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Research Article

Some Generalized Gronwall-Bellman Type Impulsive Integral Inequalities and Their Applications

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This paper investigates some generalized Gronwall-Bellman type impulsive integral inequalities containing integration on infinite intervals. Some new results are obtained, which generalize some existing conclusions. Our result is also applied to study a boundary value problem of differential equations with impulsive terms.

1. Introduction

It is well known that Gronwall-Bellman type integral inequalities involving functions of one and more than one independent variables play important roles in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of the theory of differential and integral equations. A lot of contributions to its generalization have been archived by many researchers (see [1–14]). Pachpatte [15] especially studied the following inequality:

$$u(x) \le a(x) + \int_x^\infty b(s)\, u(s)\, ds \tag{1}$$

containing integration on infinite integral and used it in the study of terminal value problems for Gronwall-Bellman type differential equations. Then, Cheung and Ma [16] generalized it into two independent variables with a nonlinear term:

$$u(x, y) \le a(x, y) + c(x, y) \int_x^\infty \int_y^\infty d(s, t)\, \omega(u(s, t))\, ds\, dt. \tag{2}$$

Along the development of the theory of impulsive differential systems, more and more attention is paid to generalizations of Gronwall-Bellman's results for discontinuous functions (that is, impulsive integral inequalities) and their applications (see [17–25]). Among them, one of the important things is that Samoilenko and Perestyuk [17] considered

$$u(x) \le c + \int_{x_0}^x f(s)\, u(s)\, ds + \sum_{x_0 < x_i < x} \beta_i u(x_i - 0) \tag{3}$$

about the nonnegative piecewise continuous function $u(x)$ where c, β_i are nonnegative constants, $f(x)$ is a positive function, and x_i are the first kind discontinuity points of the function $u(x)$. Then Borysenko [18] investigated integral inequalities with two independent variables:

$$\begin{aligned} u(x, y) \le{}& a(x, y) + \int_{x_0}^x \int_{y_0}^y \tau(s, t)\, u(s, t)\, ds\, dt \\ &+ \sum_{(x_0, y_0) < (x_i, y_i) < (x, y)} \beta_i u(x_i - 0, y_i - 0). \end{aligned} \tag{4}$$

Here $u(x, y)$ is an unknown nonnegative continuous function with the exception of the points (x_i, y_i) where there is a finite jump $u(x_i - 0, y_i - 0) \ne u(x_i + 0, y_i + 0)$ for $i = 1, 2, \ldots$.

In 2013, Zheng [25] considered the following delay integral inequalities containing integration on infinite intervals:

$$u(x) \le c + \int_x^\infty f_1(x,s)\, u(\tau(s))\, ds + \int_x^\infty f_2(x,s)\, \omega(u(\tau(s)))\, ds + \sum_{x<x_i<\infty} \beta_i u(x_i - 0), \tag{5}$$

$$u(x,y) \le c + \int_x^\infty \int_y^\infty f_1(s,t)\, u(\sigma(s),\tau(t))\, ds\, dt + \int_x^\infty \int_y^\infty f_2(s,t)\, \omega(u(\sigma(s),\tau(t)))\, ds\, dt + \sum_{x<x_i<\infty, y<y_i<\infty} \beta_i u(x_i - 0, y_i - 0) \tag{6}$$

with one general nonlinear term $\omega(u)$. They assumed that $\omega \in \wp$ where the class $\wp$ consists of all nonnegative, nondecreasing, and continuous functions $\omega(u)$ on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(\alpha u) \le \omega(\alpha)\omega(u)$ for all $\alpha > 0$ and $u \ge 0$. Actually, when we study behaviors of solutions of differential equations with impulsive terms, ω may not satisfy the following condition: $\omega \in \wp$. For example, $\omega(u) = e^u$ does not belong to the class $\wp$ for any $\alpha > 1$ and large $u > 0$. Thus, it is very interesting to avoid such conditions. Our main aim here, motivated by the work above, is to discuss the following much more general integral inequality:

$$u(x) \le a(x) + \sum_{k=1}^{2} \int_x^\infty f_k(x,s)\, \omega_k(u(\sigma_k(s)))\, ds + \sum_{x<x_i<\infty} \beta_i u^m(x_i - 0), \quad m > 0, \tag{7}$$

$$u(x,y) \le a(x,y) + \sum_{k=1}^{2} \int_x^\infty \int_y^\infty f_k(x,y,s,t) \times \omega_k(u(\sigma_k(s),\tau_k(t)))\, ds\, dt + \sum_{x<x_i<\infty, y<y_i<\infty} \beta_i u^m(x_i - 0, y_i - 0), \quad m > 0 \tag{8}$$

with two nonlinear terms $\omega_1(u)$ and $\omega_2(u)$ where we do not restrict ω_1 and ω_2 to the class $\wp$. Moreover, our main results are applied to estimate the bounds of solutions of differential equations with impulsive terms.

2. Main Results

In what follows, $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}_+ = [0,\infty)$, and $D_1 z(x,y)$ denotes the first-order partial derivative of $z(x,y)$ with respect to x.

Consider (7) and assume that

(H_1) $f_k(x,s)$ ($k = 1,2$) is a continuous and nonnegative function for $x, s \in \mathbf{R}_+$ and is bounded in $x \in \mathbf{R}_+$ for each fixed $s \in \mathbf{R}_+$;

(H_2) $\omega_1(u)$ and $\omega_2(u)$ are continuous and nonnegative functions on $[0,\infty)$ and positive on $(0,\infty)$ such that $\omega_2(u)/\omega_1(u)$ is nondecreasing;

(H_3) $u(x)$ is a nonnegative and continuous function defined on $\mathbf{R}_+$ with the first kind of discontinuities at the points x_i where $i = 1,2,\ldots,n$ and $0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty$;

(H_4) $a(x)$ is a continuous and bounded function for $x \in \mathbf{R}_+$ and $a(\infty) \ne 0$; β_i is a nonnegative constant for any positive integer i;

(H_5) $\sigma_1(x)$ and $\sigma_2(x)$ are continuous and nonnegative functions on $\mathbf{R}_+$ such that $\sigma_k(x) \ge x$ and $\sigma_k(x) \le x_i$ for $x \in [x_{i-1}, x_i)$, $i = 1,2,\ldots,n+1$, and $k = 1,2$.

Let $W_j(u) = \int_{\tilde{u}_j}^{u} (dz/\omega_j(z))$ for $u \ge \tilde{u}_j$ and $j = 1,2$ where $\tilde{u}_j$ is a given positive constant. Clearly, W_j is strictly increasing so its inverse W_j^{-1} is well defined, continuous, and increasing in its corresponding domain.

Theorem 1. *Suppose that (H_1)–(H_5) hold and $u(x)$ satisfies (7) for a positive constant m. If one lets $u_{i-1}(x) = u(x)$ for $x \in [x_{i-1}, x_i)$, $i = 1,2,\ldots,n+1$, then the estimate of $u(x)$ is recursively given by*

$$u_{i-1}(x) \le W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_{i-1}(x)) + \int_x^{x_i} \tilde{f}_1(x,s)\, ds\right) + \int_x^{x_i} \tilde{f}_2(x,s)\, ds\right] \tag{9}$$

for $x \in [x_{i-1}, x_i)$ and $i = 1,2,\ldots,n+1$, where

$$r_n(x) = \sup_{x \le \tau < \infty} |a(\tau)|,$$
$$\tilde{f}_k(x,s) = \sup_{x \le \tau < \infty} f_k(\tau,s), \quad k = 1,2,$$
$$r_{i-1}(x) = r_n(x) + \sum_{j=i}^{n} \sum_{k=1}^{2} \int_{x_j}^{x_{j+1}} f_k(x,s)\, \omega_k\left(u_j(\sigma(s))\right) ds + \sum_{j=i}^{n} \beta_j u_j^m\left(x_j - 0\right), \quad i = 1,2,\ldots,n, \tag{10}$$

provided that

$$W_1\left(r_{i-1}(x)\right)+\int_x^{x_i}\tilde{f}_1(x,s)\,ds\le\int_{\tilde{u}_1}^{\infty}\frac{dz}{\omega_1(z)},$$

$$W_2\circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right)+\int_x^{x_i}\tilde{f}_1(x,s)\,ds\right)+\int_x^{x_i}\tilde{f}_2(x,s)\,ds\le\int_{\tilde{u}_2}^{\infty}\frac{dz}{\omega_2(z)}. \tag{11}$$

Proof. From the assumptions, we know that $r_n(x)$ and $\tilde{f}_k(x,s)$ $(k=1,2)$ are well defined. Moreover, $r_n(x)$ is nonnegative and nonincreasing in x and $\tilde{f}_k(x,s)$ is nonnegative and nonincreasing in x and satisfies $a(x)\le r_n(x)$, $f_k(x,s)\le\tilde{f}_k(x,s)$.

Case 1. If $x\in[x_n,\infty)$ (in fact, $x_{n+1}=\infty$), from the definition of σ_k, we have $\sigma_k(x)\in[x_n,\infty)$ $(k=1,2)$. According to (7) and (10) we get

$$u(x)\le r_n(x)+\sum_{k=1}^{2}\int_x^{\infty}\tilde{f}_k(x,s)\,\omega_k\left(u\left(\sigma_k(s)\right)\right)ds. \tag{12}$$

Take any fixed $T\in[x_n,\infty)$, and we investigate the following inequality:

$$u(x)\le r_n(T)+\sum_{k=1}^{2}\int_x^{\infty}\tilde{f}_k(T,s)\,\omega_k\left(u\left(\sigma_k(s)\right)\right)ds \tag{13}$$

for $x\in[T,\infty)$. Let

$$z(x)=\sum_{k=1}^{2}\int_x^{\infty}\tilde{f}_k(T,s)\,\omega_k\left(u\left(\sigma_k(s)\right)\right)ds \tag{14}$$

and let $z(\infty)=0$. Hence, $u(x)\le r_n(T)+z(x)$. Clearly, $z(x)$ is a nonnegative, nonincreasing, and differentiable function for $x\in[T,\infty)$. The assumption $a(\infty)\ne0$ yields that $r_n(T)+z(x)>0$. Thus

$$\begin{aligned}
&\frac{z'(x)}{\omega_1\left(r_n(T)+z(x)\right)}\\
&\quad=\frac{-\tilde{f}_1(T,x)\,\omega_1\left(u\left(\sigma_1(x)\right)\right)-\tilde{f}_2(T,x)\,\omega_2\left(u\left(\sigma_2(x)\right)\right)}{\omega_1\left(r_n(T)+z(x)\right)}\\
&\quad\ge\Big(-\tilde{f}_1(T,x)\,\omega_1\left(r_n(T)+z\left(\sigma_1(x)\right)\right)\\
&\qquad\quad-\tilde{f}_2(T,x)\,\omega_2\left(r_n(T)+z\left(\sigma_2(x)\right)\right)\Big)\\
&\qquad\times\left(\omega_1\left(r_n(T)+z(x)\right)\right)^{-1}\\
&\quad\ge-\frac{\tilde{f}_1(T,x)\,\omega_1\left(r_n(T)+z(x)\right)}{\omega_1\left(r_n(T)+z(x)\right)}\\
&\qquad-\frac{\tilde{f}_2(T,x)\,\omega_2\left(r_n(T)+z(x)\right)}{\omega_1\left(r_n(T)+z(x)\right)}\\
&\quad\ge-\tilde{f}_1(T,x)-\frac{\tilde{f}_2(T,x)\,\omega_2\left(r_n(T)+z(x)\right)}{\omega_1\left(r_n(T)+z(x)\right)}.
\end{aligned} \tag{15}$$

Integrating both sides of the above inequality from x to ∞, we obtain

$$\begin{aligned}
&W_1\left(r_n(T)\right)-W_1\left(r_n(T)+z(x)\right)\\
&\quad\ge-\int_x^{\infty}\tilde{f}_1(T,s)\,ds\\
&\qquad-\int_x^{\infty}\tilde{f}_2(T,s)\,\phi\left(r_n(T)+z(s)\right)ds
\end{aligned} \tag{16}$$

for $x\in[T,\infty)$, where $\phi(x)=\omega_2(x)/\omega_1(x)$, so

$$\begin{aligned}
W_1\left(r_n(T)+z(x)\right)\le{}&W_1\left(r_n(T)\right)+\int_x^{\infty}\tilde{f}_1(T,s)\,ds\\
&+\int_x^{\infty}\tilde{f}_2(T,s)\,\phi\left(r_n(T)+z(s)\right)ds
\end{aligned} \tag{17}$$

or, equivalently,

$$\begin{aligned}
\xi(x)\le{}&W_1\left(r_n(T)\right)+\int_x^{\infty}\tilde{f}_1(T,s)\,ds\\
&+\int_x^{\infty}\tilde{f}_2(T,s)\,\phi\left(W_1^{-1}\left(\xi(s)\right)\right)ds\triangleq z_1(x),
\end{aligned} \tag{18}$$

where

$$\xi(x)=W_1\left(r_n(T)+z(x)\right). \tag{19}$$

It is easy to check that $\xi(x)\le z_1(x)$, $z_1(\infty)=W_1(r_n(T))$ and $z_1(x)$ is differentiable, positive, and nonincreasing on $[T,\infty)$. Since $\phi(W_1^{-1}(u))$ is nondecreasing, from the assumption (H_2), we have

$$\begin{aligned}
&\frac{z_1'(x)}{\phi\left(W_1^{-1}\left(z_1(x)\right)\right)}\\
&\quad=-\frac{\tilde{f}_1(T,x)}{\phi\left(W_1^{-1}\left(z_1(x)\right)\right)}-\frac{\tilde{f}_2(T,x)\,\phi\left(W_1^{-1}\left(\xi(x)\right)\right)}{\phi\left(W_1^{-1}\left(z_1(x)\right)\right)}\\
&\quad\ge-\frac{\tilde{f}_1(T,x)}{\phi\left[W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_x^{\infty}\tilde{f}_1(T,s)\,ds\right)\right]}-\tilde{f}_2(T,x).
\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
&\int_x^{\infty}\frac{z_1'(s)}{\phi\left(W_1^{-1}\left(z_1(s)\right)\right)}ds\\
&\quad=\int_x^{\infty}\frac{\omega_1\left(W_1^{-1}\left(z_1(s)\right)\right)z_1'(s)}{\omega_2\left(W_1^{-1}\left(z_1(s)\right)\right)}ds\\
&\quad=W_2\circ W_1^{-1}\left(z_1(\infty)\right)-W_2\circ W_1^{-1}\left(z_1(x)\right)\\
&\quad=W_2\left(r_n(T)\right)-W_2\circ W_1^{-1}\left(z_1(x)\right).
\end{aligned} \tag{21}$$

Integrating both sides of (20) from x to ∞, we obtain

$$\begin{aligned} & W_2\left(r_n(T)\right)-W_2 \circ W_1^{-1}\left(z_1(x)\right) \\ & \quad=\int_x^{\infty} \frac{z_1^{\prime}(s)}{\phi\left(W_1^{-1}\left(z_1(s)\right)\right)} d s \\ & \quad \geq-\int_x^{\infty} \frac{\widetilde{f}_1(T, s)}{\phi\left[W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_s^{\infty} \widetilde{f}_1(T, \tau) d \tau\right)\right]} d s \\ & \qquad-\int_x^{\infty} \widetilde{f}_2(T, s) d s \\ & \quad \geq-W_2 \circ W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_x^{\infty} \widetilde{f}_1(T, s) d s\right) \\ & \qquad+W_2\left(r_n(T)\right)-\int_x^{\infty} \widetilde{f}_2(T, s) d s . \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} & W_2 \circ W_1^{-1}\left(z_1(x)\right) \\ & \quad \leq W_2 \circ W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_x^{\infty} \widetilde{f}_1(T, s) d s\right) \\ & \qquad+\int_x^{\infty} \widetilde{f}_2(T, s) d s . \end{aligned} \tag{23}$$

We have by (11)

$$\begin{aligned} u(x) & \leq z(x)+r_n(T) \\ & \leq W_1^{-1}(\xi(x)) \leq W_1^{-1}\left(z_1(x)\right) \\ & \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_x^{\infty} \widetilde{f}_1(T, s) d s\right)\right. \\ & \left.\qquad+\int_x^{\infty} \widetilde{f}_2(T, s) d s\right] . \end{aligned} \tag{24}$$

Since the inequality above is true for any $x \in [T, \infty)$, we obtain

$$\begin{aligned} u(T) \leq W_2^{-1} & {\left[W_2 \circ W_1^{-1}\left(W_1\left(r_n(T)\right)+\int_T^{\infty} \widetilde{f}_1(T, s) d s\right)\right.} \\ & \left.+\int_T^{\infty} \widetilde{f}_2(T, s) d s\right] . \end{aligned} \tag{25}$$

Replacing T by x and ∞ by x_{n+1} yields

$$\begin{aligned} u(x) \leq W_2^{-1} & {\left[W_2 \circ W_1^{-1}\left(W_1\left(r_n(x)\right)+\int_x^{x_{n+1}} \widetilde{f}_1(x, s) d s\right)\right.} \\ & \left.+\int_x^{x_{n+1}} \widetilde{f}_2(x, s) d s\right] . \end{aligned} \tag{26}$$

This means that (9) is true for $x \in [x_n, \infty)$ if we replace $u(x)$ with $u_n(x)$.

Case 2. If $x \in [x_{n-1}, x_n)$, (7) becomes

$$\begin{aligned} u(x) \leq & r_n(x) \\ & +\sum_{k=1}^2 \int_{x_n}^{x_{n+1}} f_k(x, s) \omega_k\left(u_n\left(\sigma_k(s)\right)\right) d s \\ & +\beta_n u_n^m\left(x_n-0\right) \\ & +\sum_{k=1}^2 \int_x^{x_n} f_k(x, s) \omega_k\left(u\left(\sigma_k(s)\right)\right) d s \\ \leq & r_{n-1}(x)+\sum_{k=1}^2 \int_x^{x_n} f_k(x, s) \omega_k\left(u\left(\sigma_k(s)\right)\right) d s, \end{aligned} \tag{27}$$

where the definition of $r_{n-1}(x)$ is given in (10), which is similar to (12). Then, we obtain

$$\begin{aligned} u(x) \leq W_2^{-1} & {\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{n-1}(x)\right)+\int_x^{x_n} \widetilde{f}_1(x, s) d s\right)\right.} \\ & \left.+\int_x^{x_n} \widetilde{f}_2(x, s) d s\right] . \end{aligned} \tag{28}$$

This implies that (9) is true for $x \in [x_{n-1}, x_n)$ if we replace $u(x)$ by $u_{n-1}(x)$.

Case 3. If (7) is true for $x \in [x_i, x_{i+1})$, that is,

$$\begin{aligned} u_i(x) \leq W_2^{-1} & {\left[W_2 \circ W_1^{-1}\left(W_1\left(r_i(x)\right)+\int_x^{x_{i+1}} \widetilde{f}_1(x, s) d s\right)\right.} \\ & \left.+\int_x^{x_{i+1}} \widetilde{f}_2(x, s) d s\right], \end{aligned} \tag{29}$$

then, for $x \in [x_{i-1}, x_i)$, (7) becomes

$$\begin{aligned} u(x) \leq & r_n(x) \\ & +\sum_{j=i}^n \sum_{k=1}^2 \int_{x_j}^{x_{j+1}} f_k(x, s) \omega_k\left(u_j\left(\sigma_k(s)\right)\right) d s \\ & +\sum_{j=i}^n \beta_j u_j^m\left(x_j-0\right) \\ & +\sum_{k=1}^2 \int_x^{x_i} f_k(x, s) \omega_k\left(u\left(\sigma_k(s)\right)\right) d s \\ \leq & r_{i-1}(x)+\sum_{k=1}^2 \int_x^{x_i} f_k(x, s) \omega_k\left(u\left(\sigma_k(s)\right)\right) d s, \end{aligned} \tag{30}$$

where we use the fact that the estimate of $u(x)$ is already known for $x \in [x_j, x_{j+1})$, $j = i, i+1, \dots, n$. By assumption

(29), again (30) is the same as (27) if we replace $r_{n-1}(x)$ by $r_{i-1}(x)$ and x_n by x_i. Thus, by (28), we have

$$u(x) \le W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds\right) + \int_x^{x_i} \widetilde{f}_2(x,s)\,ds\right]. \tag{31}$$

This completes the proof of Theorem 1 by mathematical induction. □

Remark 2. Zheng [25] investigated (5) which is the special case of (7). His results are under the assumptions that $a(x) = c$, $f_1(x,s)$, $f_2(x,s)$ are decreasing in s for every fixed s and $\omega \in \wp$. In our result, these assumptions are avoided.

Consider the inequality

$$\varphi(u(x)) \le a(x) + \sum_{k=1}^{2}\int_x^{\infty} f_k(x,s)\,\omega_k\left(u\left(\sigma_k(s)\right)\right)ds + \sum_{x<x_i<\infty}\beta_i\psi\left(u\left(x_i-0\right)\right), \tag{32}$$

which looks much more complicated than (7).

Corollary 3. *In addition to the assumptions* (H_1)–(H_5), *suppose that* $\psi(u)$ *is positive on* $(0,\infty)$, $\varphi(u)$ *is positive and strictly increasing on* $(0,\infty)$, *and* $u(x)$ *satisfies* (32). *If one lets* $u_{i-1}(x) = u(x)$ *for* $x \in [x_{i-1}, x_i)$, $i = 1,2,\ldots,n+1$, *then the estimate of* $u(x)$ *is recursively given by*

$$u_{i-1}(x) \le \varphi^{-1}\left\{W_2^{-1}\left[W_2 \circ W_1^{-1} \times\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds\right) + \int_x^{x_i} \widetilde{f}_2(x,s)\,ds\right]\right\}, \tag{33}$$

where $W_j(u) = \int_{\widetilde{u}_j}^{u}\left(dz/\omega_j\left(\varphi^{-1}(z)\right)\right)$, $r_n(x)$, *and* $\widetilde{f}_k(x,s)$ *are given in Theorem 1 and* $r_{i-1}(x)$ *is defined as follows:*

$$r_{i-1}(x) = r_n(x) + \sum_{j=i}^{n}\sum_{k=1}^{2}\int_{x_j}^{x_{j+1}} f_k(x,s)\,\omega_k\left(u_j\left(\sigma_k(s)\right)\right)ds + \sum_{j=i}^{n}\beta_j\psi\left(u_j\left(x_j-0\right)\right), \quad i = 1,2,\ldots,n, \tag{34}$$

provided that

$$\begin{gathered} W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds \le \int_{\widetilde{u}_1}^{\infty}\frac{dz}{\omega_1(z)}, \\ W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds\right) + \int_x^{x_i} \widetilde{f}_2(x,s)\,ds \le \int_{\widetilde{u}_2}^{\infty}\frac{dz}{\omega_2(z)}. \end{gathered} \tag{35}$$

Proof. Let $\varphi(u(x)) = h(x)$. Since the function φ is strictly increasing on $[0,\infty)$, its inverse φ^{-1} is well defined. And (32) becomes

$$h(x) \le a(x) + \sum_{k=1}^{2}\int_x^{\infty} f_k(x,s)\,\omega_k\left(\varphi^{-1}\left(h\left(\sigma_k(s)\right)\right)\right)ds + \sum_{x<x_i<\infty}\beta_i\psi\left(\varphi^{-1}\left(h\left(x_i-0\right)\right)\right). \tag{36}$$

Let $\widetilde{\omega}_k = \omega_k \circ \varphi^{-1}$ and $\widetilde{\psi} = \psi \circ \varphi^{-1}$; (36) becomes

$$h(x) \le a(x) + \sum_{k=1}^{2}\int_x^{\infty} f_k(x,s)\,\widetilde{\omega}_k\left(h\left(\sigma_k(s)\right)\right)ds + \sum_{x<x_i<\infty}\beta_i\widetilde{\psi}\left(h\left(x_i-0\right)\right). \tag{37}$$

It is easy to see that $\widetilde{\psi}(u) > 0$, $\widetilde{\omega}_1(u)$ and $\widetilde{\omega}_2(u)$ are continuous and nonnegative functions on $[0,\infty)$, and $\widetilde{\omega}_2(u)/\widetilde{\omega}_1(u)$ is nondecreasing on $(0,\infty)$. Even though $\widetilde{\psi}(u)$ is much more general, using the same way in Theorem 1, for $x \in [x_{i-1}, x_i)$, $i = 1,2,\ldots,n+1$, we can obtain the estimate of $u(x)$:

$$u_{i-1}(x) \le \varphi^{-1}\left\{W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\left(r_{i-1}(x)\right) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds\right) + \int_x^{x_i} \widetilde{f}_2(x,s)\,ds\right]\right\}. \tag{38}$$

This completes the proof of Corollary 3. □

If $\varphi(u) = u^{\lambda}$ where $\lambda > 0$ is a constant, we can study the inequality

$$u^{\lambda}(x) \le a(x) + \sum_{k=1}^{2}\int_x^{\infty} f_k(x,s)\,\omega_k\left(u\left(\sigma_k(s)\right)\right)ds + \sum_{x<x_i<\infty}\beta_i\psi\left(u\left(x_i-0\right)\right). \tag{39}$$

According to Corollary 3, we have the following result.

Corollary 4. *In addition to the assumptions* (H_1)–(H_5), *suppose that* $\psi(u)$ *is positive on* $(0,\infty)$ *and* $u(x)$ *satisfies* (39).

If one lets $u_{i-1}(x) = u(x)$ *for* $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n+1$, *then the estimate of* $u(x)$ *is recursively given by*

$$u_{i-1}(x) \leq \left\{W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1(r_{i-1}(x)) + \int_x^{x_i} \widetilde{f}_1(x,s)\,ds\right) + \int_x^{x_i} \widetilde{f}_2(x,s)\,ds\right]\right\}^{1/\lambda}, \tag{40}$$

where $W_j(u) = \int_{\widetilde{u}_j}^u (dz/\omega(z^{1/\lambda}))$, $r_n(x)$, $r_{i-1}(x)$, *and* $\widetilde{f}_k(x,s)$ *are given in Corollary 3.*

Let

$$\Omega = \cup_{i,j\geq 1}\Omega_{ij},$$
$$\Omega_{ij} = \left\{(x,y) : x_{i-1} \leq x < x_i,\ y_{j-1} \leq y < y_j\right\}, \tag{41}$$

for $i, j = 1, 2, \ldots, n+1$, $0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = \infty$, and $0 < y_0 < y_1 < y_2 < \cdots < y_n < y_{n+1} = \infty$.

Consider (8) and assume that

(C_1) $f_k(x,y,s,t)$ $(k = 1, 2)$ is continuous and nonnegative on $\Omega \times \Omega$ and bounded in $(x,y) \in \Omega$ for each fixed $(s,t) \in \Omega$ and satisfies $f_k(x,y,s,t) = 0$ $(k = 1,2)$ if $(s,t) \in \Omega_{ij}$, $i \neq j$ for arbitrary $i, j = 1, 2, \ldots, n+1$;

(C_2) $\omega_1(u)$ and $\omega_2(u)$ are continuous and nonnegative functions on $[0,\infty)$ and are positive on $(0,\infty)$ such that $\omega_2(u)/\omega_1(u)$ is nondecreasing;

(C_3) $u(x,y)$ is nonnegative and continuous on Ω with the exception of the points (x_i, y_i) where there is a finite jump: $u(x_i - 0, y_i - 0) \neq u(x_i + 0, y_i + 0)$, $i = 1, 2, \ldots, n$;

(C_4) $a(x,y)$ is continuous and bounded for $(x,y) \in \Omega$ and $a(\infty,\infty) \neq 0$; β_i is a nonnegative constant for any positive integer i;

(C_5) $\sigma_k(x)$ and $\tau_k(y)$ $(k = 1,2)$ are continuous and nonnegative such that $\sigma_k(x) \geq x$ and $\sigma_k(x) \leq x_i$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \ldots, n+1$, and $\tau_k(y) \geq y$ and $\tau_k(y) \leq y_i$ for $y \in [y_{i-1}, y_i)$, $i = 1, 2, \ldots, n+1$.

Theorem 5. *Suppose that* (C_1)–(C_5) *hold and* $u(x,y)$ *satisfies* (8) *for a positive constant* m. *If one lets* $u_i(x,y) = u(x,y)$ *for* $(x,y) \in \Omega_{ii}$, $i = 1, 2, \ldots, n$, *then the estimate of* $u(x,y)$ *is recursively given by*

$$\begin{aligned} u_{i-1}(x,y) &\leq W_2^{-1}\Big[W_2 \circ W_1^{-1} \\ &\quad \times \left(W_1(r_{i-1}(x,y)) + \int_x^{x_i}\int_y^{y_i} \widetilde{f}_1(x,y,s,t)\,ds\,dt\right) \\ &\quad + \int_x^{x_i}\int_y^{y_i} \widetilde{f}_2(x,y,s,t)\,ds\,dt\Big], \end{aligned} \tag{42}$$

for $(x,y) \in \Omega_{ii}$, $i = 1, 2, \ldots, n+1$, *where*

$$\begin{aligned} r_n(x,y) &= \sup_{x\leq\xi<\infty}\ \sup_{y\leq\eta<\infty} |a(\xi,\eta)|, \\ \widetilde{f}_k(x,y,s,t) &= \sup_{x\leq\xi<\infty}\ \sup_{y\leq\eta<\infty} f_k(\xi,\eta,s,t), \\ r_{i-1}(x,y) &= r_n(x,y) \\ &\quad + \sum_{j=i}^n\sum_{k=1}^2 \int_{x_j}^{x_{j+1}}\int_{y_j}^{y_{j+1}} f_k(x,y,s,t)\,\omega_k(u_j(s,t))\,ds\,dt \\ &\quad + \sum_{j=i}^n \beta_j u_j^m(x_j - 0, y_j - 0), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{43}$$

provided that

$$\begin{aligned} &W_1(r_{i-1}(x,y)) + \int_x^{x_i}\int_y^{y_i} \widetilde{f}_1(x,y,s,t)\,ds\,dt \leq \int_{\widetilde{u}_1}^\infty \frac{dz}{\omega_1(z)}, \\ &W_2 \circ W_1^{-1}\left[W_1(r_{i-1}(x,y)) + \int_x^{x_i}\int_y^{y_i} \widetilde{f}_1(x,y,s,t)\,ds\,dt\right] \\ &\quad + \int_x^{x_i}\int_y^{y_i} \widetilde{f}_2(x,y,s,t)\,ds\,dt \leq \int_{\widetilde{u}_2}^\infty \frac{dz}{\omega_2(z)}. \end{aligned} \tag{44}$$

Proof. Obviously, for any $(x,y) \in \Omega$, $r_n(x,y)$ is positive and nonincreasing with respect to x and y; $\widetilde{f}_k(x,y,s,t)$ $(k = 1,2)$ is nonnegative and nonincreasing with respect to x and y for each fixed s and t. They satisfy $a(x,y) \leq r_n(x,y)$ and $f_k(x,y,s,t) \leq \widetilde{f}_k(x,y,s,t)$.

Case 1. If $(x,y) \in \Omega_{n+1,n+1} = \{(x,y) : x_n \leq x < x_{n+1},\ y_n \leq y < y_{n+1}\}$, we have from (8)

$$\begin{aligned} u(x,y) &\leq r_n(x,y) \\ &\quad + \sum_{k=1}^2 \int_x^\infty\int_y^\infty \widetilde{f}_k(x,y,s,t)\,\omega_k \\ &\qquad \times (u(\sigma_k(s), \tau_k(t)))\,ds\,dt. \end{aligned} \tag{45}$$

Take any fixed $\widetilde{x} \in [x_n, \infty)$, $\widetilde{y} \in [y_n, \infty)$, and for arbitrary $x \in [\widetilde{x}, \infty)$, $y \in [\widetilde{y}, \infty)$, we get

$$\begin{aligned} u(x,y) &\leq r_n(\widetilde{x},\widetilde{y}) \\ &\quad + \sum_{k=1}^2 \int_x^\infty\int_y^\infty \widetilde{f}_k(\widetilde{x},\widetilde{y},s,t)\,\omega_k \\ &\qquad \times (u(\sigma_k(s), \tau_k(t)))\,ds\,dt. \end{aligned} \tag{46}$$

Let

$$z(x,y) = r_n(\tilde{x},\tilde{y}) + \sum_{k=1}^{2}\int_x^\infty\int_y^\infty \tilde{f}_k(\tilde{x},\tilde{y},s,t)\,\omega_k \times (u(\sigma_k(s),\tau_k(t)))\,ds\,dt \tag{47}$$

and let $z(\infty, y) = r_n(\tilde{x},\tilde{y})$. Hence, $u(x,y) \le z(x,y)$. Clearly, $z(x,y)$ is a nonnegative, nonincreasing, and differentiable function for $x \in [\tilde{x},\infty)$ and $y \in [\tilde{y},\infty)$. Since $a(\infty,\infty) \ne 0$ and $\omega_1(z(x,y)) > 0$, we have

$$\begin{aligned}
\frac{D_1 z(x,y)}{\omega_1(z(x,y))} &= -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,\omega_1(u(\sigma_1(x),\tau_1(t)))\,dt}{\omega_1(z(x,y))} \\
&\quad - \frac{\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\omega_2(u(\sigma_2(x),\tau_2(t)))\,dt}{\omega_1(z(x,y))} \\
&\ge -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,\omega_1(z(\sigma_1(x),\tau_1(t)))\,dt}{\omega_1(z(x,y))} \\
&\quad - \frac{\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\omega_2(z(\sigma_2(x),\tau_2(t)))\,dt}{\omega_1(z(x,y))} \\
&\ge -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,\omega_1(z(x,t))\,dt}{\omega_1(z(x,y))} \\
&\quad - \frac{\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\omega_2(z(x,t))\,dt}{\omega_1(z(x,y))} \\
&\ge -\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,dt \\
&\quad - \int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\frac{\omega_2(z(x,t))}{\omega_1(z(x,t))}dt.
\end{aligned} \tag{48}$$

Integrating both sides of the above inequality from x to ∞, we obtain

$$\begin{aligned}
&W_1(z(\infty,y)) - W_1(z(x,y)) \\
&\quad \ge -\int_x^\infty\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},s,t)\,ds\,dt \\
&\qquad - \int_x^\infty\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},s,t)\,\frac{\omega_2(z(s,t))}{\omega_1(z(s,t))}ds\,dt.
\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}
W_1(z(x,y)) &\le W_1(r_n(\tilde{x},\tilde{y})) \\
&\quad + \int_x^\infty\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},s,t)\,ds\,dt \\
&\quad + \int_x^\infty\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},s,t)\,\phi(z(s,t))\,ds\,dt
\end{aligned} \tag{50}$$

for $\tilde{x} \le x < \infty$ and $\tilde{y} \le y < \infty$, where $\phi(u) = \omega_2(u)/\omega_1(u)$, or equivalently

$$\begin{aligned}
\xi(x,y) &\le W_1(r_n(\tilde{x},\tilde{y})) \\
&\quad + \int_x^\infty\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},s,t)\,ds\,dt \\
&\quad + \int_x^\infty\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},s,t)\,\phi\left(W_1^{-1}(\xi(s,t))\right)ds\,dt \\
&\triangleq z_1(x,y),
\end{aligned} \tag{51}$$

where

$$\xi(x,y) = W_1(z(x,y)). \tag{52}$$

It is easy to check that $\xi(x,y) \le z_1(x,y)$, $z_1(\infty,y) = W_1(r_n(\tilde{x},\tilde{y}))$, and $z_1(x,y)$ is differentiable, positive, and nonincreasing on $[\tilde{x},\infty)$ and $[\tilde{y},\infty)$. Since $\phi(W_1^{-1}(u))$ is nondecreasing, from assumption (C_2), we have

$$\begin{aligned}
&\frac{D_1 z_1(x,y)}{\phi(W_1^{-1}(z_1(x,y)))} \\
&= -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,dt}{\phi(W_1^{-1}(z_1(x,y)))} \\
&\quad - \frac{\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\phi\left(W_1^{-1}(\xi(x,t))\right)dt}{\phi(W_1^{-1}(z_1(x,y)))} \\
&\ge -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,dt}{\phi\left[W_1^{-1}\left(W_1(r_n(\tilde{x},\tilde{y})) + \int_x^\infty\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},s,t)\,ds\,dt\right)\right]} \\
&\quad - \frac{\int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,\phi\left(W_1^{-1}(z_1(x,t))\right)dt}{\phi(W_1^{-1}(z_1(x,y)))} \\
&\ge -\frac{\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},x,t)\,dt}{\phi\left[W_1^{-1}\left(W_1(r_n(\tilde{x},\tilde{y})) + \int_x^\infty\int_y^\infty \tilde{f}_1(\tilde{x},\tilde{y},s,t)\,ds\,dt\right)\right]} \\
&\quad - \int_y^\infty \tilde{f}_2(\tilde{x},\tilde{y},x,t)\,dt.
\end{aligned} \tag{53}$$

Note that

$$\begin{aligned}
&\int_x^\infty \frac{D_1 z_1(s,y)}{\phi(W_1^{-1}(z_1(s,y)))}ds \\
&\quad = \int_x^\infty \frac{D_1 z_1(s,y)\,\omega_1\left(W_1^{-1}(z_1(s,y))\right)}{\omega_2(W_1^{-1}(z_1(s,y)))}ds \\
&\quad = \int_{W_1^{-1}(z_1(x,y))}^{W_1^{-1}(z_1(\infty,y))} \frac{du}{\omega_2(u)}
\end{aligned}$$

$$\begin{aligned}
&= W_2 \circ W_1^{-1}(z_1(\infty, y)) - W_2 \circ W_1^{-1}(z_1(x, y)) \\
&= W_2 \circ W_1^{-1}(W_1(r_n(\tilde{x}, \tilde{y}))) - W_2 \circ W_1^{-1}(z_1(x, y)) \\
&= W_2(r_n(\tilde{x}, \tilde{y})) - W_2 \circ W_1^{-1}(z_1(x, y)).
\end{aligned} \tag{54}$$

Integrating both sides of (53) from x to ∞, we obtain

$$\begin{aligned}
&W_2(r_n(\tilde{x}, \tilde{y})) - W_2 \circ W_1^{-1}(z_1(x, y)) \\
&= \int_x^\infty \frac{D_1 z_1(s, y)}{\phi(W_1^{-1}(z_1(s, y)))} ds \\
&\geq -\int_x^\infty \frac{\int_y^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, s, t)\,dt}{\phi\left[W_1^{-1}\left(W_1(r_n(\tilde{x}, \tilde{y})) + \int_s^\infty \int_y^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, \tau, t)\,d\tau\,dt\right)\right]} ds \\
&\quad - \int_x^\infty \int_y^\infty \tilde{f}_2(\tilde{x}, \tilde{y}, s, t)\,ds\,dt \\
&\geq -W_2 \circ W_1^{-1}\left[W_1(r_n(\tilde{x}, \tilde{y})) + \int_x^\infty \int_y^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\right] \\
&\quad + W_2(r_n(\tilde{x}, \tilde{y})) - \int_x^\infty \int_y^\infty \tilde{f}_2(\tilde{x}, \tilde{y}, s, t)\,ds\,dt.
\end{aligned} \tag{55}$$

Thus,

$$\begin{aligned}
&W_2 \circ W_1^{-1}(z_1(x, y)) \\
&\quad \leq W_2 \circ W_1^{-1}\left[W_1(r_n(\tilde{x}, \tilde{y})) + \int_x^\infty \int_y^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\right] \\
&\qquad + \int_x^\infty \int_y^\infty \tilde{f}_2(\tilde{x}, \tilde{y}, s, t)\,ds\,dt.
\end{aligned} \tag{56}$$

Hence,

$$\begin{aligned}
u(x, y) &\leq z(x, y) \leq W_1^{-1}(\xi(x, y)) \leq W_1^{-1}(z_1(x, y)) \\
&\leq W_2^{-1}\Bigg[W_2 \circ W_1^{-1} \\
&\qquad \times \left(W_1(r_n(\tilde{x}, \tilde{y})) + \int_x^\infty \int_y^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\right) \\
&\qquad + \int_x^\infty \int_y^\infty \tilde{f}_2(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\Bigg].
\end{aligned} \tag{57}$$

Since the above inequality is true for any $x \in [\tilde{x}, \infty)$, $y \in [\tilde{y}, \infty)$, we obtain

$$\begin{aligned}
u(\tilde{x}, \tilde{y}) &\leq W_2^{-1}\Bigg[W_2 \circ W_1^{-1} \\
&\qquad \times \left(W_1(r_n(\tilde{x}, \tilde{y})) + \int_{\tilde{x}}^\infty \int_{\tilde{y}}^\infty \tilde{f}_1(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\right) \\
&\qquad + \int_{\tilde{x}}^\infty \int_{\tilde{y}}^\infty \tilde{f}_2(\tilde{x}, \tilde{y}, s, t)\,ds\,dt\Bigg].
\end{aligned} \tag{58}$$

Replacing $\tilde{x}$, $\tilde{y}$, and ∞ by x, y, and x_{n+1}, respectively, yields

$$\begin{aligned}
&u(x, y) \\
&\leq W_2^{-1}\Bigg[W_2 \circ W_1^{-1} \\
&\qquad \times \left(W_1(r_n(x, y)) + \int_x^{x_{n+1}} \int_y^{y_{n+1}} \tilde{f}_1(x, y, s, t)\,ds\,dt\right) \\
&\qquad + \int_x^{x_{n+1}} \int_y^{y_{n+1}} \tilde{f}_2(x, y, s, t)\,ds\,dt\Bigg].
\end{aligned} \tag{59}$$

This means that (42) is true for $(x, y) \in \Omega_{n+1,n+1}$ and $i = n$ if we replace $u(x, y)$ with $u_n(x, y)$.

Case 2. If $(x, y) \in \Omega_{n,n} = \{(x, y) : x_{n-1} \leq x < x_n,\ y_{n-1} \leq y < y_n\}$, (8) becomes

$$\begin{aligned}
u(x, y) &\leq r_n(x, y) \\
&\quad + \sum_{k=1}^{2} \int_{x_n}^{x_{n+1}} \int_{y_n}^{y_{n+1}} f_k(x, y, s, t)\,\omega_k \\
&\qquad \times (u_n(\sigma_k(s), \tau_k(t)))\,ds\,dt \\
&\quad + \beta_n u_n^m(x_n - 0, y_n - 0) \\
&\quad + \sum_{k=1}^{2} \int_x^{x_n} \int_y^{y_n} f_k(x, y, s, t)\,\omega_k \\
&\qquad \times (u(\sigma_k(s), \tau_k(t)))\,ds\,dt \\
&\leq r_{n-1}(x, y) \\
&\quad + \sum_{k=1}^{2} \int_x^{x_n} \int_y^{y_n} f_k(x, y, s, t)\,\omega_k \\
&\qquad \times (u(\sigma_k(s), \tau_k(t)))\,ds\,dt,
\end{aligned} \tag{60}$$

where the definition of $r_{n-1}(x, y)$ is given in (43). Note that the estimate of $u_n(x, y)$ is known. Clearly, (60) is the same as (45) if we replace $r_n(x,y)$ and (∞,∞) by $r_{n-1}(x,y)$ and (x_n,y_n). Thus, by (59), we have

$$u(x,y) \le W_2^{-1}\Big[W_2\circ W_1^{-1}\Big(W_1(r_{n-1}(x,y))+\int_x^{x_n}\int_y^{y_n}\widetilde{f}_1(x,y,s,t)\,ds\,dt\Big)+\int_x^{x_n}\int_y^{y_n}\widetilde{f}_2(x,y,s,t)\,ds\,dt\Big]. \quad (61)$$

This implies that (42) is true for $(x,y)\in\Omega_{n,n}$ and $i=n-1$ if we replace $u(x,y)$ by $u_{n-1}(x,y)$.

Case 3. Assume that (42) is true for $(x,y)\in\Omega_{i+1,i+1}=\{(x,y): x_i\le x<x_{i+1},\ y_i\le y<y_{i+1}\}$. Then for $(x,y)\in\Omega_{i,i}=\{(x,y): x_{i-1}\le x<x_i,\ y_{i-1}\le y<y_i\}$, (8) becomes

$$\begin{aligned} u(x,y) &\le r_n(x,y)+\sum_{j=i}^{n}\sum_{k=1}^{2}\int_{x_i}^{x_{i+1}}\int_{y_i}^{y_{i+1}} f_k(x,y,s,t)\,\omega_k\big(u_j(\sigma_k(s),\tau_k(t))\big)\,ds\,dt \\ &\quad+\sum_{j=i}^{n}\beta_j u_j^m(x_j-0,y_j-0) +\sum_{k=1}^{2}\int_x^{x_i}\int_y^{y_i} f_k(x,y,s,t)\,\omega_k(u(\sigma_k(s),\tau_k(t)))\,ds\,dt \\ &\le r_{i-1}(x,y)+\sum_{k=1}^{2}\int_x^{x_i}\int_y^{y_i} f_k(x,y,s,t)\,\omega_k(u(\sigma_k(s),\tau_k(t)))\,ds\,dt, \end{aligned} \quad (62)$$

where we use the fact that the estimate of $u(x,y)$ is already known for $(x,y)\in\Omega_{jj}$, $j=i,\dots,n$. Again, (62) is the same as (60) if we replace $r_{n-1}(x,y)$ and (x_n,y_n) by $r_{i-1}(x,y)$ and (x_i,y_i). Thus, by (61), we have

$$u(x,y) \le W_2^{-1}\Big[W_2\circ W_1^{-1}\Big(W_1(r_{i-1}(x,y))+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_1(x,y,s,t)\,ds\,dt\Big)+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_2(x,y,s,t)\,ds\,dt\Big]. \quad (63)$$

This yields that (42) is true for $(x,y)\in\Omega_{i,i}$ if we replace $u(x,y)$ by $u_{i-1}(x,y)$. By mathematical induction, we know that (42) holds for $(x,y)\in\Omega_{i,i}$ for any nonnegative integer i. This completes the proof of Theorem 5. □

Remark 6. (1) If $a(x,y)$ is nonincreasing in each variable $x,y\in\mathbf{R}_+$ and we take $f_1(x,y,s,t)=b(x,y)c(s,t)$, $f_2(x,y,s,t)=0$, $\sigma_k(x)=x$, $\tau_k(y)=y$, and $u(x,y)$ being continuous on $\mathbf{R}_+^2$, then (8) reduces to (2) and Theorem 1 becomes Theorem 2.2 in [16].

(2) Zheng [25] investigated (6) which is the special case of (8). His results are under the assumptions that $a(x,y)=c$, $f_k(x,y,s,t)=f_k(s,t)$, and $\omega\in\wp$. In our results, these assumptions are avoided.

Consider the inequality

$$\begin{aligned}\varphi(u(x,y)) &\le a(x,y)+\sum_{k=1}^{2}\int_x^{\infty}\int_y^{\infty} f_k(x,y,s,t)\,\omega_k(u(\sigma_k(s),\tau_k(t)))\,ds\,dt \\ &\quad+\sum_{x<x_i<\infty,\,y<y_i<\infty}\beta_i\psi(u(x_i-0,y_i-0)),\end{aligned} \quad (64)$$

which looks much more complicated than (8).

Corollary 7. *In addition to the assumptions (C_1)–(C_5), suppose that $\psi(u)$ is positive on $(0,\infty)$, $\varphi(u)$ is positive and strictly increasing on $(0,\infty)$, and $u(x,y)$ satisfies (64) for a positive constant m. If one lets $u_i(x,y)=u(x,y)$ for $(x,y)\in\Omega_{ii}$, then the estimate of $u(x,y)$ is recursively given by*

$$u_{i-1}(x,y)\le\varphi^{-1}\Big\{W_2^{-1}\Big[W_2\circ W_1^{-1}\Big(W_1(r_{i-1}(x,y))+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_1(x,y,s,t)\,ds\,dt\Big)+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_2(x,y,s,t)\,ds\,dt\Big]\Big\}, \quad (65)$$

where $W_j(u)=\int_{\widetilde{u}_j}^{u}(dz/\omega_j(\varphi^{-1}(z)))$, $r_n(x,y)$, and $\widetilde{f}_k(x,y,s,t)$ are given in Theorem 5; $r_{i-1}(x,y)$ is defined as follows:

$$\begin{aligned} r_{i-1}(x,y) &= r_n(x,y)+\sum_{j=i}^{n}\sum_{k=1}^{2}\int_{x_j}^{x_{j+1}}\int_{y_j}^{y_{j+1}} f_k(x,y,s,t)\,\omega_k\big(u_j(s,t)\big)\,ds\,dt \\ &\quad+\sum_{j=i}^{n}\beta_j\psi\big(u_j(x_j-0,y_j-0)\big),\quad i=1,2,\dots,n, \end{aligned} \quad (66)$$

provided that

$$\begin{aligned}
&W_1\left(r_{i-1}(x,y)\right)+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_1(x,y,s,t)\,ds\,dt\le\int_{\widetilde{u}_1}^{\infty}\frac{dz}{\omega_1(z)},\\
&W_2\circ W_1^{-1}\left[W_1\left(r_{i-1}(x,y)\right)+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_1(x,y,s,t)\,ds\,dt\right]\\
&\quad+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_2(x,y,s,t)\,ds\,dt\le\int_{\widetilde{u}_2}^{\infty}\frac{dz}{\omega_2(z)}.
\end{aligned}\tag{67}$$

The proof is similar to Corollary 3.

If $\varphi(u)=u^{\lambda}$, where $\lambda>0$ is a constant, we can study the inequality

$$\begin{aligned}
u^{\lambda}(x,y)&\le a(x,y)\\
&\quad+\sum_{k=1}^{2}\int_x^{\infty}\int_y^{\infty}f_k(x,y,s,t)\\
&\qquad\times\omega_k\left(u\left(\sigma_k(s),\tau_k(t)\right)\right)ds\,dt\\
&\quad+\sum_{x<x_i<\infty,y<y_i<\infty}\beta_i\psi\left(u\left(x_i-0,y_i-0\right)\right).
\end{aligned}\tag{68}$$

According to Corollary 7, we have the following result.

Corollary 8. *In addition to the assumptions (C_1)–(C_5), suppose that $\psi(u)$ is positive on $(0,\infty)$ and $u(x,y)$ satisfies (68) for a positive constant m. If one lets $u_i(x,y)=u(x,y)$ for $(x,y)\in\Omega_{ii}$, then the estimate of $u(x,y)$ is recursively given by*

$$\begin{aligned}
u_{i-1}(x,y)\le\Bigg\{W_2^{-1}\Bigg[W_2\circ W_1^{-1}&\\
\times\Bigg(W_1\left(r_{i-1}(x,y)\right)&\\
+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_1(x,y,s,t)\,ds\,dt\Bigg)&\\
+\int_x^{x_i}\int_y^{y_i}\widetilde{f}_2(x,y,s,t)\,ds\,dt\Bigg]\Bigg\}^{1/\lambda}&,
\end{aligned}\tag{69}$$

where $W_j(u)=\int_{\widetilde{u}_j}^{u}(dz/\omega(z^{1/\lambda}))$, $r_n(x,y)$, $r_{i-1}(x,y)$, and $\widetilde{f}_k(x,y,s,t)$ are given in Corollary 7.

3. Applications

Example 9. Consider the following impulsive differential equation:

$$\frac{dg}{dx}=F(x,g),\quad x\ne x_i,\tag{70}$$

$$\Delta g|_{x=x_i}=I_i(x),\qquad g(\infty)=\theta\ne0,\tag{71}$$

where $g:\mathbf{R}\to\mathbf{R}$, $F:\mathbf{R}^2\to\mathbf{R}$, $I_i:\mathbf{R}\to\mathbf{R}$ and $i=1,2,\ldots,n$, $0<x_0<x_1<x_2<\cdots<x_n<x_{n+1}=\infty$. Here, θ is a constant.

Assume that

(A_1) $|F(x,g)|\le h_1(x)e^{|g|}+h_2(x)e^{2|g|}$ where h_1 and h_2 are nonnegative, bounded, and continuous on $\mathbf{R}^+$;

(A_2) $|I_i(g)|\le\beta_i|g|^m$ where β_i and m are nonnegative constants.

Theorem 10. *Suppose that (A_1) and (A_2) hold. If one lets $g_{i-1}(x)=g(x)$ for $x\in[x_{i-1},x_i)$, $i=1,2,\ldots,n+1$, then the solution of (70) has an estimate for $x\in[x_{i-1},x_i)$:*

$$\begin{aligned}
&\left|g_{i-1}(x)\right|\\
&\quad\le-\frac{1}{2}\ln\left[\left(e^{-r_{i-1}(x)}-\int_x^{x_i}h_1(s)\,ds\right)^2-2\int_x^{x_i}h_2(s)\,ds\right],
\end{aligned}\tag{72}$$

where $r_n(x)=|\theta|$ and

$$\begin{aligned}
r_{i-1}(x)&=r_n(x)\\
&\quad+\sum_{j=i}^{n}\int_{x_j}^{x_{j+1}}h_1(s)e^{|g_j(s)|}ds\\
&\quad+\sum_{j=i}^{n}\int_{x_j}^{x_{j+1}}h_2(s)e^{2|g_j(s)|}ds\\
&\quad+\sum_{j=i}^{n}\beta_j\left|g_j\left(x_j-0\right)\right|^m,\quad i=1,2,\ldots,n,
\end{aligned}\tag{73}$$

$$\left(e^{-r_{i-1}(x)}-2\int_x^{x_i}h_1(s)\,ds\right)^2-2\int_x^{x_i}h_2(s)\,ds>0.$$

Proof. Integrating (70) from x to ∞ and using the initial conditions (71), we get

$$g(x)=\theta-\int_x^{\infty}F(s,g)\,ds-\sum_{x<x_i<\infty}I_i\left(g\left(x_i-0\right)\right),\tag{74}$$

which implies that

$$\begin{aligned}
|g(x)|&\le|\theta|\\
&\quad+\int_x^{\infty}h_1(s)e^{|g(s)|}ds+\int_x^{\infty}h_2(s)e^{2|g(s)|}ds\\
&\quad+\sum_{x<x_i<\infty}\beta_i\left|g\left(x_i-0\right)\right|^m.
\end{aligned}\tag{75}$$

Let

$$\begin{aligned}
&u(x)=|g(x)|,\qquad a(x)=|\theta|,\qquad\sigma_1(x)=\sigma_2(x)=x,\\
&f_1(x,s)=h_1(s),\qquad f_2(x,s)=h_2(s),\qquad\omega_1(u)=e^{u},\\
&\omega_2(u)=e^{2u}.
\end{aligned}\tag{76}$$

Thus, (75) is the same as (7). It is easy to obtain that for any positive constants $\tilde{u}_1$ and $\tilde{u}_2$

$$r_n(x) = |\theta|, \qquad \tilde{f}_1(x,s) = h_1(s), \qquad \tilde{f}_2(x,s) = h_2(s),$$
$$W_1(u) = \int_{\tilde{u}_1}^{u} \frac{dz}{\omega_1(z)} = \int_{\tilde{u}_1}^{u} e^{-z}dz = e^{-\tilde{u}_1} - e^{-u},$$
$$W_1^{-1}(u) = -\ln\left(e^{-\tilde{u}_1} - u\right),$$
$$W_2(u) = \int_{\tilde{u}_2}^{u} \frac{dz}{\omega_2(z)} = \int_{\tilde{u}_2}^{u} e^{-2z}dz = \frac{1}{2}\left(e^{-2\tilde{u}_2} - e^{-2u}\right),$$
$$W_2^{-1}(u) = -\frac{1}{2}\ln\left(e^{-2\tilde{u}_2} - 2u\right),$$
$$r_{i-1}(x) = r_n(x) + \sum_{j=i}^{n}\int_{x_j}^{x_{j+1}} h_1(s)\,e^{|g_j(s)|}ds + \sum_{j=i}^{n}\int_{x_j}^{x_{j+1}} h_2(s)\,e^{2|g_j(s)|}ds + \sum_{j=i}^{n}\beta_j\left|g_j\left(x_j-0\right)\right|^m. \tag{77}$$

Therefore, for any nonnegative i and $x \in [x_{i-1}, x_i)$

$$\left|g_{i-1}(x)\right| \le -\frac{1}{2}\ln\left[\left(e^{-r_{i-1}(x)} - \int_x^{x_i} h_1(s)\,ds\right)^2 - 2\int_x^{x_i} h_2(s)\,ds\right], \tag{78}$$

provided that

$$\left(e^{-r_{i-1}(x)} - 2\int_x^{x_i} h_1(s)\,ds\right)^2 - 2\int_x^{x_i} h_2(s)\,ds > 0. \tag{79}$$

□

Example 11. Consider the following partial differential equation with an impulsive term:

$$\begin{gathered}\frac{\partial^2 v(x,y)}{\partial x\partial y} = H(x,y,v(x,y)),\\ (x,y)\in\Omega_{ii}, \quad x\ne x_i, \quad y\ne y_i,\\ \Delta v|_{x=x_i,y=y_i} = I_i(v),\\ v(x,\infty)=\phi_1(x), \qquad v(\infty,y)=\phi_2(y),\\ \phi_1(\infty)=\phi_2(\infty)\ne 0,\end{gathered} \tag{80}$$

where $v : \mathbf{R}^2 \to \mathbf{R}$, $H : \mathbf{R}^3 \to \mathbf{R}$, $I_i : \mathbf{R} \to \mathbf{R}$, and $i = 1,2,\dots,n+1$.

Assume that

(B_1) $|H(x,y,v(x,y))| \le h_1(x,y)e^{|v(x,y)|} + h_2(x,y)e^{2|v(x,y)|}$ where h_1, h_2 are nonnegative, bounded, and continuous on Ω, $h_1(x,y) = 0$, $h_2(x,y) = 0$ for $(x,y) \in \Omega_{ij}$, $i \ne j$, $i,j = 1,2,\dots,n+1$;

(B_2) $|I_i(v)| \le \beta_i|v|^m$ where β_i and m are nonnegative constants.

Theorem 12. *Suppose that (B_1) and (B_2) hold. If one lets $v_i(x,y) = v(x,y)$ for $(x,y)\in\Omega_{ii}$, then the solution of system (80) has an estimate for $(x,y)\in\Omega_{ii}$:*

$$\left|v_i(x,y)\right| \le -\frac{1}{2}\ln\left[\left(e^{-r_{i-1}(x,y)} - \int_x^{x_i}\int_y^{y_i} h_1(s,t)\,ds\,dt\right)^2 - 2\int_x^{x_i}\int_y^{y_i} h_2(s,t)\,ds\,dt\right], \tag{81}$$

where

$$\begin{gathered}r_n(x,y) = \sup_{x\le\xi<\infty}\sup_{y\le\eta<\infty}\left|\phi_1(\xi)+\phi_2(\eta)-\phi_1(\infty)\right| > 0,\\ r_{i-1}(x,y) = r_n(x,y) + \sum_{j=i}^{n}\int_{x_j}^{x_{j+1}}\int_{y_j}^{y_{j+1}} h_1(s,t)\,e^{|v_j(s,t)|}ds\,dt + \sum_{j=i}^{n}\int_{x_j}^{x_{j+1}}\int_{y_j}^{y_{j+1}} h_2(s,t)\,e^{2|v_j(s,t)|}ds\,dt + \sum_{j=i}^{n}\beta_j\left|v_j\left(x_j-0,y_j-0\right)\right|^m,\\ \left(e^{-r_{i-1}(x,y)} - \int_x^{x_i}\int_y^{y_i} h_1(s,t)\,ds\,dt\right)^2 - 2\int_x^{x_i}\int_y^{y_i} h_2(s,t)\,ds\,dt > 0.\end{gathered} \tag{82}$$

Proof. The solution of (80) with an initial value is given by

$$\begin{aligned}v(x,y) &= v(x,\infty)+v(\infty,y)-v(\infty,\infty) + \int_x^{\infty}\int_y^{\infty} H(s,t,v(s,t))\,ds\,dt + \sum_{x<x_i<\infty,y<y_i<\infty} I_i\left(v\left(x_i-0,y_i-0\right)\right)\\ &= \phi_1(x)+\phi_2(y)-\phi_1(\infty) + \int_x^{\infty}\int_y^{\infty} H(s,t,v(s,t))\,ds\,dt + \sum_{x<x_i<\infty,y<y_i<\infty} I_i\left(v\left(x_i-0,y_i-0\right)\right),\end{aligned} \tag{83}$$

which implies that

$$\begin{aligned}|v(x,y)| \le{}& |\phi_1(x) + \phi_2(y) - \phi_1(\infty)| \\ &+ \int_x^\infty \int_y^\infty h_1(s,t)\, e^{|v(s,t)|} ds\, dt \\ &+ \int_x^\infty \int_y^\infty h_2(s,t)\, e^{2|v(s,t)|} ds\, dt \\ &+ \sum_{x<x_i<\infty, y<y_i<\infty} \beta_i |v(x_i - 0, y_i - 0)|^m.\end{aligned} \tag{84}$$

Similar to Theorem 10, we can obtain, for $(x, y) \in \Omega_{ii}$,

$$\begin{aligned}|v_i(x,y)| \le -\frac{1}{2} \ln \Bigg[\left(e^{-r_{i-1}(x,y)} - \int_x^{x_i} \int_y^{y_i} h_1(s,t)\, ds\, dt \right)^2 \\ -2 \int_x^{x_i} \int_y^{y_i} h_2(s,t)\, ds\, dt \Bigg].\end{aligned} \tag{85}$$

□

Remark 13. From Examples 9 and 11, we know that $\omega_1(u) = e^u$. Clearly, $\omega_1(2u) = e^{2u} \le \omega_1(2)\omega_1(u) = e^2 e^u$ does not hold for large $u > 0$. Thus, $\omega_1(u) = e^u$ does not belong to class $\wp$ in [25]. Hence, the results in [25] can not be applied to inequality (75).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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