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Research Article

Periodic Solutions of a Stage-Structured Plant-Hare Model with Toxin-Determined Functional Responses

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The purpose of this paper is to obtain some sufficient conditions for the global existence of multiple positive periodic solutions of a delayed stage-structured plant-hare model with a toxin-determined functional response. Some novel estimation techniques to construct two open subsets for a priori bounds are employed.

1. Introduction

A lot of classical predator-prey models have been well studied (e.g., see [1–12]). Recently, Gao et al. [13] considered a nonautonomous plant-hare dynamical system with a toxin-determined functional response given by

$$\begin{aligned}\dot{N}(t) &= r(t)N(t)\left[1-\frac{N(t)}{K}\right]-C(N(t))P(t),\\ \dot{P}(t) &= B(t)C(N(t))P(t)-d(t)P(t),\end{aligned}\tag{1}$$

$$\begin{aligned}C(N(t)) &= f(N(t))\left[1-\frac{f(N(t))}{4G}\right],\\ f(N(t)) &= \frac{e\delta N(t)}{1+he\delta N(t)},\end{aligned}\tag{2}$$

where $N(t)$ denotes the density of plant at time t and $P(t)$ denotes the herbivore biomass at time t.

On the other hand, many experts argued that the predator-prey models should be modified to fit the more realistic environment. They suggested that one should take the stage structure factor into consideration. Because it is very unrealistic to assume that each individual predator admits the same ability of attacking in the classical predator-prey models. They divided the individuals into two stages in life history, namely, immature and mature stages, where the rate of the immature predator attacking the prey and the reproductive rate can be ignored, while the mature predators are responsible for the prey. For example, one can refer to [14, 15] and the references cited therein. To discuss the effects of Holling type IV functional responses on a stage-structured model, the authors in [16] proposed the following delayed system:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\left[r_1(t)-a_1(t)\vphantom{\int}\right.\\ &\quad\times\int_{-\infty}^{t}K(t-s)x(s)\,ds\\ &\quad\left.-\frac{a_2(t)y_2(t)}{x^2(t)/m+x(t)+a}\right],\\ \frac{dy_1(t)}{dt} &= \frac{b_1(t)x(t)y_2(t)}{x^2(t)/m+x(t)+a}-\beta(t)y_1(t)\\ &\quad-b_1(t-\tau)\exp\left(-\int_{t-\tau}^{t}\beta(s)\,ds\right)\\ &\quad\times\frac{x(t-\tau)y_2(t-\tau)}{x^2(t-\tau)/m+x(t-\tau)+a},\\ \frac{dy_2(t)}{dt} &= b_1(t-\tau)\exp\left(-\int_{t-\tau}^{t}\beta(s)\,ds\right)\\ &\quad\times\frac{x(t-\tau)y_2(t-\tau)}{x^2(t-\tau)/m+x(t-\tau)+a}\\ &\quad-r_2(t)y_2(t).\end{aligned}\tag{3}$$

However, Holling IV type functional response is not appropriate for the plant-hare model if we explore the impact of plant toxicity on the dynamics of plant-hare interactions. Because such kind of plant can produce toxicity to protect itself. Therefore, in the present paper, we discuss the stage-structured plant-hare model with toxin-determined functional response as follows:

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\Bigg[r_1(t) - a(t)x(t) \\
&\qquad - \frac{4Ge\delta y_2(t) + (4Gh-1)e^2\delta^2 x(t) y_2(t)}{4G(1+he\delta x(t))^2}\Bigg], \\
\frac{dy_1(t)}{dt} &= -\beta(t) y_1(t) \\
&\quad + \Big(\big(b_1(t) 4Ge\delta Bx(t) y_2(t) \\
&\qquad + (4Gh-1)e^2\delta^2 Bx^2(t) y_2(t)\big) \\
&\quad \times \left(4G(1+he\delta x(t))^2\right)^{-1}\Big) \\
&\quad - b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\} \\
&\quad \times \Big(\big(4Ge\delta Bx(t-\tau) y_2(t-\tau) \\
&\qquad + (4Gh-1)e^2\delta^2 Bx^2(t-\tau) y_2(t-\tau)\big) \\
&\quad \times \left(4G(1+he\delta x(t-\tau))^2\right)^{-1}\Big), \\
\frac{dy_2(t)}{dt} &= b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\} \\
&\quad \times \Big(\big(4Ge\delta Bx(t-\tau) y_2(t-\tau) \\
&\qquad + (4Gh-1)e^2\delta^2 Bx^2(t-\tau) y_2(t-\tau)\big) \\
&\quad \times \left(4G(1+he\delta x(t-\tau))^2\right)^{-1}\Big) \\
&\quad - r_2(t) y_2(t),
\end{aligned} \tag{4}$$

where $x(t)$ denotes the density of the plant at time t, $y_1(t)$ is the density of immature individual hares at time t, and $y_2(t)$ denotes the density of mature individual hares at time t, respectively; $r_1(t)$, $r_2(t)$, $a(t)$, $b_1(t)$, and $\beta(t)$ are continuously positive periodic functions with period ω. B is the conversion rate, e is the encounter rate per hare, δ is the fraction of food items encountered that the hares ingest, G measures the toxicity level, and h is the time for handing one unit of plant. e, δ, G, and h are positive real constants. $r_1(t)$ is the intrinsic growth rate of the prey, $a(t)$ is the density-dependent coefficient of the plant, and $r_2(t)$ is the death rate of the mature hares.

For any continuous ω-periodic function $f(t)$, we always adopt the following notations throughout this paper:

$$\begin{gathered}
\bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\,dt, \qquad f^L = \min_{t\in[0,\omega]} f(t), \\
f^M = \max_{[0,\omega]} f(t),
\end{gathered} \tag{5}$$

where f is a continuous ω-periodic function.

The purpose of this paper is to obtain some sufficient conditions for the global existence of multiple positive periodic solutions of system (4). Our method is based on Mawhin's coincidence degree and novel estimation techniques for a priori bounds of unknown solutions to $Lx = \lambda Nx$. To the best of our knowledge, it is the first time that a delayed stage-structured plant-hare dynamical system with a toxin-determined functional response has been proposed and studied by using this method.

Remark 1. The term $\exp\{-\int_{t-\tau}^{t}\beta(s)ds\}(x(t-\tau)y_2(t-\tau)/(x^2(t-\tau)/m + x(t-\tau) + a))$ in the third equation of (4) involves $x(t-\tau)y_2(t-\tau)$ instead of $x(t-\tau)y_2(t)$; the method used in [13] cannot be applied to system (4) directly. Thus, novel estimation techniques must be employed for a priori bounds of unknown solutions to the operator equation $Lx = \lambda Nx$. More specifically, integrating the second equation of system (1) over $[0,\omega]$, the authors in [13] obtained

$$\begin{aligned}
\int_0^\omega &\Big(\big(4Ge\delta B(t)\exp\{u_1(t)\} \\
&\quad + (4Gh-1)e^2\delta^2 B(t)\exp\{2u_1(t)\}\big) \\
&\quad \times \left(4G(1+he\delta\exp\{u_1(t)\})^2\right)^{-1}\Big)\,dt \\
&= \bar{d}\omega.
\end{aligned} \tag{6}$$

It follows that

$$\bar{d}\omega \ge \int_0^\omega \frac{4Ge\delta B(t)\exp\{u_1(t)\}}{4G(1+he\delta\exp\{u_1(t)\})^2}\,dt. \tag{7}$$

By some arguments, this inequality then leads them to

$$\begin{aligned}
&\bar{d}h^2e^2\delta^2\exp(2u_1(\eta_1)) \\
&\quad - \left(e\delta\bar{B}\exp\{-2\bar{r}\omega\} - 2he\delta\bar{d}\right) \\
&\quad \times \exp\{u_1(\eta_1)\} + \bar{d} > 0,
\end{aligned} \tag{8}$$

which implies that

$$x_1(\eta_1) < \ln h_-^0, \qquad x_1(\eta_1) > \ln h_+^0, \tag{9}$$

where

$$\begin{aligned}
h_\pm^0 &= \frac{\left(e\delta\bar{B}\exp\{-2\bar{r}\omega\} - 2he\delta\bar{d}\right) \pm \sqrt{\Delta_1}}{2\bar{d}h^2e^2\delta^2}, \\
\Delta_1 &= \left[e\delta\bar{B}\exp\{-2\bar{r}\omega\} - 2he\delta\bar{d}\right]^2 - 4\bar{d}^2h^2e^2\delta^2.
\end{aligned} \tag{10}$$

It should be noted that it is possible to construct two open subsets Ω_1 and Ω_2 due to (9). The essential reason to obtain (9) is the inequality (7). In inequality (7), there is no variable $u_2(t)$ and only one variable $u_1(t)$.

However, since the term $x(t-\tau)y_2(t-\tau)$ is in the third equation of (4), by same arguments in [13], we will see that

$$\begin{aligned}\int_0^\omega & b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\}\\ &\times\exp\{u_2(t-\tau)-u_2(t)\}\\ &\times\Big(\big(4Ge\delta B\exp\{u_1(t-\tau)\}\\ &\qquad+(4Gh-1)e^2\delta^2B\exp\{2u_1(t-\tau)\}\big)\\ &\quad\times\left(4G(1+he\delta\exp\{u_1(t-\tau)\})^2\right)^{-1}\Big)\,dt\\ &=\bar{r}_2\omega.\end{aligned}\tag{11}$$

Note that both u_1 and u_2 appear simultaneously in the above equality. If we were to use the same ideas in [13], then the above equality does not lead us anywhere. Thus, some new arguments should be employed to obtain a priori bounds for u_1. To see how to overcome this difficulty, the reader can refer to (33)–(56) in Section 2.

Remark 2. It should be noted that the standard estimation techniques used in [16] are not applicable to the system (4) either, due to the term $C(N(t))$. If we were to use the standard arguments in [16], we can not obtain two positive roots of $\exp(u_1(\xi_1))$. Consequently, we can not construct two open subsets. Thus, we can not obtain two positive solutions in these two open subsets.

2. Existence of Multiple Positive Periodic Solutions

In this section, we will study the existence of multiple periodic solutions of (4). We recall a few concepts and results from [17].

Lemma 3 (see [17]). *Let $\Omega\subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and NL-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda\in(0,1)$, $x\in\partial\Omega\cap\operatorname{Dom}L$, $Lx\neq\lambda Nx$;*

(b) *for each $x\in\partial\Omega\cap\operatorname{Ker}L$, $QNx\neq 0$;*

(c) $\deg\{JQN,\Omega\cap\operatorname{Ker}L,0\}\neq 0$.

Then $Lx=Nx$ has at least one solution in $\overline{\Omega}\cap\operatorname{Dom}L$.

Lemma 4. *If $\beta(t)$ and $g(t)$ are ω-periodic functions, then the system*

$$\frac{dy(t)}{dt}=\beta(t)\,y(t)+g(t)\tag{12}$$

has a unique ω-periodic solution which can be represented as $y(t)=\int_{-\infty}^{t}\exp(\int_s^t\beta(\sigma)d\sigma)g(s)ds$.

Throughout, we assume the following:

(A_1) $1/4h<G<1/3h$;

(A_2) $4hr_2^M\exp\{\tau\beta^M\}\exp\{2\bar{r}_1\omega\}/b_1^L<B<4Gr_2^Lh^2(b_1^M)^{-1}\exp\{\tau\beta^L\}/(4Gh-1)$.

We further introduce six positive numbers which will be used later as follows:

$$\begin{aligned}h_\pm&=\left(\left(b_1^Le\delta B\exp\{-\tau\beta^M\}\exp\{-2\bar{r}_1\omega\}-2he\delta r_2^M\right)\pm\sqrt{\Delta_1}\right)\times\left(2r_2^Mh^2e^2\delta^2\right)^{-1},\\ l_\pm&=\Big(\Big[4Gh^2e\delta B\exp\{2\bar{r}_1\omega\}\\ &\qquad-2he\delta\left(4Gh^2r_2^L\left(b_1^M\right)^{-1}\exp\{\tau\beta^L\}-(4Gh-1)B\right)\Big]\\ &\qquad\pm\sqrt{\Delta_2}\Big)\\ &\quad\times\Big(2h^2e^2\delta^2\Big[4Gh^2r_2^L\left(b_1^M\right)^{-1}\exp\{\tau\beta^L\}\\ &\qquad-(4Gh-1)B\Big]\Big)^{-1},\\ u_\pm&=\frac{\left(4Ge\delta B-8Ghe\delta\bar{r}_2\bar{b}^{-1}\right)\pm\sqrt{\Delta_3}}{2\left[4G\bar{r}_2\bar{b}^{-1}h^2e^2\delta^2-(4Gh-1)e^2\delta^2B\right]},\end{aligned}\tag{13}$$

where

$$\begin{aligned}\Delta_1&=\left[b_1^Le\delta B\exp\{-\tau\beta^M\}\exp\{-2\bar{r}_1\omega\}-2he\delta r_2^M\right]^2\\ &\quad-4\left(r_2^M\right)^2h^2e^2\delta^2,\\ \Delta_2&=\Big[4Gh^2e\delta B\exp\{2\bar{r}_1\omega\}\\ &\quad-2he\delta\left(4Gh^2r_2^L\left(b_1^M\right)^{-1}\exp\{\tau\beta^L\}-(4Gh-1)B\right)\Big]^2\\ &\quad-4h^2e^2\delta^2\left[4Gh^2r_2^L\left(b_1^M\right)^{-1}\exp\{\tau\beta^L\}-(4Gh-1)B\right]^2,\\ \Delta_3&=\left(4Ge\delta B-8Ghe\delta\bar{r}_2\bar{b}^{-1}\right)^2\\ &\quad-16G\bar{r}_2\bar{b}^{-1}\left[4G\bar{r}_2\bar{b}^{-1}h^2e^2\delta^2-(4Gh-1)e^2\delta^2B\right],\end{aligned}\tag{14}$$

$\bar{b}=(1/\omega)\int_0^\omega b_1(t)\exp\{-\int_t^{t+\tau}\beta(s)\}dsdt$. Under assumptions (A_1) and (A_2), it is not difficult to show that

$$l_-<u_-<h_-<h_+<u_+<l_+.\tag{15}$$

Theorem 5. *In addition to (A_1) and (A_2), suppose that*

(A_3) $\bar{r}_1-\bar{a}\exp\{\ln l_++2\bar{r}\omega\}>0$.

Then system (4) has at least two positive ω-periodic solutions.

Proof. Note that the first equation and the third equation of (4) can be separated from the whole system. Consider the following subsystem:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Bigg[r_1(t) - a(t)x(t) \\ &\qquad - \frac{4Ge\delta y_2(t) + (4Gh-1)e^2\delta^2 x(t)y_2(t)}{4G(1+he\delta x(t))^2}\Bigg], \\ \frac{dy_2(t)}{dt} &= b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\} \\ &\quad\times\Big(\big(4Ge\delta Bx(t-\tau)y_2(t-\tau) + (4Gh-1)e^2\delta^2 \\ &\qquad\times B(t)x^2(t-\tau)y_2(t-\tau)\big) \\ &\quad\times\left(4G(1+he\delta x(t-\tau))^2\right)^{-1}\Big) \\ &\quad - r_2(t)y_2(t).\end{aligned} \tag{16}$$

Make the change of variables

$$x(t) = \exp\{u_1(t)\}, \qquad y(t) = \exp\{u_2(t)\}; \tag{17}$$

then system (16) can be rewritten as

$$\begin{aligned}\dot{u}_1(t) &= r_1(t) - a(t)\exp\{u_1(t)\} \\ &\quad - \Big(\big(4Ge\delta\exp\{u_2(t)\} + (4Gh-1)e^2\delta^2 \\ &\qquad\times\exp\{u_1(t)+u_2(t)\}\big) \\ &\qquad\times\left(4G(1+he\delta\exp\{u_1(t)\})^2\right)^{-1}\Big) := f_1(t,u), \\ \dot{u}_2(t) &= -r_2(t) + b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\} \\ &\quad\times\Big(\big(4Ge\delta B\exp\{u_1(t-\tau)\} + (4Gh-1)e^2\delta^2 B \\ &\qquad\times\exp\{2u_1(t-\tau)\}\big) \\ &\qquad\times\left(4G(1+he\delta\exp\{u_1(t-\tau)\})^2\right)^{-1}\Big) \\ &\quad\times\exp\{u_2(t-\tau)-u_2(t)\} := f_2(t,u).\end{aligned} \tag{18}$$

Take

$$X = Y = \left\{x = (u_1,u_2)^T \in C\left(\mathbb{R},\mathbb{R}^2\right) \mid x(t+\omega) = x(t)\right\} \tag{19}$$

and define

$$\begin{gathered}\|x\| = \max_{t\in[0,\omega]}\left|u_1(t)\right| + \max_{t\in[0,\omega]}\left|u_2(t)\right|, \\ x = (u_1,u_2)^T \in X \text{ or } Y;\end{gathered} \tag{20}$$

here $|\cdot|$ denotes the Euclidean norm. Then X and Y are Banach spaces with the norm $\|\cdot\|$. Set

$$\begin{gathered}L : \operatorname{Dom} L \cap X, \\ L(u_1(t),u_2(t))^T = \left(\frac{du_1(t)}{dt}, \frac{du_2(t)}{dt}\right)^T,\end{gathered} \tag{21}$$

where $\operatorname{Dom} L = \{(u_1(t),u_2(t))^T \in C^1(\mathbb{R},\mathbb{R}^2)\}$. Further, $N : X \to X$ is defined by

$$N\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}f_1(t,u)\\f_2(t,u)\end{pmatrix}. \tag{22}$$

Define

$$\begin{gathered}P\begin{pmatrix}u_1\\u_2\end{pmatrix} = Q\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\dfrac{1}{\omega}\displaystyle\int_0^{\omega}u_1(t)\,dt\\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}u_2(t)\,dt\end{pmatrix}, \\ \begin{pmatrix}u_1\\u_2\end{pmatrix} \in X = Y.\end{gathered} \tag{23}$$

It is not difficult to show that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) K_p : $\operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists. Standard arguments show that N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Now, we will search for two appropriate open bounded subsets in order to apply the continuation theorem.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned}\dot{u}_1(t) &= \lambda r_1(t) \\ &\quad - \lambda\Big[a(t)\exp\{u_1(t)\} \\ &\qquad - \Big(\big(4Ge\delta\exp\{u_2(t)\} + (4Gh-1)e^2\delta^2 \\ &\qquad\quad\times\exp\{u_1(t)+u_2(t)\}\big) \\ &\qquad\quad\times\left(4G(1+he\delta\exp\{u_1(t)\})^2\right)^{-1}\Big)\Big],\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{u}_2(t) &= -\lambda r_2(t) \\ &\quad + \lambda\Big[b_1(t-\tau)\exp\left\{-\int_{t-\tau}^{t}\beta(s)\,ds\right\} \\ &\qquad\times\Big(\big(4Ge\delta B\exp\{u_1(t-\tau)\} + (4Gh-1)e^2\delta^2 B \\ &\qquad\quad\times\exp\{2u_1(t-\tau)\}\big) \\ &\qquad\quad\times\left(4G(1+he\delta\exp\{u_1(t-\tau)\})^2\right)^{-1}\Big) \\ &\qquad\times\exp\{u_2(t-\tau)-u_2(t)\}\Big].\end{aligned} \tag{25}$$

Suppose $x = (u_1(t), u_2(t))^T \in X$ is a solution of (24) and (25) for a certain $\lambda \in (0,1)$. Integrating (24), (25) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}
&\int_0^\omega a(t)\exp\{u_1(t)\}\,dt \\
&\quad + \int_0^\omega \Big(\big(4Ge\delta\exp\{u_2(t)\} + (4Gh-1)e^2\delta^2 \\
&\qquad \times \exp\{u_1(t)+u_2(t)\}\big) \\
&\qquad \times \big(4G(1+he\delta\exp\{u_1(t)\})^2\big)^{-1}\Big)\,dt \\
&= \overline{r}_1\omega, \\
&\int_0^\omega b_1(t-\tau)\exp\Big\{-\int_{t-\tau}^t \beta(s)\,ds\Big\} \\
&\quad \times \exp\{u_2(t-\tau)-u_2(t)\} \\
&\quad \times \Big(\big(4Ge\delta B\exp\{u_1(t-\tau)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad \times \exp\{2u_1(t-\tau)\}\big) \\
&\qquad \times \big(4G(1+he\delta\exp\{u_1(t-\tau)\})^2\big)^{-1}\Big)\,dt \\
&= \overline{r}_2\omega.
\end{aligned} \tag{26, 27}$$

It follows from (A_1), (24), and (26) that

$$\begin{aligned}
&\int_0^\omega |\dot{u}_1(t)|\,dt \\
&= \lambda\int_0^\omega \Big| r_1(t) - a(t)\exp\{u_1(t)\} \\
&\qquad - \Big(\big(4Ge\delta\exp\{u_2(t)\} + (4Gh-1)e^2\delta^2 \\
&\qquad\quad \times \exp\{u_1(t)+u_2(t)\}\big) \\
&\qquad\quad \times \big(4G(1+he\delta\exp\{u_1(t)\})^2\big)^{-1}\Big)\Big|\,dt \\
&< \int_0^\omega r_1(t)\,dt + \int_0^\omega a(t)\exp\{u_1(t)\}\,dt \\
&\quad + \int_0^\omega \Big(\big(4Ge\delta\exp\{u_2(t)\} + (4Gh-1)e^2\delta^2 \\
&\qquad \times \exp\{u_1(t)+u_2(t)\}\big) \\
&\qquad \times \big(4G(1+he\delta\exp\{u_1(t)\})^2\big)^{-1}\Big)\,dt \\
&= \int_0^\omega r_1(t)\,dt + \overline{r}_1\omega = 2\overline{r}_1\omega;
\end{aligned} \tag{28}$$

that is,

$$\int_0^\omega |\dot{u}_1(t)|\,dt < 2\overline{r}_1\omega. \tag{29}$$

Similarly, it follows from (A_1), (25), and (27) that

$$\int_0^\omega |\dot{u}_2(t)|\,dt < 2\overline{r}_2\omega. \tag{30}$$

Since $(u_1(t), u_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t), \quad i = 1,2. \tag{31}$$

Multiplying (25) by $\exp\{u_2(t)\}$ and integrating over $[0,\omega]$, we obtain

$$\begin{aligned}
&\int_0^\omega r_2(t)\exp\{u_2(t)\}\,dt \\
&= \int_0^\omega \Big[b_1(t-\tau)\exp\Big\{-\int_{t-\tau}^t \beta(s)\,ds\Big\} \\
&\qquad \times \Big(\big(4Ge\delta B\exp\{u_1(t-\tau)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad\quad \times \exp\{2u_1(t-\tau)\}\big) \\
&\qquad\quad \times \big(4G(1+he\delta\exp\{u_1(t-\tau)\})^2\big)^{-1}\Big) \\
&\qquad \times \exp\{u_2(t-\tau)\}\Big]\,dt \\
&= \int_{-\tau}^{\omega-\tau} \Big[b_1(\sigma)\exp\Big\{-\int_\sigma^{\sigma+\tau} \beta(s)\,ds\Big\} \\
&\qquad \times \Big(\big(4Ge\delta B\exp\{u_1(\sigma)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad\quad \times \exp\{2u_1(\sigma)\}\big) \\
&\qquad\quad \times \big(4G(1+he\delta\exp\{u_1(\sigma)\})^2\big)^{-1}\Big) \\
&\qquad \times \exp\{u_2(\sigma)\}\Big]\,dt \\
&= \int_0^\omega \Big[b_1(t)\exp\Big\{-\int_t^{t+\tau} \beta(s)\,ds\Big\} \\
&\qquad \times \Big(\big(4Ge\delta B\exp\{u_1(t)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad\quad \times \exp\{2u_1(t)\}\big) \\
&\qquad\quad \times \big(4G(1+he\delta\exp\{u_1(t)\})^2\big)^{-1}\Big) \\
&\qquad \times \exp\{u_2(t)\}\Big]\,dt;
\end{aligned} \tag{32}$$

that is,

$$\begin{aligned}
&\int_0^\omega r_2(t)\exp(u_2(t))\,dt \\
&= \int_0^\omega \Bigg[b_1(t)\exp\left\{-\int_t^{t+\tau}\beta(s)\,ds\right\} \\
&\quad \times \Big(\big(4Ge\delta B\exp\{u_1(t)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad \times \exp\{2u_1(t)\}\big) \\
&\qquad \times \left(4G(1+he\delta\exp\{u_1(t)\})^2\right)^{-1}\Big) \\
&\quad \times \exp\{u_2(t)\}\Bigg]\,dt.
\end{aligned} \tag{33}$$

It follows from (27), (33), and (A_1); we see that

$$\begin{aligned}
&r_2^L \int_0^\omega \exp\{u_2(t)\}\,dt \\
&\le \int_0^\omega r_2(t)\exp\{u_2(t)\}\,dt \\
&= \int_0^\omega \Bigg[b_1(t)\exp\left\{-\int_t^{t+\tau}\beta(s)\,ds\right\} \\
&\quad \times \Big(\big(4Ge\delta B\exp\{u_1(t)\} + (4Gh-1)e^2\delta^2 B \\
&\qquad \times \exp\{2u_1(t)\}\big) \\
&\qquad \times \left(4G(1+he\delta\exp\{u_1(t)\})^2\right)^{-1}\Big) \\
&\quad \times \exp\{u_2(t)\}\Bigg]\,dt \\
&\le \int_0^\omega b_1(t)\exp\left\{-\int_t^{t+\tau}\beta(s)\,ds\right\} \\
&\quad \times \left[\frac{4Ge\delta B\exp\{u_1(t)\}}{4G(1+he\delta\exp\{u_1(t)\})^2}\right. \\
&\qquad \left. + \frac{(4Gh-1)e^2\delta^2 B\exp\{2u_1(t)\}}{4Gh^2e^2\delta^2\exp\{2u_1(t)\}}\right] \\
&\quad \times \exp\{u_2(t)\}\,dt \\
&\le b_1^M \exp\left\{-\tau\beta^L\right\} \\
&\quad \times \left[\frac{e\delta B\exp\{u_1(\eta_1)\}}{(1+he\delta\exp\{u_1(\xi_1)\})^2}\right. \\
&\qquad \left. + \frac{(4Gh-1)B}{4Gh^2}\right]\int_0^\omega \exp\{u_2(t)\}\,dt,
\end{aligned} \tag{34}$$

which implies

$$\begin{aligned}
r_2^L &\le b_1^M \exp\left\{-\tau\beta^L\right\} \\
&\quad \times \left[\frac{e\delta B\exp\{u_1(\eta_1)\}}{(1+he\delta\exp\{u_1(\xi_1)\})^2} + \frac{(4Gh-1)B}{4Gh^2}\right].
\end{aligned} \tag{35}$$

So

$$\begin{aligned}
u_1(\eta_1) \ge \ln &\Big(\Big(\Big[4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\Big] \\
&\times (1+he\delta\exp\{u_1(\xi_1)\})^2\Big) \\
&\times \left(4Gh^2 e\delta B\right)^{-1}\Big).
\end{aligned} \tag{36}$$

This, combined with (29), gives

$$\begin{aligned}
u_1(t) &\ge u_1(\eta_1) - \int_0^\omega |\dot{u}_1(t)|\,dt \\
&> \ln\Big(\Big(\Big[4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\Big] \\
&\qquad \times (1+he\delta\exp\{u_1(\xi_1)\})^2\Big) \\
&\qquad \times \left(4Gh^2 e\delta B\right)^{-1}\Big) - 2\bar{r}_1\omega.
\end{aligned} \tag{37}$$

In particular, we have

$$\begin{aligned}
u_1(\xi_1) > \ln &\Big(\Big(\Big[4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\Big] \\
&\times (1+he\delta\exp\{u_1(\xi_1)\})^2\Big) \\
&\times \left(4Gh^2 e\delta B\right)^{-1}\Big) - 2\bar{r}_1\omega,
\end{aligned} \tag{38}$$

or

$$\begin{aligned}
&\Big[4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\Big] h^2e^2\delta^2 \\
&\quad \times \exp\{2u_1(\xi_1)\} \\
&\quad - \Big[4Gh^2 e\delta B\exp\{2\bar{r}\omega\} \\
&\qquad -2he\delta\left(4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\right)\Big] \\
&\quad \times \exp\{u_1(\xi_1)\} \\
&\quad + \Big[4Gh^2 r_2^L\big(b_1^M\big)^{-1}\exp\left\{\tau\beta^L\right\} - (4Gh-1)B\Big] < 0.
\end{aligned} \tag{39}$$

In view of (A_2), we have

$$\ln l_- < u_1(\xi_1) < \ln l_+. \tag{40}$$

Similarly, it follows from (33) that

$$\begin{aligned} & b_1^L \exp\left\{-\tau\beta^M\right\} \\ & \quad \times \frac{4Ge\delta B \exp\{u_1(\xi_1)\} + (4Gh-1)e^2\delta^2 B \exp\{2u_1(\xi_1)\}}{4G(1+he\delta\exp\{u_1(\eta_1)\})^2} \\ & \quad \times \int_0^\omega \exp\{u_2(t)\}\,dt \\ & \leq \int_0^\omega b_1(t)\exp\left\{-\int_t^{t+\tau}\beta(s)\,ds\right\} \\ & \qquad \times \frac{4Ge\delta B \exp\{u_1(t)\} + (4Gh-1)e^2\delta^2 B \exp\{2u_1(t)\}}{4G(1+he\delta\exp\{u_1(t)\})^2} \\ & \qquad \times \exp\{u_2(t)\}\,dt \\ & = \int_0^\omega r_2(t)\exp\{u_2(t)\}\,dt \\ & \leq r_2^M \int_0^\omega \exp\{u_2(t)\}\,dt, \end{aligned} \tag{41}$$

which implies

$$\begin{aligned} & b_1^L \exp\left\{-\tau\beta^M\right\} \\ & \quad \times \frac{4Ge\delta B \exp\{u_1(\xi_1)\} + (4Gh-1)e^2\delta^2 B \exp\{2u_1(\xi_1)\}}{4G(1+he\delta\exp\{u_1(\eta_1)\})^2} \\ & \leq r_2^M; \end{aligned} \tag{42}$$

that is,

$$b_1^L \exp\left\{-\tau\beta^M\right\} ds \frac{e\delta B\exp\{u_1(\xi_1)\}}{(1+he\delta\exp\{u_1(\eta_1)\})^2} \leq r_2^M. \tag{43}$$

So

$$u_1(\xi_1) \leq \ln \frac{r_2^M(1+he\delta\exp\{u_1(\eta_1)\})^2}{b_1^L e\delta B\exp\{-\tau\beta^M\}}. \tag{44}$$

This, combined with (29), gives

$$\begin{aligned} u_1(t) & \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|\,dt \\ & < \ln \frac{r_2^M(1+he\delta\exp\{u_1(\eta_1)\})^2}{b_1^L e\delta B\exp\{-\tau\beta^M\}} \\ & \quad + 2\overline{r}_1\omega. \end{aligned} \tag{45}$$

In particular, we have

$$\begin{aligned} u_1(\eta_1) < {} & \ln \frac{r_2^M(1+he\delta\exp\{u_1(\eta_1)\})^2}{b_1^L e\delta B\exp\{-\tau\beta^M\}} \\ & + 2\overline{r}_1\omega, \end{aligned} \tag{46}$$

or

$$\begin{aligned} & r_2^M h^2 e^2 \delta^2 \exp\{2u_1(\eta_1)\} \\ & \quad - \left[b_1^L e\delta B\exp\left\{-\tau\beta^M\right\}\exp\{-2\overline{r}_1\omega\} - 2he\delta r_2^M\right] \\ & \quad \times \exp\{u_1(\eta_1)\} + r_2^M > 0. \end{aligned} \tag{47}$$

It follows from (A_2) that

$$u_1(\eta_1) < \ln h_- \quad \text{or} \quad u_1(\eta_1) > \ln h_+. \tag{48}$$

From (29) and (40), we find

$$\begin{aligned} u_1(t) & \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|\,dt \\ & < \ln l_+ + 2\overline{r}_1\omega \triangleq H_{11}. \end{aligned} \tag{49}$$

On the other hand, it follows from (A_1), (26), and (49) that

$$\overline{r}_1\omega \geq \int_0^\omega \frac{4Ge\delta\exp\{u_2(\xi_2)\}}{4G(1+he\delta\exp\{\ln l_+ + 2\overline{r}\omega\})^2}dt, \tag{50}$$

$$\begin{aligned} \overline{r}_1\omega \leq {} & \int_0^\omega a(t)\exp\{\ln l_+ + 2\overline{r}_1\omega\}\,dt \\ & + \int_0^\omega e\delta\exp\{u_2(\eta_2)\}\,dt \\ & + \int_0^\omega \frac{e\delta\exp\{u_2(\eta_2)\}}{2}dt. \end{aligned} \tag{51}$$

It follows from (50) that

$$u_2(\xi_2) \leq \ln \frac{\overline{r}_1(1+he\delta\exp\{\ln l_+ + 2\overline{r}_1\omega\})^2}{e\delta}. \tag{52}$$

This, combined with (30), gives

$$\begin{aligned} u_2(t) & \leq u_2(\xi_2) + \int_0^\omega |\dot{u}_2(t)|\,dt \\ & < \ln \frac{\overline{r}_1(1+he\delta\exp\{\ln l_+ + 2\overline{r}_1\omega\})^2}{e\delta} \\ & \quad + 2\overline{r}_2\omega \triangleq H_{21}. \end{aligned} \tag{53}$$

Moreover, because of (A_3), it follows from (51) that

$$u_2(\eta_2) \geq \ln \frac{2(\overline{r}_1 - \overline{a}\exp\{\ln l_+ + 2\overline{r}\omega\})}{3e\delta}. \tag{54}$$

This, combined with (30) again, gives

$$\begin{aligned} u_2(t) & \geq u_2(\eta_2) - \int_0^\omega |\dot{u}_2(t)|\,dt \\ & > \ln \frac{2(\overline{r}_1 - \overline{a}\exp\{\ln l_+ + 2\overline{r}\omega\})}{3e\delta} \\ & \quad - 2\overline{r}_2\omega \triangleq H_{22}. \end{aligned} \tag{55}$$

It follows from (53) and (55) that

$$\max_{t\in[0,\omega]} u_2(t) < \max\{|H_{21}|, |H_{22}|\} \triangleq H_2. \quad (56)$$

Now, let us consider QNx with

$$QN(u_1,u_2)^T = \left(\bar{r}_1\omega - \bar{a}\omega\exp\{u_1\} - w \times \frac{4Ge\delta\exp\{u_2\} + (4Gh-1)e^2\delta^2\exp\{u_1+u_2\}}{4G(1+he\delta\exp\{u_1\})^2}, -\bar{r}_2\omega + \bar{b}\omega \times \frac{4Ge\delta B\exp\{u_1\} + (4Gh-1)e^2\delta^2 B\exp\{2u_1\}}{4G(1+he\delta\exp\{u_1\})^2}\right)^T. \quad (57)$$

In view of (A_1), (A_2), and (A_3), we can show that the equation $QN(u_1,u_2)^T = 0$ has two distinct solutions

$$\tilde{u} = \left(\ln u_-,\ \ln\frac{4G(\bar{r}_1 - \bar{a}u_-)(1+he\delta u_-)^2}{4Ge\delta + (4Gh-1)e^2\delta^2 u_-}\right),$$
$$\hat{u} = \left(\ln u_+,\ \ln\frac{4G(\bar{r}_1 - \bar{a}u_+)(1+he\delta u_+)^2}{4Ge\delta + (4Gh-1)e^2\delta^2 u_+}\right). \quad (58)$$

Choose $C > 0$ such that

$$C > \max\left\{\left|\ln\frac{4G(\bar{r}_1 - \bar{a}u_-)(1+he\delta u_-)^2}{4Ge\delta + (4Gh-1)e^2\delta^2 u_-}\right|, \left|\ln\frac{4G(\bar{r}_1 - \bar{a}u_+)(1+he\delta u_+)^2}{4Ge\delta + (4Gh-1)e^2\delta^2 u_+}\right|\right\}. \quad (59)$$

We define two open bounded subsets. Let

$$\Omega_1 = \left\{x = (u_1,u_2)^T \in X \mid u_1(t) \in (\ln l_-, \ln h_-),\ \max_{t\in[0,\omega]}|u_2(t)| < H_2 + C\right\},$$
$$\Omega_2 = \left\{x = (u_1,u_2)^T \in X \mid \min_{t\in[0,\omega]} u_1(t) \in (\ln l_-, \ln l_+),\ \max_{t\in[0,\omega]} u_1(t) \in (\ln h_+, H_{11}),\ \max_{t\in[0,\omega]}|u_2(t)| < H_2 + C\right\}. \quad (60)$$

Then both Ω_1 and Ω_2 are bounded open subsets of X. It follows from (16) and (59) that $\tilde{u} \in \Omega_1$ and $\hat{u} \in \Omega_2$. With the help of (16), (40), (48), (49), (56), and (59), it is easy to see that $\Omega_1 \cap \Omega_2 = \phi$, and Ω_i satisfies the requirement (a) in Lemma 3 for $i = 1, 2$. Moreover, $QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2$. A direct computation gives $\deg\{JQN, \Omega_i \cap \operatorname{Ker} L, 0\} \neq 0$. Here, J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 3. Hence, (16) has at least two ω-periodic solutions $u^*(t)$ and $u^+(t)$ with $u^* \in \operatorname{Dom} L \cap \overline{\Omega}_1$ and $u^+ \in \operatorname{Dom} L \cap \overline{\Omega}_2$. Obviously, u^* and u^+ are different. Let $x^*(t) = \exp(u_1^*(t))$, $y_2^*(t) = \exp(u_2^*(t))$ and $x^+(t) = \exp(u_1^+(t))$, $y_2^+(t) = \exp(u_2^+(t))$. Then, by (18), $(x^*(t), y_2^*(t))$ and $(x^+(t), y_2^+(t))$ are two different positive ω-periodic solutions of (4). By the periodicity of the coefficients of system (4), it is not difficult to verify that

$$\begin{aligned} g^*(t) = {} & -\beta(t)y_1(t) \\ & + \Big(\big(b_1(t)4Ge\delta Bx^*(t)y_2^*(t) \\ & \quad + (4Gh-1)e^2\delta^2 B(x^*)^2(t)y_2^*(t)\big) \\ & \quad \times \big(4G(1+he\delta x^*(t))^2\big)^{-1}\Big) \\ & - b_1(t-\tau)\exp\left(-\int_{t-\tau}^{t}\beta(s)\,ds\right) \\ & \times \Big(\big(4Ge\delta Bx^*(t-\tau)y_2^*(t-\tau) \\ & \quad + (4Gh-1)e^2\delta^2 B(x^*)^2(t-\tau)y_2^*(t-\tau)\big) \\ & \quad \times \big(4G(1+he\delta x^*(t-\tau))^2\big)^{-1}\Big) \end{aligned} \quad (61)$$

is also ω-periodic. Then, from Lemma 4, we know that

$$\frac{dy_1(t)}{dt} = -\beta(t)y_1(t) + g^*(t) \quad (62)$$

has a unique ω-periodic solution denoted by $y_1^*(t)$. And

$$\frac{dy_2(t)}{dt} = -\beta(t)y_1(t) + g^+(t) \quad (63)$$

has a unique ω-periodic solution denoted by $y_1^+(t)$. Therefore, $(x^*(t), y_1^*(t), y_2^*(t))$ and $(x^+(t), y_1^+(t), y_2^+(t))$ are two different ω-periodic solutions of system (3). This completes the proof of Theorem 5. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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