Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 354053, 7 pages
http://dx.doi.org/10.1155/2014/354053

Research Article

Asymptotic Estimates for r-Whitney Numbers of the Second Kind

Cristina B. Corcino,[1] Roberto B. Corcino,[2] and Nestor Acala[2]

[1] *Institute of Mathematics, University of the Philippines, Diliman, 1101 Quezon City, Philippines*
[2] *Department of Mathematics, Mindanao State University, 9700 Marawi City, Philippines*

Correspondence should be addressed to Roberto B. Corcino; rcorcino@yahoo.com

Received 13 November 2013; Accepted 6 January 2014; Published 12 February 2014

Academic Editor: Nicola Mastronardi

The r-Whitney numbers of the second kind are a generalization of all the Stirling-type numbers of the second kind which are in line with the unified generalization of Hsu and Shuie. In this paper, asymptotic formulas for r-Whitney numbers of the second kind with integer and real parameters are obtained and the range of validity of each formula is established.

1. Introduction

The r-Whitney numbers of the second kind, denoted by $W_{\beta,r}(n,m)$, have been introduced by Mezo [1] to obtain a new formula for Bernoulli polynomials. These numbers are equivalent to the numbers considered by Rucinski and Voigt [2] and the (r,β)-Stirling numbers [3]. They are considered as a generalization of all the Stirling-type numbers of the second kind which satisfy

$$\frac{1}{\beta^m (m!)} e^{rz}\left(e^{\beta z}-1\right)^m = \sum_{n=m}^{\infty} W_{\beta,r}(n,m)\frac{z^n}{n!}, \tag{1}$$

where n and m are positive integers. More properties of r-Whitney numbers of the second kind can be found in [1, 3–7]. For instance, the index $\widehat{K}_{\beta,r}(n)$ for which the sequence $\{W_{\beta,r}(n,k)\}_{k=0}^{n}$ assumes its maximum value satisfies

$$\begin{gathered}\widehat{K}_{\beta,r}(n) < \frac{n}{\log n - \log\log n}, \qquad n \geq 3, \\ \frac{n}{\beta\log n} - \frac{r}{\beta} < \widehat{K}_{\beta,r}(n), \qquad n \geq \max\left\{n_\beta, \frac{\log 2\beta}{\log(1+\beta/r)}\right\}.\end{gathered} \tag{2}$$

This sequence was also shown in [3] to be unimodal for fixed $n \geq 3$ with $k \leq n$ and further shown to be asymptotically normal in the sense that

$$\sum_{j=1}^{x_n} \frac{1}{G_{n,r,\beta}} W_{\beta,r}(n,k) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt \quad \text{as } n \longrightarrow \infty, \tag{3}$$

where

$$\begin{gathered}x_n = \sqrt{\frac{G_{n+2,r,\beta}}{G_{n,r,\beta}} - \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}}\right)^2}\, x + \left(\frac{G_{n+1,r,\beta}}{G_{n,r,\beta}} - 1\right), \\ G_{n,r,\beta} = \sum_{k=0}^{n} W_{\beta,r}(n,k)\end{gathered} \tag{4}$$

represents the generalized Bell numbers.

The r-Whitney numbers of the second kind can be interpreted combinatorially as follows [5].

Consider $k+1$ distinct cells the first k of which each has β distinct compartments and the last cell with r distinct compartments. Suppose we distribute n distinct balls into the $k+1$ cells one ball at a time such that

(A1) the capacity of each compartment is unlimited;

(B1) the first k cells are nonempty.

Table 1

	Exact value	Approximate value	Relative error
$W_{7,4}(100, 5)$	5.685×10^{152}	6.335×10^{152}	0.11428
$W_{7,4}(100, 10)$	7.728×10^{171}	8.169×10^{171}	0.05713
$W_{7,4}(100, 15)$	8.411×10^{178}	8.731×10^{178}	0.03804
$W_{7,4}(100, 30)$	5.604×10^{174}	5.706×10^{174}	0.01824
$W_{7,4}(100, 60)$	7.399×10^{122}	7.446×10^{122}	0.00641
$W_{7,4}(100, 80)$	1.275×10^{70}	1.279×10^{70}	0.00263
$W_{7,4}(100, 90)$	2.208×10^{38}	2.211×10^{38}	0.00123

Let Ω be the set of all possible ways of distributing n balls under restriction (A1). Then $|\Omega| = (\beta k + r)^n$ and the number of outcomes in Ω satisfying (B1) is $\beta^k k! W_{\beta,r}(n, k)$ with $\beta, r \geq 0$.

Recently, Cheon and Jung [8] gave certain combinatorial interpretation for the r-Whitney numbers over the Dowling lattice and derived some algebraic identities for such numbers. Moreover, they defined r-Dowling polynomials as

$$D_{m,r}(n, x) = \sum_{k=0}^{n} W_{\beta,r}(n, k) x^k, \tag{5}$$

which give the above generalized Bell numbers $G_{n,r,\beta}$ as particular case. That is, $G_{n,r,\beta} = D_{m,r}(n, 1)$. It is worth mentioning that Rahmani [9] obtained more combinatorial identities in relation to r-Dowling polynomials. On the other hand, Belbachir and Bousbaa [10] defined, combinatorially, certain translated r-Whitney numbers in terms of permutations and partitions under some conditions and obtained some properties parallel to those of r-Whitney numbers.

In a separate paper [11], an asymptotic formula has been obtained for r-Whitney numbers of the second kind, also called generalized Stirling numbers of the second kind, using saddle-point method. More precisely,

$$W_{\beta,r}(n, m) \approx \frac{n!}{m!} \frac{e^{\mu R}\left(e^R - 1\right)^m}{2\beta^{m-n} R^n \sqrt{\pi m R H}} \left[1 + \frac{I}{m R \sqrt{\pi}}\right], \tag{6}$$

which is valid for $m > (1/4)n(r/\beta)$, $n > 4$ such that $n - m \to \infty$ as $n \to \infty$, where $\mu = r/\beta$, $\beta \neq 0$, and R is the unique positive solution to the equation

$$\begin{gathered} R\left(\mu + \frac{y}{1 - e^{-R}}\right) - x = 0, \\ H = \frac{\mu}{2y} + \frac{e^R\left(e^R - R - 1\right)}{2\left(e^R - 1\right)^2}. \end{gathered} \tag{7}$$

Table 1 displays the exact and approximate values of $W_{\beta,r}(n, m)$ for $n = 100$, $r = 4$, $\beta = 7$.

The approximation should be good for $m > 15$ following the restriction $m > (1/4)n(r/\beta)$. The computed approximate values for $m = 15, 30, 60, 80, 90$ confirm this.

In this paper, another asymptotic formula for the r-Whitney numbers of the second kind $W_{\beta,r}(n, n - m)$ with integral values of m and n is obtained using a similar analysis as that in [12], which is proved to be valid when m is in the range $n - o(\sqrt{n}) \leq n - m \leq n$. This can be considered as the final range since it covers the right most tail of the interval $0 < m \leq n$. Since these subranges overlap, the present formula also counterchecks the other and may be used as an alternative formula for better computation. Moreover, it is shown that the formula obtained is valid in the given range when n and m are real numbers.

2. Derivation of the Asymptotic Formula

Applying Cauchy integral formula to (1), the following integral representation is obtained:

$$W_{\beta,r}(n, m) = \frac{n!}{2\pi i \beta^m (m!)} \int_C \frac{e^{rz}\left(e^{\beta z} - 1\right)^m}{z^{n+1}} dz, \tag{8}$$

where C is a circle about the origin. Using this representation with m being replaced by $n - m$, we have

$$W_{\beta,r}(n, n - m) = \binom{n}{m} \frac{m! \beta^m}{2\pi i} \int_C \frac{e^{\nu u}\left(e^u - 1\right)^{n-m}}{u^{n+1}} du, \tag{9}$$

where $u = \beta z$, $\nu = r/\beta$.

With $f(u) = ((e^u - 1)/u) - 1$, $du = \beta dz$, and $e^u - 1 = u[f(u) + 1]$, (9) can be written as

$$W_{\beta,r}(n, n - m) = \binom{n}{m} \frac{m! \beta^m}{2\pi i} \int_C \frac{e^{\nu u}\left[f(u) + 1\right]^{n-m}}{u^{m+1}} du. \tag{10}$$

We let $q = 2/(n - m)$ and introduce the new variable $qw = u$. Then $du = q dw$ and (10) can further be written as

$$\begin{aligned} &W_{\beta,r}(n, n - m) \\ &\quad = \binom{n}{m} \frac{m! \beta^m}{2\pi i q^m} \int_C e^{\nu q w}\left[f(qw) + 1\right]^{2/q} w^{-(m+1)} dw. \end{aligned} \tag{11}$$

Let

$$T(q, w, \nu) = \exp\left\{-w + \nu q w + \frac{2}{q} \log\left(f(qw) + 1\right)\right\}, \tag{12}$$

where the logarithm is to the base e. Then

$$W_{\beta,r}(n, n - m) = \binom{n}{m} \frac{m! \beta^m}{2\pi i q^m} \int_C \frac{T(q, w, \nu)}{w^{m+1}} dw. \tag{13}$$

Consider $h(q, w) = e^{qw} - 1$. The Maclaurin series of $h(q, w)$ is given by

$$h(q, w) = \sum_{k=0}^{\infty} \frac{(qw)^k}{k!} - 1 = \sum_{k=1}^{\infty} \frac{q^k w^k}{k!}. \tag{14}$$

Thus,

$$\begin{aligned} f(qw) &= \frac{h(q, w)}{qw} - 1 = \sum_{k=1}^{\infty} \frac{q^{k-1} w^{k-1}}{k!} - 1 \\ &= \sum_{k=2}^{\infty} \frac{w^{k-1}}{k!} q^{k-1}. \end{aligned} \tag{15}$$

Let $G(q,w)=\log[f(qw)+1]$. Then

$$G(q,w)=\frac{w}{2}q+\frac{w^2}{24}q^2+0q^3-\frac{w^4}{2880}q^4+\cdots, \quad (16)$$

$$H(q,w)=\frac{G(q,w)}{q}=\frac{w}{2}+\frac{w^2}{24}q+0q^2-\frac{w^4}{2880}q^3+\cdots. \quad (17)$$

Note that $T(q,w,\nu)=\exp[F(q,w,\nu)]$, where $F(q,w,\nu)=-w+\nu qw+2H(q,w)$ and $F(0,w,\nu)=0$.

Writing $T_k(w,\nu)=T^k(0,w,\nu)/k!$, we have

$$T(q,w,\nu)=\sum_{k=0}^{\infty}T_k(w,\nu)q^k=1+\sum_{k=1}^{\infty}T_k(w,\nu)q^k. \quad (18)$$

We prove the following lemma.

Lemma 1. *$T_k(w,\nu)$ is a polynomial in w whose lowest power in w is at least k.*

Proof. Let $E(q,w)=e^{-w+2H(q,w)}$ and $L(q,w)=e^{\nu qw}$. Then

$$\begin{aligned}T(q,w,\nu)&=e^{-w+\nu qw}\exp\left[\frac{2}{q}\log[f(qw)+1]\right]\\&=E(q,w)L(q,w).\end{aligned} \quad (19)$$

By Leibniz Rule,

$$\begin{aligned}\left[\frac{d^kT}{dq^k}\right]_{q=0}&=\left[E(q,w)L^k(q,w)\right]_{q=0}\\&\quad+\left[\sum_{p=1}^{k}\binom{k}{p}E^{(p)}(q,w)L^{(k-p)}(q,w)\right]_{q=0},\end{aligned} \quad (20)$$

where $E^{(p)}(q,w)$ denotes the pth derivative of $E(q,w)$ with respect to q and $L^{(k-p)}(q,w)$ denotes the $(k-p)$th derivative of $L(q,w)$ with respect to q and $E^{(0)}(q,w)=E(q,w)$.

Denote the lowest power of w in a polynomial $P(w)$ by $\eta[P(w)]$. From the computations above, $\eta[f(qw)]=1$; $\eta[f^{(k-1)}(qw)]_{q=0}=k-1$; $\eta[H(q,w)]=1$; $\eta[H^{(k)}(0,w)]=k+1$. With $h(q)=-w+2H(q,w)$, $h^{(k)}(q)=2H^{(k)}(q,w)$. Hence, $\eta[h^{(k)}(0,w)]=k+1$.

To find $\eta[E^{(k)}(0,w)]$, note that the concern is only the power of w, so we omit the details of the constant coefficients in the formula. With $E(q,w)=e^{h(q,w)}$ and applying Faa di Bruno's formula on the mth derivative of a composite function, the following will be obtained:

$$\begin{aligned}&\left[E^{(k)}(q,w)\right]_{q=0}\\&=\left[e^{-w+2H(q,w)}\sum c_i\left(h'(q)\right)^{b_1}\left(h''(q)\right)^{b_2}\cdots\left(h^{(k)}(q)\right)^{b_k}\right]_{q=0},\end{aligned} \quad (21)$$

where c_i denotes the constant coefficient.

The factor $e^{-w+2H(q,w)}$ in the above expression for $[E^{(k)}(q,w)]_{q=0}$ does not contribute to the resulting power of w because $H(0,w)=w/2$; and hence at $q=0$, $e^{-w+2H(0,w)}=e^0=1$. Thus, we only need to count the power of w in each term of the sum. Each $h^{(j)}(0)$, if it does occur as a factor in a term, contributes at least $(j+1)b_j$ in the power of w. Hence, the lowest power of w in $E^{(k)}(0,w)$ is $k+i$, where $1\le i\le k$. The least i is 1; thus; the least power of w in $E^k(0,w)$ is $k+1$. Using the greatest value of i which is k, we get $2k$ as the greatest power of w in $E^{(k)}(0,w)$. Now, we have

$$\begin{aligned}&\eta\left[E(q,w)L^{(k)}(q,w)\right]\\&\qquad=\eta\left[E(q,w)\right]+\eta\left[L^{(k)}(q,w)\right]=0+k=k,\end{aligned} \quad (22)$$

while

$$\eta\left[E^{(p)}(q,w)L^{(k-p)}(q,w)\right]=p+1+k-p=k+1. \quad (23)$$

Note that

$$\begin{aligned}\left[\frac{d^kT}{dq^k}\right]_{q=0}&=\left[E(q,w)L^{(k)}(q,w)\right]_{q=0}\\&\quad+\left[\sum_{p=1}^{k}\binom{k}{p}E^{(p)}(\mathrm{q},w)L^{(k-p)}(q,w)\right]_{q=0}.\end{aligned} \quad (24)$$

Hence, $T_k(w,\nu)$ is a polynomial in w whose lowest power in w is at least k. □

In particular, for $k=1,2,3$, the computation for $T_k(w,\nu)$ gives

$$\begin{aligned}T_1(w,\nu)&=\nu w+\frac{w^2}{12},\\T_2(w,\nu)&=\nu^2w^2+\frac{\nu w^3}{6}+\frac{w^4}{144},\\T_3(w,\nu)&=\nu^3w^3+\left(\frac{\nu^2}{4}-\frac{1}{240}\right)w^4\\&\quad+\frac{\nu w^5}{48}+\frac{w^6}{1728}.\end{aligned} \quad (25)$$

Continuing in the derivation of the formula, we see that

$$\begin{aligned}&W_{\beta,r}(n,n-m)\\&=\binom{n}{m}\left(\frac{\beta}{q}\right)^m\left[\left(\frac{d^m}{dw^m}e^w\right)_{w=0}\right.\\&\qquad\left.+\left(\frac{d^m}{dw^m}\sum_{k=1}^{m}T_k(w,\nu)q^ke^w\right)_{w=0}\right].\end{aligned} \quad (26)$$

Note that the upper limit of the sum is replaced by m because, for $k > m$, the mth derivative of the sum evaluated at $w = 0$ is 0. Hence

$$W_{\beta,r}(n, n-m) = \binom{n}{m}\left(\frac{\beta}{q}\right)^m \left[1 + \sum_{k=1}^{m} q^k \left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0}\right]. \tag{27}$$

To find the first few terms of the sum in (27), we solve $((d^m/dw^m)T_k(w,\nu)e^w)_{w=0}$ for $k = 1, 2, 3, \ldots$ using the formula

$$\begin{aligned}\left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0} &= \sum_{j=0}^{m} \binom{m}{j} \left(\frac{d^j}{dw^j} T_k(w,\nu)\right)_{w=0} \left(\frac{d^{m-j}}{dw^{m-j}} e^w\right)_{w=0} \\ &= \sum_{j=0}^{m} \frac{(m)_j}{j!} \left(\frac{d^j}{dw^j} T_k(w,\nu)\right)_{w=0}.\end{aligned} \tag{28}$$

It follows from the preceding lemma that, for $j < k$,

$$\left(\frac{d^j}{dw^j} T_k(w,\nu)\right)_{w=0} = 0. \tag{29}$$

Moreover, for $j \geq k$,

$$\left(\frac{d^j}{dw^j} T_k(w,\nu)\right)_{w=0} = j!\left[w^j\right], \tag{30}$$

where $[w^j]$ is the coefficient of w^j in $T_k(w,\nu)$. Thus,

$$\left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0} = \sum_{j=k}^{m} (m)_j \left[w^j\right]. \tag{31}$$

The first few terms of (27) are given as follows:

$$\begin{aligned}W_{\beta,r}(n, n-m) &= \binom{n}{m}\left(\frac{\beta}{q}\right)^m \left\{\left\{m\nu + \frac{(m)_2}{12}\right\} q \right. \\ &\quad + \frac{1}{2}\left\{\nu^2 (m)_2 + \frac{\nu (m)_3}{6} + \frac{(m)_4}{144}\right\} q^2 \\ &\quad + \frac{1}{6}\left\{\nu^3 (m)_3 + \left[\frac{\nu^2}{4} - \frac{1}{240}\right](m)_4 \right. \\ &\quad \left.\left. + \frac{\nu (m)_5}{48} + \frac{(m)_6}{1728}\right\} q^3 + \cdots \right\}.\end{aligned} \tag{32}$$

Table 2

	Exact value	Approximate value
$W_{2,1}(100, 90)$	7.896×10^{32}	7.895×10^{32}
$W_{2,1}(100, 94)$	9.223×10^{20}	9.223×10^{20}
$W_{2,1}(100, 95)$	6.350×10^{17}	6.350×10^{17}
$W_{2,1}(100, 96)$	3.542×10^{14}	3.542×10^{14}

When $\nu = 0$, (32) will reduce to the formula obtained in [12]. Substituting $q = 2/(n-m)$ in (32) will yield

$$\begin{aligned}W_{\beta,r}(n, n-m) &= \binom{n}{m}\left(\frac{\beta(n-m)}{2}\right)^m \\ &\quad \times \left[1 + \frac{1}{n-m}\left\{2m\nu + \frac{(m)_2}{6}\right\} \right. \\ &\quad + \frac{1}{(n-m)^2}\left\{2\nu^2 (m)_2 + \frac{\nu (m)_3}{3} + \frac{(m)_4}{72}\right\} \\ &\quad + \frac{1}{(n-m)^3}\left\{\frac{4}{3}\nu^3 (m)_3 + \left[\frac{\nu^2}{3} - \frac{1}{180}\right](m)_4 \right. \\ &\quad \left.\left. + \frac{\nu (m)_5}{36} + \frac{(m)_6}{1296}\right\} + \cdots \right].\end{aligned} \tag{33}$$

The formula in (33) gives values correct up to even the 3rd digit for $m = 10, 6, 5, 4$; $r = 1$, $\beta = 2$ as shown in Table 2.

3. The Range of Validity of the Formula

To be able to use (33) as an exact formula beyond $m = 3$ requires finding

$$\left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0}, \tag{34}$$

for $k = 4, 5, \ldots$. Such computation is quite tedious considering that $T(q, w, \nu)$ is a composition of a number of functions. Hence, we need to establish the range of m for which (33) behaves as an asymptotic approximation for large n.

Write (27) in the form

$$\begin{aligned}W_{\beta,r}(n, m) &= \binom{n}{m}\left(\frac{\beta}{q}\right)^m \\ &\quad \times \left[1 + \sum_{k=1}^{s} q^k \left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0} + E_S\right],\end{aligned} \tag{35}$$

where

$$E_S = \sum_{k=s+1}^{m} q^k \left(\frac{d^m}{dw^m} T_k(w,\nu) e^w\right)_{w=0}. \tag{36}$$

Let

$$A_m^k = \left(\frac{d^m}{dw^m} T_k(w,\nu) e^w \right)_{w=0}. \tag{37}$$

Then, by Leibniz's rule,

$$A_m^k = \sum_{j=0}^{m} \binom{m}{j} T_k^{(j)}(0,\nu), \tag{38}$$

where $T_k^{(j)}(0,\nu)$ is the jth derivative of $T_k(w,\nu)$ evaluated at $w = 0$. Because $T_k(w,\nu)$ is a polynomial in w whose lowest power in w is at least k, we may write

$$A_m^k = \sum_{j=k}^{m} \binom{m}{j} T_k^{(j)}(0,\nu). \tag{39}$$

Consider the Maclaurin expansion of $h(q,w)$ in (16) and note that

$$\lim_{k\to\infty} \left| \frac{q^{k+1} w^{k+1}}{(k+1)!} \cdot \frac{k!}{q^k w^k} \right| = \lim_{k\to\infty} \left| \frac{qw}{k+1} \right| = 0. \tag{40}$$

Thus, by ratio test, the expansion of $h(q,w)$ in (14) is absolutely convergent. In particular, it is absolutely convergent if $|qw| < 1$.

Similarly, the series expansion of $f(q,w)$ in (15) is absolutely convergent if $|qw| < 1$. These imply that $f(q,w)$ and $h(q,w)$ are both analytic in the interior to the circle $|w| = 1$ and consequently, so is $T(q,w,\nu)$ as defined in (12). Moreover, by the maximum modulus principle $T_k(w,\nu)$ takes its maximum on the circle and not inside the circle. By Cauchy's inequality, we have

$$\left| T^{(j)}(0,\nu) \right| \le Mj!, \tag{41}$$

where M is the maximum value of $T_k(w,\nu)$ on the circle $|w| = 1$. Hence,

$$\begin{aligned} \left| A_m^k \right| &\le \sum_{j=k}^{m} \binom{m}{j} Mj! = M \sum_{j=k}^{m} \frac{m!\,j!}{j!\,(m-j)!} \\ &\le Mm! \sum_{n=0}^{\infty} \frac{1}{n!} \le Mm!e \le M(m)^{2k} e, \end{aligned} \tag{42}$$

where e denotes the natural number $e = 2.71828\ldots$. The last inequality above is justified by $m! = (m)_m \le (m)^m \le (m)^{2k}$, because $m \le 2k$ and the degree of $T_k(w)$ is at most $2k$.

An estimate of E_S is given by

$$\sum_{k=s+1}^{m} \left[Me \frac{2m^2}{n-m} \right]^k. \tag{43}$$

Note that the right hand side of the last inequality is a geometric series with common ratio

$$\rho = \frac{2m^2 Me}{n-m}. \tag{44}$$

Table 3

	Exact value	Approximate value	Relative error
$W_{7,4}(100,15)$	8.411×10^{178}	3.962×10^{168}	1.00000
$W_{7,4}(100,30)$	5.604×10^{174}	3.648×10^{170}	0.99993
$W_{7,4}(100,60)$	7.399×10^{122}	3.700×10^{122}	0.50000
$W_{7,4}(100,80)$	1.275×10^{70}	1.269×10^{70}	0.00539
$W_{7,4}(100,90)$	2.208×10^{38}	2.211×10^{38}	0.00118
$W_{7,4}(100,92)$	2.613×10^{31}	2.615×10^{31}	0.00095
$W_{7,4}(100,93)$	7.249×10^{27}	7.255×10^{27}	0.00083
$W_{7,4}(100,95)$	3.358×10^{20}	3.360×10^{20}	0.00058
$W_{7,4}(100,97)$	6.617×10^{12}	6.619×10^{12}	0.00035
$W_{7,4}(100,98)$	597867963	598005375	0.00023

If $m^2 = o(n-m)$, for sufficiently large n,

$$\left| E_s \right| \le 2 \left[2Me \frac{m^2}{n-m} \right]^{s+1}, \tag{45}$$

where M a finite constant. Therefore, (33) behaves as an asymptotic approximation for large values of n provided that $\lim_{n\to\infty}(m^2/(n-m)) = 0$. In other words, $m = o(\sqrt{n-m}) \le o(\sqrt{n})$. Thus, we have the following theorem.

Theorem 2. *The formula*

$$\begin{aligned} & W_{\beta,r}(n,n-m) \\ &= \binom{n}{m} \left(\frac{\beta(n-m)}{2} \right)^m \\ &\quad \times \left[1 + \frac{1}{n-m} \left\{ 2m\nu + \frac{(m)_2}{6} \right\} \right. \\ &\qquad + \frac{1}{(n-m)^2} \left\{ 2\nu^2 (m)_2 + \frac{\nu(m)_3}{3} + \frac{(m)_4}{72} \right\} \\ &\qquad + \frac{1}{(n-m)^3} \left\{ \frac{4}{3}\nu^3 (m)_3 + \left[\frac{\nu^2}{3} - \frac{1}{180} \right] (m)_4 \right. \\ &\qquad\qquad \left. \left. + \frac{\nu(m)_5}{36} + \frac{(m)_6}{1296} \right\} + \cdots \right] \end{aligned} \tag{46}$$

behaves as an asymptotic approximation as $n \to \infty$ for $n-m$ in the range $n - o(\sqrt{n}) \le n-m \le n$.

Table 3 displays the exact and approximate values of $W_{\beta,r}(n,n-m)$ and their corresponding relative errors when $\beta = 7$, $r = 4$, and $n = 100$.

We observe that the asymptotic formula for $W_{\beta,r}(n,m)$ in (6) is valid when $m > (1/4)n(r/\beta)$ such that $n - m \to \infty$ as $n \to \infty$. On the other hand, the asymptotic formula in Theorem 2 for $W_{\beta,r}(n,n-m)$ is valid when $n - o(\sqrt{n}) \le n - m \le n$ as $n \to \infty$. These two asymptotic formulas are complimentary to each other since, for large values of n, the former will give a good approximation when m is not close to n, while the preceding will work efficiently when m is close to n. However, the two asymptotic formulas will fail when $m \le (1/4)n(r/\beta)$. Hence, it is interesting to establish the asymptotic formula for $W_{\beta,r}(n,m)$ when $m \leq (1/4)n(r/\beta)$. Meantime, the formula in [11] which is obtained using saddle point method can be used for this range.

4. Asymptotic Formula with Real Parameters

Following Flajolet and Prodinger [13], we define

$$W_{\beta,r}(y,x) = \frac{y!}{2\pi i \beta^x x!} \int_{\mathscr{H}} \frac{e^{rz}\left(e^{\beta z}-1\right)^x}{z^{y+1}} dz, \quad (47)$$

where x and y are positive real numbers, $y!$ and $x!$ are generalized factorials defined via the gamma function as

$$y! = \Gamma(y+1), \qquad x! = \Gamma(x+1), \quad (48)$$

and $\mathscr{H}$ is the Hankel contour which starts at $-\infty$, circles the origin, and goes back to $-\infty$ subject to $|\Im z| < 2\pi$. The integral in (47) may be written in the form

$$W_{\beta,r}(y,x) = \beta^{y-x} \frac{y!}{2\pi i x!} \int_{\mathscr{H}} e^{\nu u}\left(e^u-1\right)^x \frac{du}{u^{y+1}}. \quad (49)$$

Consequently,

$$W_{\beta,r}(y,y-x) = \binom{y}{x} \frac{x!\beta^x}{2\pi i} \int_{\mathscr{H}} \frac{e^{\nu u}\left(e^u-1\right)^x}{u^{y+1}} du. \quad (50)$$

Then the computations from (10) up to the lemma are valid. Equation (50) becomes

$$\begin{aligned} W_{\beta,r}(y,y-x) &= \binom{y}{x} \frac{x!\beta^x}{2\pi i q^x} \int_{\mathscr{H}} \frac{T(q,w,\nu)}{w^{x+1}} dw \\ &= \binom{y}{x} \frac{x!\beta^x}{2\pi i q^x} \int_{\mathscr{H}} \frac{1+\sum_{k=1}^{\infty} T_k(w,\nu)}{w^{x+1}} dw, \end{aligned} \quad (51)$$

where $T_k(w,\nu)$ is a polynomial in w in Lemma 1. Then

$$\begin{aligned} W_{\beta,r}(y,y-x) &= \beta^x \binom{y}{x} \left(\frac{y-x}{2}\right)^x \left[\frac{x!}{2\pi i} \int_{\mathscr{H}} \frac{e^w}{w^{x+1}} \right. \\ &\quad \left. + \frac{x!}{2\pi i} \int_{\mathscr{H}} \sum_{k=1}^{\infty} \frac{T_k(w,\nu) q^k e^w}{w^{x+1}} dw \right]. \end{aligned} \quad (52)$$

To compute the first few terms of the integrals in (52), we use the computed value of T_1, T_2, and T_3 obtained in Section 2 and apply the following classical identity due to Hankel:

$$\frac{1}{2\pi i} \int_{\mathscr{H}} e^w w^{-x-1} dw = \frac{1}{\Gamma(x+1)}. \quad (53)$$

Computation yields

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathscr{H}} \frac{T_1(w,\nu) q^1 e^w}{w^{x+1}} dw \\ &\quad = q\left[\frac{\nu}{(x-1)!} + \frac{1}{12} + \frac{1}{(x-2)!}\right], \\ &\frac{1}{2\pi i} \int_{\mathscr{H}} \frac{T_2(w,\nu) q^2 e^w}{w^{x+1}} dw \\ &\quad = q^2\left[\frac{\nu^2}{(x-2)!} + \frac{\nu}{6} \cdot \frac{1}{(x-3)!} + \frac{1}{144} \cdot \frac{1}{(x-4)!}\right], \\ &\frac{1}{2\pi i} \int_{\mathscr{H}} \frac{T_3(w,\nu) q^3 e^w}{w^{x+1}} dw \\ &\quad = q^3\left[\frac{\nu^3}{(x-3)!} + \left(\frac{\nu^2}{4} - \frac{1}{240}\right) \frac{1}{(x-4)!} \right. \\ &\qquad \left. + \frac{\nu}{48} \cdot \frac{1}{(x-5)!} + \frac{1}{1728} \cdot \frac{1}{(x-6)!}\right]. \end{aligned} \quad (54)$$

Substituting to (52) gives the following asymptotic formula:

$$\begin{aligned} &W_{\beta,r}(y,y-x) \\ &= \binom{y}{x} \left(\frac{\beta(y-x)}{2}\right)^x \\ &\quad \times \left[1 + \frac{1}{y-x}\left\{2x\nu + \frac{(x)_2}{6}\right\} \right. \\ &\qquad + \frac{1}{(y-x)^2}\left\{2\nu^2(x)_2 + \frac{\nu(x)_3}{3} + \frac{(x)_4}{72}\right\} \\ &\qquad + \frac{1}{(y-x)^3}\left\{\frac{4}{3}\nu^3(x)_3 + \left[\frac{\nu^2}{3} - \frac{1}{180}\right](x)_4 \right. \\ &\qquad \left. \left. + \frac{\nu(x)_5}{36} + \frac{(x)_6}{1296}\right\} + \cdots \right], \end{aligned} \quad (55)$$

which is analogous to the asymptotic formula in Theorem 2.

For the range of validity of this formula, we observe that, in (52), we can let

$$E_s = \frac{x!}{2\pi i} \int_{\mathscr{H}} \sum_{k=s+1}^{\infty} \frac{T_k(w,\nu) q^k e^w}{w^{x+1}} dw. \quad (56)$$

Since the series is convergent, we can interchange the order of integration and summation. Thus,

$$E_s = \frac{x!}{2\pi i} \sum_{k=s+1}^{\infty} q^k \int_{\mathscr{H}} \frac{T_k(w,\nu) e^w}{w^{x+1}} dw. \quad (57)$$

Note that

$$\frac{x!}{2\pi i}\int_{\mathscr{H}}\frac{T_k(w,\nu)\,q^k e^w}{w^{x+1}}dw = \frac{x!}{2\pi i}\int_{\mathscr{H}}\frac{q^k e^w \sum_{j=k}^{2k} a_j w^j}{w^{x+1}}dw = x!q^k\sum_{j=k}^{2k}a_j\frac{1}{2\pi i}\int_{\mathscr{H}} w^{-(x+1-j)}e^w dw. \quad (58)$$

Using the classical identity of Hankel, we obtain

$$\frac{x!}{2\pi i}\int_{\mathscr{H}}\frac{T_k(w,\nu)\,q^k e^w}{w^{x+1}}dw = x!q^k\sum_{j=k}^{2k}a_j\frac{1}{\Gamma(x+1-j)} = q^k\sum_{j=k}^{2k}a_j\frac{\Gamma(x+1)}{\Gamma(x+1-j)}. \quad (59)$$

Since a_j is finite for all $j = k, k+1, \ldots, 2k$, there is a constant M such that $|a_j| \le M$. Thus, we have

$$\left|\frac{x!}{2\pi i}\int_{\mathscr{H}}\frac{T_k(w,\nu)\,q^k e^w}{w^{x+1}}dw\right| \le M\left|\sum_{j=k}^{2k}q^k\frac{\Gamma(x+1)}{\Gamma(x+1-j)}\right|. \quad (60)$$

It is known that $\Gamma(x+1)/\Gamma(x+1-j) \sim x^j$. Hence,

$$\left|\frac{x!}{2\pi i}\int_{\mathscr{H}}\frac{T_k(w,\nu)\,q^k e^w}{w^{x+1}}dw\right| \le M\left|q^k\sum_{j=k}^{2k}x^j\right| = M\left|q^k x^{2k}\sum_{j=1}^{k}\frac{1}{x^j}\right| \le M\left|q^k x^{2k}\frac{1}{1-1/x}\right|. \quad (61)$$

Then,

$$\left|E_s\right| \le \sum_{k=s+1}^{\infty}M\left|q^k x^{2k}\frac{1}{1-1/x}\right| \le 2^k M\left|\frac{1}{1-1/x}\right|\sum_{k=s+1}^{\infty}\left|\left(\frac{x^2}{y-x}\right)^k\right| \le 2^k M\left|\frac{1}{1-1/x}\right|\left(\frac{x^2}{y-x}\right)^{s+1}\sum_{k=0}^{\infty}\left|\left(\frac{x^2}{y-x}\right)^k\right|. \quad (62)$$

The series

$$\sum_{k=0}^{\infty}\left|\left(\frac{x^2}{y-x}\right)^k\right| \quad (63)$$

is bounded provided $x^2 = o(y-x)$. Moreover, the factor $1/(1-1/x)$ is bounded for $x \ne 1$. Hence, the expansion for $W_{\beta,r}(y, y-x)$ behaves as an asymptotic formula when $x = o(\sqrt{y-x}) \le o(\sqrt{y})$, that is, when $y - o(\sqrt{y}) \le y - x \le y$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank Mindanao State University-Main Campus, Marawi City, Philippines for partially funding this research. The authors also wish to thank the referees for their corrections and suggestions that improved the clarity of this paper.

References

[1] I. Mezo, "A new formula for the bernoulli polynomials," *Results in Mathematics*, vol. 58, no. 3, pp. 329–335, 2010.

[2] A. Rucinski and B. Voigt, "A local limit theorem for generalized stirling numbers," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 35, no. 2, pp. 161–172, 1990.

[3] R. B. Corcino, C. B. Corcino, and R. Aldema, "Asymptotic normality of the (r, β)-stirling numbers," *Ars Combinatoria*, vol. 81, pp. 81–96, 2006.

[4] R. B. Corcino and C. B. Corcino, "On the maximum of generalized stirling numbers," *Utilitas Mathematica*, vol. 86, pp. 241–256, 2011.

[5] R. B. Corcino and R. Aldema, "Some combinatorial and statistical applications of (r, β)—stirling numbers," *Matimy'as Mat*, vol. 25, no. 1, pp. 19–29, 2002.

[6] M. Merca, "A convolution for complete and elementary symmetric functions," *Aequationes Mathematicae*, vol. 86, no. 3, pp. 217–229, 2013.

[7] M. Merca, "A note on the r-Whitney numbers of Dowling lattices," *Comptes Rendus de l'Académie des Sciences I*, vol. 351, no. 16-17, pp. 649–655, 2013.

[8] G.-S. Cheon and J.-H. Jung, "r -Whitney numbers of Dowling lattices," *Discrete Mathematics*, vol. 312, no. 15, pp. 2337–2348, 2012.

[9] M. Rahmani, "Some results on Whitney numbers of Dowling lattices," *Arab Journal of Mathematical Sciences*, vol. 20, no. 1, pp. 11–27, 2013.

[10] H. Belbachir and I. E. Bousbaa, "Translated whitney and r-whitney numbers: a combinatorial approach," *Journal of Integer Sequences*, vol. 16, article 13.8.6, 2013.

[11] C. B. Corcino and R. B. Corcino, "Asymptotic estimates for second kind generalized stirling numbers," *Journal of Applied Mathematics*, vol. 2013, Article ID 592818, 7 pages, 2013.

[12] L. Moser and M. Wyman, "Stirling numbers of the second kind," *Duke Mathematics*, vol. 25, pp. 29–43, 1958.

[13] P. Flajolet and H. Prodinger, "On stirling numbers for complex arguments and Hankel contours," *SIAM Journal on Discrete Mathematics*, vol. 12, no. 2, pp. 155–159, 1999.