u + \nabla p + \Lambda^{2\alpha} u = -\nabla d \cdot \Delta d, \tag{1}$$

$$d_t + u \cdot \nabla d + \Lambda^{2\beta} d = -f(d), \tag{2}$$

$$\operatorname{div} u = 0, \tag{3}$$

$$(u, d)|_{t=0} = (u_0, d_0), \tag{4}$$

where $\alpha \geq 0$, $\beta \geq 0$ are real parameters and u is the velocity, d is a vectorial function modeling the orientation of the crystal molecules, and p is the scalar pressure. Here $f(d) := (|d|^2 - 1)d$ and $\Lambda = (-\Delta)^{1/2}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi| \widehat{f}(\xi). \tag{5}$$

When $\alpha = \beta = 1$, it has been shown that (1)–(4) has unique global weak and smooth solutions [1–3]. In [4], global regularity for this system with mixed partial viscosity is proved. Some regularity criteria are established for the system with zero dissipation in [5].

The aim of this paper is to establish the following global regularity for the 2D liquid crystal model with fractional diffusion.

Theorem 1. *Assume $(u_0, d_0) \in H^3(\mathbb{R}^2) \times H^4(\mathbb{R}^2)$. Let (u, d) be the local strong solution to the problem (1)–(4). If α and β satisfy $\beta = 0$, $\alpha \geq 2$, then the 2D liquid crystal model has a unique global classical solution (u, d) satisfying*

$$u \in L^\infty\left(0, T; H^3\left(\mathbb{R}^2\right)\right), \qquad u \in L^2\left(0, T; H^{3+\alpha}\left(\mathbb{R}^2\right)\right),$$
$$d \in L^\infty\left(0, T; H^4\left(\mathbb{R}^2\right)\right). \tag{6}$$

Remark 2. This work is partially motivated by the recent progress on the 2D incompressible MHD system with fractional diffusion; we refer to [6–10] and references therein. In [7], Tran et al. obtained the global regularity of 2D GMHD equations for the following three cases: (1) $\alpha \geq 1$, $\beta \geq 1$; (2) $0 \leq \alpha < 1/2$, $2\alpha + \beta > 2$; (3) $\alpha \geq 2$, $\beta = 0$. Combining them with the result in [10], we know that if $\alpha + \beta \geq 2$, 2D incompressible MHD system with fractional diffusion possesses a global smooth solution. Fan et al. [8] proved the global existence of smooth solutions with $\alpha > 0$, $\beta = 1$. Global regularity for the case $\alpha = 0$, $\beta > 1$ was established by Jiu and Zhao [9] which improves the result in [6]. Very recently, the authors improved the case $\alpha = 0$, $\beta > 1$ for the 2D liquid crystal model in [11].

2. Proof of Theorem 1

It is sufficient to prove Theorem 1 with $\alpha = 2$, $\beta = 0$.

We will prove Theorem 1 if we can demonstrate the boundedness of $\|u\|_{H^3}^2 + \|d\|_{H^4}^2$. In order to reach our purpose, we will show this by contradiction: assume

$$\lim_{t \to T} \sup \|u\|_{H^3}^2 + \|d\|_{H^4}^2 = \infty \tag{7}$$

for some finite time $T > 0$. Our thought is that when T_0 is close enough to T, $\|u\|_{H^3}^2+\|d\|_{H^4}^2$ remains uniformly bounded for $T_0 < t < T$ under such assumption, thus reaching a contradiction.

First, we do L^2 estimate for d. Multiplying (2) by d and using (3), after integration by parts, we see that

$$\frac{1}{2}\frac{d}{dt}\|d\|_{L^2}^2 + \|d\|_{L^4}^4 = \|d\|_{L^2}^2. \tag{8}$$

By using the Gronwall inequality, we have

$$\|d\|_{L^2} + \int_0^T \|d\|_{L^4}^4 d\tau \le C. \tag{9}$$

Then, we will show the L^2 estimate for u and ∇d. Multiplying (1) and (2) by u and $-\Delta d$, respectively, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \left\|\Lambda^2 u\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \nabla f(d)\, \nabla d\, dx \\ &\le -3\int_{\mathbb{R}^2} |d|^2 |\nabla d|^2 dx + \|\nabla d\|_{L^2}^2.\end{aligned} \tag{10}$$

Thanks to Gronwall's inequality and (9), we have

$$\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \int_0^T \left\|\Lambda^2 u\right\|_{L^2}^2 d\tau \le C, \tag{11}$$

which means $\nabla u \in L^2(0,T;\mathrm{BMO})$.

The H^1 estimate for u and H^2 estimate for d will be shown as follows. Multiplying (1) by Δu, applying Δ to (2), multiplying by Δd, and then summing them up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) + \|\Delta\nabla u\|_{L^2}^2 \\ &\le \int_{\mathbb{R}^2} \nabla d\cdot\Delta d\cdot\Delta u - \Delta(u\cdot\nabla d)\cdot\Delta d \\ &\qquad - \Delta f(d)\cdot\Delta d\, dx \\ &\le C\|\Delta d\|_{L^2}^2\|\nabla u\|_{L^\infty} \\ &\quad + \int_{\mathbb{R}^2} -3|d|^2|\Delta d|^2 - d|\nabla d|^2\Delta d + |\Delta d|^2 dx \\ &\le C\|\Delta d\|_{L^2}^2\|\nabla u\|_{L^\infty} - 2\|d\Delta d\|_{L^2}^2 + C\|\nabla d\|_{L^4}^4 + \|\Delta d\|_{L^2}^2 \\ &\le C\|\Delta d\|_{L^2}^2\left(\|\nabla u\|_{L^\infty}+1\right) - 2\|d\Delta d\|_{L^2}^2 \\ &\le C\|\Delta d\|_{L^2}^2\left(\|\nabla u\|_{L^\infty}+1\right).\end{aligned} \tag{12}$$

Let us introduce the following commutator and bilinear estimates established in [12, 13]:

$$\begin{aligned}&\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \\ &\quad\le C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^{s-1}g\right\|_{L^{q_1}} + \|g\|_{L^{p_1}}\|\Lambda^s f\|_{L^{q_1}}\right), \\ &\|\Lambda^s(fg)\|_{L^p} \\ &\quad\le C\left(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right),\end{aligned} \tag{13}$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Now, we do the H^2 estimate for u and H^3 estimate for d. Applying Λ^2 to (1), multiplying by $\Lambda^2 u$, and dealing with (2) in the same way by Λ^3 and $\Lambda^3 d$, after summing them up, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right) + \left\|\Lambda^4 u\right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^2} -\Lambda^2(u\cdot\nabla u)\Lambda^2 u - \Lambda^2(\nabla d\cdot\Delta d)\Lambda^2 u \\ &\qquad - \Lambda^3(u\cdot\nabla d)\Lambda^3 d - \Lambda^3 f(d)\Lambda^3 d\, dx \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{14}$$

Using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and (13), we have the following estimates:

$$\begin{aligned}|I_1| &= \left|\int_{\mathbb{R}^2}\left(\Lambda^2(u\cdot\nabla u) - u\cdot\nabla\Lambda^2 u\right)\Lambda^2 u\, dx\right| \\ &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^2 u\right\|_{L^2}^2, \\ |I_2| &\le C\left\|\Lambda^4 u\right\|_{L^2}\|\nabla d\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4} \\ &\le C\left\|\Lambda^4 u\right\|_{L^2}\|\nabla d\|_{L^2}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le \frac{1}{4}\left\|\Lambda^4 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2, \\ |I_3| &\le C\left|\int_{\mathbb{R}^2}\Lambda^3(u\cdot\nabla d)\Lambda^3 d - u\cdot\nabla\Lambda^3 d\Lambda^3 d\, dx\right| \\ &\le C\int_{\mathbb{R}^2}\left|\Lambda^3 u\right||\Lambda d|\left|\Lambda^3 d\right| + \left|\Lambda^2 u\right|\left|\Lambda^2 d\right|\left|\Lambda^3 d\right| \\ &\qquad + |\Lambda u|\left|\Lambda^3 d\right|^2 dx \\ &=: II_1 + II_2 + II_3.\end{aligned} \tag{15}$$

Now we estimate II_1, II_2, and II_3 one by one:

$$\begin{aligned}II_1 &\le C\left\|\Lambda^3 u\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/4}\left\|\Lambda^4 u\right\|_{L^2}^{3/4}\|\Lambda d\|_{L^2}^{3/4}\left\|\Lambda^3 d\right\|_{L^2}^{5/4} \\ &\le \frac{1}{8}\left\|\Lambda^4 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{2/5}\left\|\Lambda^3 d\right\|_{L^2}^2, \\ II_2 &\le C\left\|\Lambda^2 u\right\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le C\left\|\Lambda^2 u\right\|_{L^2}^{3/4}\left\|\Lambda^4 u\right\|_{L^2}^{1/4}\|\Lambda d\|_{L^2}^{1/4}\left\|\Lambda^3 d\right\|_{L^2}^{7/4} \\ &\le \frac{1}{8}\left\|\Lambda^4 u\right\|_{L^2}^2 + \left\|\Lambda^2 u\right\|_{L^2}^{6/7}\left\|\Lambda^3 d\right\|_{L^2}^2, \\ II_3 &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 d\right\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}
I_4 &= \left\|\Lambda^3 d\right\|_{L^2}^2 - \int_{\mathbb{R}^2} \Lambda^3\left(|d|^2 d\right)\Lambda^3 d \\
&\le \left\|\Lambda^3 d\right\|_{L^2}^2 - 3\left\|d\Lambda^3 d\right\|_{L^2}^2 \\
&\quad + C\int_{\mathbb{R}^2}\left|\Lambda^2 d\right||\Lambda d||d|\left|\Lambda^3 d\right| + C\int_{\mathbb{R}^2}|\Lambda d|^3\left|\Lambda^3 d\right| \\
&=: \left\|\Lambda^3 d\right\|_{L^2}^2 - 3\left\|d\Lambda^3 d\right\|_{L^2}^2 + K_1 + K_2.
\end{aligned} \tag{16}$$

K_1 and K_2 can be estimated as follows:

$$\begin{aligned}
K_1 &\le C\|\Lambda d\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|d\Lambda^3 d\right\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}^{1/4}\left\|\Lambda^3 d\right\|_{L^2}^{3/4}\|\Lambda d\|_{L^2}^{3/4}\left\|\Lambda^3 d\right\|_{L^2}^{1/4}\left\|d\Lambda^3 d\right\|_{L^2} \\
&\le C\left\|\Lambda^3 d\right\|_{L^2}^2 + 3\left\|d\Lambda^3 d\right\|_{L^2}^2, \\
K_2 &\le C\|\Lambda d\|_{L^6}^3\left\|\Lambda^3 d\right\|_{L^2} \\
&\le C\left(\|\Lambda d\|_{L^2}^{2/3}\left\|\Lambda^3 d\right\|_{L^2}^{1/3}\right)^3\left\|\Lambda^3 d\right\|_{L^2} \le C\left\|\Lambda^3 d\right\|_{L^2}^2.
\end{aligned} \tag{17}$$

Combining K_1 and K_2, we have

$$I_4 \le C\left\|\Lambda^3 d\right\|_{L^2}^2. \tag{18}$$

Summing all the above estimates to (14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right) + \left\|\Lambda^4 u\right\|_{L^2}^2 \\
&\quad \le C\left(\|\nabla u\|_{L^\infty} + \left\|\Lambda^2 u\right\|_{L^2}\right)\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right).
\end{aligned} \tag{19}$$

Now, we will show the H^3 estimate for u and H^4 estimate for d. Applying Λ^3 to (1), multiplying by $\Lambda^3 u$, and dealing with (2) in the same way by Λ^4 and $\Lambda^4 d$, after summing them up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^3 u\right\|_{L^2}^2 + \left\|\Lambda^4 d\right\|_{L^2}^2\right) + \left\|\Lambda^5 u\right\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^2} -\Lambda^3(u\cdot\nabla u)\Lambda^3 u - \Lambda^3(\nabla d\cdot\Delta d)\Lambda^3 u \\
&\qquad - \Lambda^4(u\cdot\nabla d)\Lambda^4 d - \Lambda^4 f(d)\Lambda^4 d\, dx \\
&\quad =: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{20}$$

Using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and (13), we have the following estimates:

$$\begin{aligned}
\left|J_1\right| &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 u\right\|_{L^2}^2, \\
\left|J_2\right| &\le C\int_{\mathbb{R}^2}|\Lambda(\nabla d\Delta d)|\left|\Lambda^5 u\right| dx \le \|\Lambda(\nabla d\cdot\Delta d)\|_{L^2}\left\|\Lambda^5 u\right\|_{L^2} \\
&\le C\left\|\Lambda^5 u\right\|_{L^2}\left(\|\Delta d\|_{L^4}^2 + \|\Lambda d\|_{L^4}\left\|\Lambda^3 d\right\|_{L^4}\right) \\
&\le C\left\|\Lambda^5 u\right\|_{L^2}\left(\|\Lambda d\|_{L^2}\left\|\Lambda^4 d\right\|_{L^2} + \|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{1/6}\right. \\
&\qquad \left.\times\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{5/6}\right) \\
&\le C\left\|\Lambda^4 d\right\|_{L^2}^2 + \frac{1}{4}\left\|\Lambda^5 u\right\|_{L^2}^2, \\
\left|J_3\right| &= C\left|\int_{\mathbb{R}^2}\left(\Lambda^4(u\cdot\nabla d) - u\cdot\nabla\Lambda^4 d\right)\Lambda^4 d\, dx\right| \\
&\le C\int_{\mathbb{R}^2}\left|\Lambda^4 u\right||\nabla d|\left|\Lambda^4 d\right| + \left|\Lambda^3 u\right|\left|\Lambda^2 d\right|\left|\Lambda^4 d\right| \\
&\qquad + \left|\Lambda^2 u\right|\left|\Lambda^3 d\right|\left|\Lambda^4 d\right| + |\Lambda u|\left|\Lambda^4 d\right|^2 dx \\
&=: J_{31} + J_{32} + J_{33} + J_{34}.
\end{aligned} \tag{21}$$

Now we estimate J_{31}, J_{32}, J_{33}, and J_{34} one by one:

$$\begin{aligned}
\left|J_{31}\right| &\le C\left\|\Lambda^4 u\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/6}\left\|\Lambda^5 u\right\|_{L^2}^{5/6}\|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{7/6} \\
&\le \frac{1}{8}\left\|\Lambda^5 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{3/7}\left\|\Lambda^4 d\right\|_{L^2}^2, \\
\left|J_{32}\right| &\le C\left\|\Lambda^3 u\right\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/2}\left\|\Lambda^5 u\right\|_{L^2}^{1/2}\|\Lambda d\|_{L^2}^{1/2}\left\|\Lambda^4 d\right\|_{L^2}^{3/2} \\
&\le \frac{1}{8}\left\|\Lambda^5 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{2/3}\left\|\Lambda^4 d\right\|_{L^2}^2, \\
\left|J_{33}\right| &\le C\left\|\Lambda^2 u\right\|_{L^4}\left\|\Lambda^3 d\right\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^3 u\right\|_{L^2}^{5/6}\|u\|_{L^2}^{1/6}\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{11/6} \\
&\le C\left\|\Lambda^3 u\right\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{11/6}, \\
\left|J_{34}\right| &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^4 d\right\|_{L^2}^2.
\end{aligned} \tag{22}$$

The estimate for J_4 is as follows:

$$\begin{aligned}
\left|J_4\right| &= \left\|\Lambda^4 d\right\|_{L^2}^2 - \int_{\mathbb{R}^2}\Lambda^4\left(|d|^2 d\right)\Lambda^4 d \\
&\le \left\|\Lambda^4 d\right\|_{L^2}^2 - 3\int_{\mathbb{R}^2}|d|^2\left|\Lambda^4 d\right|^2 \\
&\quad + C\int_{\mathbb{R}^2}|d||\Lambda d|\left|\Lambda^3 d\right|\left|\Lambda^4 d\right| \\
&\quad + C\int_{\mathbb{R}^2}|d|\left|\Lambda^2 d\right|^2\left|\Lambda^4 d\right| \\
&\quad + C\int_{\mathbb{R}^2}|\Lambda d|^2\left|\Lambda^2 d\right|\left|\Lambda^4 d\right| \\
&=: \left\|\Lambda^4 d\right\|_{L^2}^2 - 3\left\|d\Lambda^4 d\right\|_{L^2}^2 + J_{41} + J_{42} + J_{43}.
\end{aligned} \tag{23}$$

We calculate J_{41}, J_{42}, and J_{43}:

$$\begin{aligned}
\left|J_{41}\right| &\le C\left\|\Lambda^3 d\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|d\Lambda^4 d\right\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{5/6}\|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{1/6}\left\|d\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^4 d\right\|_{L^2}^2 + \frac{3}{2}\left\|d\Lambda^4 d\right\|_{L^2}^2,
\end{aligned}$$

$$
\begin{aligned}
|J_{42}| &\le C\|\Lambda^2 d\|_{L^4}^2 \|d\Lambda^4 d\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}\|\Lambda^4 d\|_{L^2}\|d\Lambda^4 d\|_{L^2} \\
&\le C\|\Lambda^4 d\|_{L^2}^2 + \frac{3}{2}\|d\Lambda^4 d\|_{L^2}^2, \\
|J_{43}| &\le C\|\Lambda^2 d\|_{L^4}\|\Lambda d\|_{L^8}^2\|\Lambda^4 d\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}^{1/2}\|\Lambda^4 d\|_{L^2}^{1/2}\|\Lambda d\|_{L^2}^{3/2}\|\Lambda^4 d\|_{L^2}^{1/2}\|\Lambda^4 d\|_{L^2} \\
&\le C\|\Lambda^4 d\|_{L^2}^2.
\end{aligned} \tag{24}
$$

Combining J_{41}, J_{42}, and J_{43}, we get

$$
J_4 \le C\|\Lambda^4 d\|_{L^2}^2. \tag{25}
$$

Combining the above estimates to (20), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2\right) + \|\Lambda^5 u\|_{L^2}^2 \\
&\quad\le C\left(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^2 u\|_{L^2}\right)\|\Lambda^4 d\|_{L^2}^2 \\
&\qquad + C\left(1 + \|\Lambda^3 u\|_{L^2}\right)\|\Lambda^4 d\|_{L^2}^{11/6}.
\end{aligned} \tag{26}
$$

Now we estimate the term $\int_{T_0}^t \|\Lambda^3 u\|_{L^2}$ by applying the Gronwall inequality to (12):

$$
\begin{aligned}
\int_{T_0}^t \|\Lambda^3 u\|_{L^2}^2(\cdot,\tau)\,d\tau &\le \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \\
&\quad + \int_{T_0}^t \|\Delta\nabla u\|_{L^2}^2(\cdot,\tau)\,d\tau \\
&\le \left(\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2\right) \\
&\quad \times \exp\left(C\int_{T_0}^t 1 + \|\nabla u\|_{L^\infty}(\cdot,\tau)\,d\tau\right).
\end{aligned} \tag{27}
$$

Here $T_0 \in (0,T)$ will be fixed later and we denote $\nabla u_0 := \nabla u(\cdot,T_0)$, $\Delta d_0 := \Delta d(\cdot,T_0)$. Set $A(t) := \max_{\tau\in(T_0,t)}(\|u\|_{H^3}^2 + \|d\|_{H^4}^2)(\tau)$. Now applying the logarithmic inequality [14]

$$
\|\nabla u\|_{L^\infty} \le C\left(1 + \|\nabla u\|_{\mathrm{BMO}}\left(1 + \ln\left(1 + \|u\|_{H^3}^2\right)\right)\right), \tag{28}
$$

we get

$$
\begin{aligned}
&\int_{T_0}^t \|\Lambda^3 u\|_{L^2}^2(\cdot,\tau)\,d\tau \\
&\quad\le C(T_0)\exp\left(C\int_{T_0}^t \|\nabla u\|_{L^\infty}(\cdot,\tau)\,d\tau\right) \\
&\quad\le C(T_0)\exp\Big(C\int_{T_0}^t 1 + \|\nabla u\|_{\mathrm{BMO}} \\
&\qquad\qquad \times\left(1 + \ln\left(1 + \|u\|_{H^3}^2\right)\right)(\cdot,\tau)\,d\tau\Big) \\
&\quad\le C(T_0)\exp\Big(C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau) \\
&\qquad\qquad \times(1 + \ln(1 + A(t)))\,d\tau\Big) \\
&\quad\le C(T_0)\exp\Big(C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau)\,d\tau \\
&\qquad\qquad \times(1 + \ln(1 + A(t)))\Big).
\end{aligned} \tag{29}
$$

Since $\|\nabla u\|_{\mathrm{BMO}} \in L^1(T_0,T)$, we can take T_0 close enough to T, so that

$$
C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau)\,d\tau \le 2\delta \tag{30}
$$

for some small positive number δ to be fixed later. With such choice of T_0 we have

$$
\int_{T_0}^t \|\Lambda^3 u(\cdot,\tau)\|_{L^2}^2 \tau \le C(T_0)(1 + A(t))^{2\delta}. \tag{31}
$$

Hölder's inequality gives

$$
\int_{T_0}^t \|\Lambda^3 u(\cdot,\tau)\|_{L^2} \tau \le C(T_0)(1 + A(t))^{\delta}. \tag{32}
$$

Fix T_0 satisfying

$$
C\int_{T_0}^t \|\nabla u(\cdot,\tau)\|_{\mathrm{BMO}}\tau \le 2\delta, \qquad \ln(1 + A(T_0)) > 1. \tag{33}
$$

Combining the above estimates together, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^3}^2 + \|d\|_{H^4}^2\right) \\
&\quad\le C\left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12} \\
&\qquad + \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2} + 1\right)A(t) \\
&\quad\le C\Big[1 + \|\nabla u\|_{\mathrm{BMO}}(1 + \ln(1 + A(t))) \\
&\qquad + \|\nabla^2 u\|_{L^2}\Big]A(t) + C\left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12} \\
&\quad\le C(T_0)\Big[\left(\|\nabla u\|_{\mathrm{BMO}} + \|\nabla^2 u\|_{L^2} + 1\right)A(t) \\
&\qquad \times \ln(1 + A(t)) + \left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12}\Big].
\end{aligned} \tag{34}
$$

Integrating the above inequality, we have

$$A(t) \leq C(T_0) A_0 + C(T_0) \int_{T_0}^{t} 1 + \left\|\Lambda^3 u\right\|_{L^2} (\cdot, \tau)\, d\tau A(t)^{11/12} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) A(t) \times \ln(1 + A(t))\, d\tau, \tag{35}$$

where $A_0 := \|u\|_{H^3}^2(T_0) + \|d\|_{H^4}^2(T_0)$.

Taking $\delta = 1/24$, we have

$$\int_{T_0}^{t} 1 + \left\|\Lambda^3 u\right\|_{L^2} d\tau \leq C(T_0)(1 + A(t))^{1/24}. \tag{36}$$

Thus (35) tells us that

$$A(t) \leq C(T_0) A_0 + C(T_0)(A(t) + 1)^{1/24} A(t)^{11/12} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) A(t) \times \ln(1 + A(t))\, d\tau. \tag{37}$$

This in turn gives

$$1 + A(t) \leq C(T_0)(1 + A_0) + C(T_0)(A(t) + 1)^{23/24} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) \times (A(t) + 1) \ln(1 + A(t))\, d\tau. \tag{38}$$

We set $B(t) := (1 + A(t))^{1/24}$, $B_0 := (1 + A_0)^{1/24}$ and divide the above inequality by $(1 + A(t))^{23/24}$; using the monotonicity of $A(t)$ we reach

$$B(t) \leq C(T_0) \left[B_0 + 1 + \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\nabla^2 u\right\|_{L^2}\right) \times B(t) \ln B(t)\, d\tau \right]. \tag{39}$$

The standard Gronwall's inequality now gives

$$B(t) \leq \left[C(T_0)(1 + B_0)\right]^{\exp[C(T_0) \int_{T_0}^{t} 1 + \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau]}, \tag{40}$$

which leads to

$$A(t) \leq \left[C(T_0)(1 + B_0)\right]^{24 \exp[C(T_0) \int_{T_0}^{t} 1 + \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau]}. \tag{41}$$

As $\int_{T_0}^{t} \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau$ remains bounded as $t \nearrow T$, the above inequality contradicts that $A(t) \nearrow \infty$ as $t \nearrow T$, so we complete our proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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