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Research Article

Uniqueness and Existence of Solution for a System of Fractional q-Difference Equations

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We prove the existence and uniqueness of solution for a system of fractional differential equations. Our results are based on the nonlinear alternative of Leray-Schauder type and Banach's fixed-point theorem.

1. Introduction

This paper is mainly concerned with the uniqueness and existence of solution for a system of fractional q-difference equations given by

$$\begin{gathered}
{}^{C}D_q^{\alpha}u(t)=f(t,v(t)),\quad 1<\alpha\le 2,\ t\in[0,1],\\
{}^{C}D_q^{\beta}v(t)=g(t,u(t)),\quad 1<\beta\le 2,\ t\in[0,1],\\
\alpha_1u(0)-\beta_1D_qu(0)=\gamma_1u(\eta_1),\\
\alpha_2u(1)+\beta_2D_qu(1)=\gamma_2u(\eta_2),\\
\alpha_3v(0)-\beta_3D_qv(0)=\gamma_3v(\eta_3),\\
\alpha_4v(1)+\beta_4D_qv(1)=\gamma_4v(\eta_4),
\end{gathered}\tag{1}$$

where ${}^{C}D_q^{\alpha}$, ${}^{C}D_q^{\beta}$ is the fractional q-derivatives of the Caputo type, $1<\alpha,\beta\le 2$, α_i $(i=1,2,3,4)$, β_i $(i=1,2,3,4)$, γ_i $(i=1,2,3,4)$, and η_i $(i=1,2,3,4)$ are arbitrary real constants, and $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

In the last few years, fractional differential equations (in short FDEs) have been studied extensively. The motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For an extensive collection of such results, we refer the readers to the monographs by Kilbas et al. [1], Miller and Ross [2], Oldham and Spanier [3], Podlubny [4], and Samko et al. [5].

Some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala ([6–8]), Babakhani and Daftardar-Gejji ([9–11]), Bai [12], and so on. Also, there are some papers which deal with the existence and multiplicity of solutions (or positive solution) for nonlinear FDE of BVPs by using techniques of nonlinear analysis (fixed-point theorems, Leray-Shauder theory, topological degree theory, etc.)—see ([13–18]) and the references therein. The study of a coupled system of fractional order is also very significant because this kind of system can often occur in applications. The reader is referred to the papers ([19–22]) and the references cited therein.

The pioneer work on q-difference calculus or quantum calculus dates back to Jackson's papers ([23, 24]), while a systematic treatment of the subject can be found in [25, 26]. For some recent existence results on q-difference equations, see [27–29] and the references therein.

There has also been a growing interest on the subject of discrete fractional equations on the time scale $\mathbb{Z}$. Some interesting results on the topic can be found in a series of papers [30–38]. Fractional q-difference equations have recently attracted the attention of several researchers. For some earlier work on the topic, we refer to [39, 40], whereas some recent work on the existence theory of fractional

q-difference equations can be found in [41–45]. However, the study of boundary value problems of fractional q-difference equations is at its infancy and much of the work on the topic is yet to be done.

From the above works, we can see a fact, although the fractional boundary value problems have been investigated by some authors. To the best of our knowledge, there have been few papers which deal with problem (1) for nonlinear fractional differential equation. Motivated by all the works above, in this paper we discuss problem (1). Using nonlinear alternative of Leray-Schauder type, we will give the existence and uniqueness of solution for a system of fractional differential equations with Riemann-Liouville integral boundary conditions of different order (1).

The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of the main results. In Section 3, we establish the uniqueness and existence of a solution for the nonlinear fractional differential equation boundary value problem (1). In last section, we give two examples to illustrate our work.

2. Preliminaries and Lemmas

In this section, we cite some definitions and fundamental results of the q-calculus as well as of the fractional q-calculus ([46, 47]). We also give a lemma that will be used in obtaining the main results of the paper.

Let $q \in (0,1)$ and define [47]

$$[a]_q = \frac{q^a - 1}{q - 1} = a^{a-1} + \cdots + 1, \quad a \in \mathbb{R}. \tag{2}$$

The q-analogue of the power $(a-b)^n$ is

$$\begin{gathered} (a-b)^{(0)} = 1, \\ (a-b)^{(n)} = \prod_{k=0}^{n-1} \left(a - bq^k\right), \quad a, b \in \mathbb{R},\ n \in \mathbb{N}. \end{gathered} \tag{3}$$

If α is not a positive integer, then

$$(a-b)^{(\alpha)} = a^{\alpha} \prod_{i=0}^{\infty} \frac{\left(1 - (b/a)\, q^i\right)}{\left(1 - (b/a)\, q^{\alpha+i}\right)}. \tag{4}$$

Note that if $b = 0$, then $a^{(\alpha)} = a^{\alpha}$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},\ 0 < q < 1 \tag{5}$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$ (see [47]).

The q-derivative of a function f is here defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)\, x} \tag{6}$$

and q-derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x), & \text{if } n = 0, \\ D_q D_q^{n-1} f(x), & \text{if } n \in \mathbb{N}. \end{cases} \tag{7}$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$\begin{gathered} \int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n)\, q^n, \\ 0 \le |q| < 1, \quad x \in [0,b]. \end{gathered} \tag{8}$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{9}$$

Similarly, as done for derivatives, an operator I_q^n can be defined, namely, by

$$\begin{gathered} \left(I_q^0 f\right)(x) = f(x), \\ \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), \quad n \in \mathbb{N}. \end{gathered} \tag{10}$$

The fundamental theorem of calculus applies to these operators I_q and D_q; that is,

$$\left(D_q I_q f\right)(x) = f(x), \tag{11}$$

and if f is continuous at $x = 0$, then

$$\left(I_q D_q f\right)(x) = f(x) - f(0). \tag{12}$$

Basic properties of the two operators can be found in the book that is mentioned in [8]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i) [43]:

$$\begin{gathered} [a(t-s)]^{(\alpha)} = a^{\alpha} (t-s)^{(\alpha)}, \\ {}_tD_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \\ \left({}_xD_q \int_0^x f(x,t)\, d_q t\right)(x) = \int_0^x {}_xD_q f(x,t)\, d_q t + f(qx, x). \end{gathered} \tag{13}$$

Remark 1. We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [43].

Definition 2 (see [40]). Let $\alpha \ge 0$ and let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $({}_{\mathrm{RL}}I_q^0 f)(x) = f(x)$ and

$$\begin{gathered} \left({}_{\mathrm{RL}}I_q^{\alpha} f\right)(x) = \int_0^x \frac{(x-qt)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(t)\, d_q t, \\ \alpha \in \mathbb{R}^+, x \in [0,1]. \end{gathered} \tag{14}$$

Definition 3 (see [48]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $({}_{RL}D_q^0 f)(x) = f(x)$ and

$$\left({}_{RL}D_q^\alpha f\right)(x) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f\right)(x), \quad \alpha > 0, \tag{15}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Definition 4 (see [48]). The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$\left({}^C D_q^\alpha f\right)(x) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(x), \quad \alpha > 0, \tag{16}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 5. *Let $\alpha, \beta \geq 0$ and let f be a function defined on $[0,1]$. Then the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 6 (see [42]). *Let $\alpha \geq 0$ and $n \in \mathbb{N}$. Then the following equality holds:*

$$\left({}_{RL}I_q^\alpha \, {}_{RL}D_q^n f\right)(x) = {}_{RL}D_q^n \, {}_{RL}I_q^\alpha f(x) - \sum_{k=0}^{\alpha-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} \left(D_q^k f\right)(0). \tag{17}$$

Lemma 7 (see [48]). *Let $\alpha > 0$ and $n \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the following equality holds:*

$$\left(I_q^\alpha \, {}^C D_q^\alpha f\right)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^k}{\Gamma_q(k+1)} \left(D_q^k f\right)(0). \tag{18}$$

For convenience, one introduces the following notations:

$$b_1 = \frac{\gamma_1(\alpha_2-\gamma_2)}{\Delta}, \qquad b_2 = \frac{\gamma_1(\alpha_2+\beta_2-\gamma_2\eta_2)}{\Delta},$$
$$b_3 = \frac{(\alpha_1-\gamma_1)}{\Delta}, \qquad b_4 = \frac{(\beta_1+\gamma_1\eta_1)}{\Delta},$$
$$\Delta = (\gamma_2-\alpha_2)(\beta_1+\gamma_1\eta_1) + (\gamma_1-\alpha_1)(\alpha_2+\beta_2-\gamma_2\eta_2). \tag{19}$$

From Lemmas 5 and 7, we can obtain the following lemma.

Lemma 8. *Let $h \in C[0,1]$ and $\Delta \neq 0$; then the unique solution of the linear fractional boundary value problem*

$$\begin{gathered} {}^C D_q^\alpha u(t) = h(t), \quad 1 < \alpha \leq 2, \; t \in [0,1], \\ \alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1), \\ \alpha_2 u(1) + \beta_2 D_q u(1) = \gamma_2 u(\eta_2) \end{gathered} \tag{20}$$

is given by

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s \\ &+ (b_1 t + b_2) \int_0^{\eta_1} \frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s \\ &+ (b_3 t + b_4) \left[-\gamma_2 \int_0^{\eta_2} \frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s \right. \\ &+ \alpha_2 \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s \\ &\left. + \beta_2 \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} h(s)\, d_q s \right]. \end{aligned} \tag{21}$$

The following lemma is fundamental in the proofs of our main result.

Lemma 9 (see [49]; nonlinear alternative of Leray-Schauder type). *Let E be a Banach space with $M \subseteq E$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $F : \overline{U} \to C$ is continuous, compact (i.e., $F(U)$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$ or*

(ii) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F u$.*

3. Main Results

In this section, we will discuss the uniqueness and existence of solutions for boundary value problem (1).

First of all, we define the Banach space $X = \{u \mid u \in C[0,1]\}$ endowed with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. For $(u,v) \in X \times X$, let $\|(u,v)\| = \max\{\|u\|, \|v\|\}$; then $(X \times X, \|(\cdot,\cdot)\|)$ is a Banach space.

For convenience, we set

$$\blacktriangle = (\gamma_4-\alpha_4)(\beta_3+\gamma_3\eta_3) + (\gamma_3-\alpha_3)(\alpha_4+\beta_4-\gamma_4\eta_4), \tag{22}$$

and let $\blacktriangle \neq 0$. Note

$$\mathfrak{b}_1 = \frac{\gamma_3(\alpha_4-\gamma_4)}{\blacktriangle}, \qquad \mathfrak{b}_2 = \frac{\gamma_3(\alpha_4+\beta_4-\gamma_4\eta_4)}{\blacktriangle},$$
$$\mathfrak{b}_3 = \frac{(\alpha_3-\gamma_3)}{\blacktriangle}, \qquad \mathfrak{b}_4 = \frac{(\beta_3+\gamma_3\eta_3)}{\blacktriangle}. \tag{23}$$

Employing Lemma 8, system (1) can be expressed as

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s \\ &+ (b_1 t + b_2) \int_0^{\eta_1} \frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s \end{aligned}$$

$$+ \left(b_3 t + b_4\right)\left[-\gamma_2 \int_0^{\eta_2} \frac{(\eta_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s + \alpha_2 \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s + \beta_2 \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s, v(s))\, d_q s\right],$$

$$\begin{aligned} v(t) = {} & \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \left(\mathfrak{b}_1 t + \mathfrak{b}_2\right) \int_0^{\eta_3} \frac{(\eta_3 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s \\ & + \left(\mathfrak{b}_3 t + \mathfrak{b}_4\right)\left[-\gamma_4 \int_0^{\eta_4} \frac{(\eta_4 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \alpha_4 \int_0^1 \frac{(1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \beta_4 \int_0^1 \frac{(1 - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} g(s, u(s))\, d_q s\right], \end{aligned} \tag{24}$$

where b_1, b_2, b_3, b_4 are given by (21), and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{b}_4$ are given by (22)

From Lemma 8 in Section 2, we can obtain the following lemma.

Lemma 10. *Suppose that $f(t, v)$ and $g(t, u)$ are continuous; then $(u, v) \in X \times X$ is a solution of BVP (1) if and only if $(u, v) \in X \times X$ is a solution of the integral equations (24).*

Let $(u, v) \in X \times X$; define an operator $T : X \times X \to X \times X$ as

$$T(u, v)(t) = \left(T_1 v(t), T_2 u(t)\right), \tag{25}$$

where

$$\begin{aligned} T_1 v(t) = {} & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s + \left(b_1 t + b_2\right) \int_0^{\eta_1} \frac{(\eta_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s \\ & + \left(b_3 t + b_4\right)\left[-\gamma_2 \int_0^{\eta_2} \frac{(\eta_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s + \alpha_2 \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, v(s))\, d_q s + \beta_2 \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s, v(s))\, d_q s\right], \end{aligned} \tag{26}$$

$$\begin{aligned} T_2 u(t) = {} & \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \left(\mathfrak{b}_1 t + \mathfrak{b}_2\right) \int_0^{\eta_3} \frac{(\eta_3 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s \\ & + \left(\mathfrak{b}_3 t + \mathfrak{b}_4\right)\left[-\gamma_4 \int_0^{\eta_4} \frac{(\eta_4 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \alpha_4 \int_0^1 \frac{(1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} g(s, u(s))\, d_q s + \beta_4 \int_0^1 \frac{(1 - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} g(s, u(s))\, d_q s\right]; \end{aligned} \tag{27}$$

then, by Lemma 10, the fixed point of operator T coincides with the solution of system (1).

In the first result, we prove uniqueness of solution of the boundary value problem (1) via Banach's contraction principle.

Theorem 11. *Assume that $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and the following conditions hold:*

(H1) *there exist two q-integrable functions $L_1, L_2 : [0, 1] \to \mathbb{R}$ that satisfy*

$$\begin{aligned} \left|f(t, u) - f(t, v)\right| &\le L_1(t)\left|u - v\right|, \quad \forall t \in [0, 1],\ u, v \in \mathbb{R}; \\ \left|g(t, u) - g(t, v)\right| &\le L_2(t)\left|u - v\right|, \quad \forall t \in [0, 1],\ u, v \in \mathbb{R}. \end{aligned} \tag{28}$$

In addition, assume that

$$\begin{aligned} \kappa_1 &:= \left(1 + |\alpha_2|\, \delta_2 \left(I_q^{\alpha} L_1\right)(1) + |\gamma_1|\, \delta_1 \left(I_q^{\alpha} L_1\right)(\eta_1)\right) + |\gamma_2|\, \delta_2 \left(I_q^{\alpha} L_1\right)(\eta_2) + |\beta_2|\, \delta_2 \left(I_q^{\alpha-1} L_1\right)(1) < 1, \\ \kappa_2 &:= \left(1 + |\alpha_4|\, \delta_4 \left(I_q^{\beta} L_2\right)(1) + |\gamma_3|\, \delta_3 \left(I_q^{\beta} L_2\right)(\eta_3)\right) + |\gamma_4|\, \delta_4 \left(I_q^{\beta} L_2\right)(\eta_4) + |\beta_4|\, \delta_4 \left(I_q^{\beta-1} L_2\right)(1) < 1, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \delta_1 &:= \frac{|\alpha_2 - \gamma_2| + |\alpha_2 + \beta_2 - \gamma_2 \eta_2|}{|\Delta|}, \\ \delta_2 &:= \frac{|\alpha_1 - \gamma_1| + |\beta_1 - \gamma_1 \eta_1|}{|\Delta|}, \\ \delta_3 &:= \frac{|\alpha_4 - \gamma_4| + |\alpha_4 + \beta_4 - \gamma_4 \eta_4|}{|\blacktriangle|}, \\ \delta_4 &:= \frac{|\alpha_3 - \gamma_3| + |\beta_3 - \gamma_3 \eta_3|}{|\blacktriangle|}, \end{aligned} \tag{30}$$

where Δ and $\blacktriangle$ are given by (19) and (22), respectively. Then system (1) has a unique solution.

Proof. Let us set $\sup_{t\in[0,1]}|f(t,0)| = M_1 < \infty$, $\sup_{t\in[0,1]}|g(t,0)| = M_2 < \infty$,

$$A_1 = \frac{1}{\Gamma_q(\alpha+1)}\left(1+|\gamma_1|\delta_1\eta_1^{(\alpha-1)}+|\gamma_2|\delta_2\eta_2^{(\alpha-1)}+|\alpha_2|\delta_2\right) + \frac{|\beta_2|\delta_2}{\Gamma_q(\alpha)}, \tag{31}$$

$$A_2 = \frac{1}{\Gamma_q(\beta+1)}\left(1+|\gamma_3|\delta_3\eta_3^{(\beta-1)}+|\gamma_4|\delta_4\eta_4^{(\beta-1)}+|\alpha_4|\delta_4\right) + \frac{|\beta_4|\delta_4}{\Gamma_q(\beta)}. \tag{32}$$

Define

$$\begin{aligned} U &= \{(u,v)\in X\times X : \|(u,v)\|\le r\},\\ B_{r_1} &= \{u\in C[0,1] : |u|\le r_1\},\\ B_{r_2} &= \{v\in C[0,1] : |v|\le r_2\}. \end{aligned} \tag{33}$$

For $v\in B_{r_1}$, we obtain

$$\begin{aligned} |f(t,v(t))| &\le |f(t,v(t))-f(t,0)|+|f(t,0)|\\ &\le L_1(t)|v(t)|+|f(t,0)|\\ &\le L_1(t)r_1+M_1. \end{aligned} \tag{34}$$

Then, for $v\in B_{r_1}$, $t\in[0,1]$, we have

$$\begin{aligned} \|T_1v\| &\le \sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs\right.\\ &\quad+\left|\frac{\gamma_1}{\Delta}[(\alpha_2-\gamma_2)t-(\alpha_2+\beta_2-\gamma_2\eta_2)]\right|\\ &\quad\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs\\ &\quad+\left|\frac{\gamma_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs\\ &\quad+\left|\frac{\alpha_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs\\ &\quad+\left|\frac{\beta_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\left.\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,v(s))|\,d_qs\right\}\\ &\le \sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}[L_1(s)r_1+M_1]\,d_qs\right.\\ &\quad+\left|\frac{\gamma_1}{\Delta}[(\alpha_2-\gamma_2)t-(\alpha_2+\beta_2-\gamma_2\eta_2)]\right|\\ &\quad\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}[L_1(s)r_1+M_1]\,d_qs\\ &\quad+\left|\frac{\gamma_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}[L_1(s)r_1+M_1]\,d_qs\\ &\quad+\left|\frac{\alpha_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}[L_1(s)r_1+M_1]\,d_qs\\ &\quad+\left|\frac{\beta_2}{\Delta}[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)]\right|\\ &\quad\left.\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}[L_1(s)r_1+M_1]\,d_qs\right\}\\ &\le M_1\left\{\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs+|\gamma_1|\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\right.\\ &\quad+|\gamma_2|\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\\ &\quad+|\gamma_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\\ &\quad\left.+|\gamma_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}d_qs\right\}\\ &\quad+r_1\left\{\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs\right.\\ &\quad+|\gamma_1|\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs\\ &\quad+|\gamma_2|\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs\\ &\quad+|\gamma_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs\\ &\quad\left.+|\gamma_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}L_1(s)\,d_qs\right\} \end{aligned}$$

$$\leq M_1 \left\{ \frac{1}{\Gamma_q(\alpha+1)} \times \left(1 + |\gamma_1| \delta_1 \eta_1^{(\alpha-1)} + |\gamma_2| \delta_2 \eta_2^{(\alpha-1)} + |\alpha_2| \delta_2\right) + \frac{|\beta_2| \delta_2}{\Gamma_q(\alpha)} \right\}$$
$$+ r_1 \left\{ \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + |\gamma_1| \delta_1 \int_0^{\eta_1} \frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + |\gamma_2| \delta_2 \int_0^{\eta_2} \frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + |\alpha_2| \delta_2 \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + |\beta_2| \delta_2 \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} L_1(s)\, d_q s \right\}. \quad (35)$$

In view of (31), we obtain

$$\|T_1 v\| \leq M_1 A_1 + r_1 \kappa_1. \quad (36)$$

From the last estimate we deduce that $r_1 = M_1 A_1/(1-\kappa_1)$.

By a similar way as done above we have

$$\|T_2 u\| \leq M_2 \left\{ \frac{1}{\Gamma_q(\beta+1)} \times \left(1 + |\gamma_3| \delta_3 \eta_3^{(\beta-1)} + |\gamma_4| \delta_4 \eta_4^{(\beta-1)} + |\alpha_4| \delta_4\right) + \frac{|\beta_4| \delta_4}{\Gamma_q(\beta)} \right\}$$
$$+ r_2 \left\{ \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} L_2(s)\, d_q s + |\gamma_3| \delta_3 \int_0^{\eta_3} \frac{(\eta_3-qs)^{(\beta-1)}}{\Gamma_q(\beta)} L_2(s)\, d_q s + |\gamma_4| \delta_4 \int_0^{\eta_4} \frac{(\eta_4-qs)^{(\beta-1)}}{\Gamma_q(\beta)} L_2(s)\, d_q s + |\alpha_4| \delta_4 \int_0^1 \frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} L_2(s)\, d_q s + |\beta_4| \delta_4 \int_0^1 \frac{(1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} L_2(s)\, d_q s \right\} \quad (37)$$
$$\leq M_2 A_2 + r_2 \kappa_2$$

and $r_2 = M_2 A_2/(1-\kappa_2)$.

Therefore, we obtain

$$\|T(u,v)\| = \max\{(T_1 v, T_2 u)\} = \max\{\|T_1 v\|, \|T_2 u\|\} = r. \quad (38)$$

From the last estimate we can choose $r = \max\{r_1, r_2\}$; then, for every $(u,v) \in U$, we have $TU \subset U$.

In order to show that T is a contraction, let $u, v, u_1, v_1 \in X$, and, for any $t \in [0,1]$, we get

$$\|T_1 v - T_1 v_1\|$$
$$\leq \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,v(s)) - f(s,v_1(s))|\, d_q s + \left| \frac{\gamma_1}{\Delta} [(\alpha_2-\gamma_2)t - (\alpha_2+\beta_2-\gamma_2\eta_2)] \right| \times \int_0^{\eta_1} \frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,v(s)) - f(s,v_1(s))|\, d_q s + \left| \frac{\gamma_2}{\Delta} [(\alpha_1-\gamma_1)t + (\beta_1+\gamma_1\eta_1)] \right| \times \int_0^{\eta_2} \frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,v(s)) - f(s,v_1(s))|\, d_q s + \left| \frac{\alpha_2}{\Delta} [(\alpha_1-\gamma_1)t + (\beta_1+\gamma_1\eta_1)] \right| \times \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,v(s)) - f(s,v_1(s))|\, d_q s + \left| \frac{\beta_2}{\Delta} [(\alpha_1-\gamma_1)t + (\beta_1+\gamma_1\eta_1)] \right| \times \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,v(s)) - f(s,v_1(s))|\, d_q s \right\}$$
$$\leq \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + \left| \frac{\gamma_1}{\Delta} [(\alpha_2-\gamma_2)t - (\alpha_2+\beta_2-\gamma_2\eta_2)] \right| \times \int_0^{\eta_1} \frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + \left| \frac{\gamma_2}{\Delta} [(\alpha_1-\gamma_1)t + (\beta_1+\gamma_1\eta_1)] \right| \times \int_0^{\eta_2} \frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s + \left| \frac{\alpha_2}{\Delta} [(\alpha_1-\gamma_1)t + (\beta_1+\gamma_1\eta_1)] \right| \times \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_1(s)\, d_q s \right.$$

$$+\left|\frac{\beta_2}{\Delta}\left[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)\right]\right|$$
$$\left.\times\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}L_1(s)\,d_qs\right\}\|v-v_1\|$$
$$\leq\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs\right.$$
$$+|\gamma_1|\,\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs$$
$$+|\gamma_2|\,\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs$$
$$+|\alpha_2|\,\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1(s)\,d_qs$$
$$\left.+|\beta_2|\,\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}L_1(s)\,d_qs\right\}\|v-v_1\|, \tag{39}$$

which, in view of $\kappa_1<1$ and (31), implies that

$$\|T_1v(t)-T_1v_1(t)\|\leq\kappa_1\|v-v_1\|. \tag{40}$$

Similarly, we have $\|T_2u-T_2u_1\|\leq\kappa_2\|u-u_1\|$. Thus, we have

$$\begin{aligned}\|T(u,v)-T(u_1,v_1)\| &=\|(T_1v-T_1v_1,T_2u-T_2u_1)\|\\ &=\max\{\|T_1v-T_1v_1\|,\|T_2u-T_2u_1\|\}\\ &\leq\max\{\kappa_1,\kappa_2\}\max\{\|v-v_1\|,\|u-u_1\|\}\\ &=\max\{\kappa_1,\kappa_2\}\|(v-v_1,u-u_1)\|.\end{aligned} \tag{41}$$

Since $\kappa_1<1$, $\kappa_2<1$, therefore, the operator T is a contraction. Hence, by Banach's contraction principle, the operator T has a unique fixed point, which is the unique solution of the system (1). This completes the proof. □

The second result is based on the nonlinear alternative of Leray-Schauder type (Lemma 9).

Theorem 12. *Assume that $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions and the following conditions hold:*

(H2) *there exist four functions $p_i(t),q_i(t)\in L^1([0,1],\mathbb{R}^+)$, $i=1,2$, and two nondecreasing functions $\varphi,\psi:\mathbb{R}^+\to\mathbb{R}^+$, such that*

$$\begin{aligned}|f(t,x)|&\leq p_1(t)\,\varphi(\|x\|)+q_1(t),\\ |g(t,y)|&\leq p_2(t)\,\psi(\|y\|)+q_2(t),\end{aligned} \tag{42}$$

where $(t,x),(t,y)\in[0,1]\times\mathbb{R}$.

(H3) *There exists a constant $M>0$ such that*

$$\begin{aligned}\varphi(M)\,\omega_1+\omega_2&<M,\\ \psi(M)\,\varpi_1+\varpi_2&<M,\end{aligned} \tag{43}$$

where

$$\begin{aligned}\omega_1:=&\ (1+|\alpha_2|\,\delta_2)\left(I_q^{\alpha}p_1\right)(1)+|\gamma_1|\,\delta_1\left(I_q^{\alpha}p_1\right)(\eta_1)\\ &+|\gamma_2|\,\delta_2\left(I_q^{\alpha}p_1\right)(\eta_2)+|\beta_2|\,\delta_2\left(I_q^{\alpha-1}p_1\right)(1),\\ \omega_2:=&\ (1+|\alpha_2|\,\delta_2)\left(I_q^{\alpha}q_1\right)(1)+|\gamma_1|\,\delta_1\left(I_q^{\alpha}q_1\right)(\eta_1)\\ &+|\gamma_2|\,\delta_2\left(I_q^{\alpha}q_1\right)(\eta_2)+|\beta_2|\,\delta_2\left(I_q^{\alpha-1}q_1\right)(1),\\ \varpi_1:=&\ (1+|\alpha_4|\,\delta_4)\left(I_q^{\beta}p_2\right)(1)+|\gamma_3|\,\delta_3\left(I_q^{\beta}p_2\right)(\eta_3)\\ &+|\gamma_4|\,\delta_4\left(I_q^{\beta}p_2\right)(\eta_4)+|\beta_4|\,\delta_4\left(I_q^{\beta-1}p_2\right)(1),\\ \varpi_2:=&\ (1+|\alpha_4|\,\delta_4)\left(I_q^{\beta}q_2\right)(1)+|\gamma_3|\,\delta_3\left(I_q^{\beta}q_2\right)(\eta_3)\\ &+|\gamma_4|\,\delta_4\left(I_q^{\beta}q_2\right)(\eta_4)+|\beta_4|\,\delta_4\left(I_q^{\beta-1}q_2\right)(1).\end{aligned} \tag{44}$$

Then system (1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $T:X\times X\to X\times X$ defined by (25). The proof consists of several steps. As a first step, it will be shown that T maps bounded sets into bounded sets in $X\times X$. For a positive number r, let $U=\{(u,v)\in X\times X:\|(u,v)\|\leq r\}$ be bounded set in $X\times X$; then, for $(u,v)\in U$, we have

$$\|T_1v\|$$
$$\leq\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs\right.$$
$$+\left|\frac{\gamma_1}{\Delta}\left[(\alpha_2-\gamma_2)t-(\alpha_2+\beta_2-\gamma_2\eta_2)\right]\right|$$
$$\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs$$
$$+\left|\frac{\gamma_2}{\Delta}\left[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)\right]\right|$$
$$\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs$$
$$+\left|\frac{\alpha_2}{\Delta}\left[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)\right]\right|$$
$$\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,v(s))|\,d_qs$$
$$+\left|\frac{\beta_2}{\Delta}\left[(\alpha_1-\gamma_1)t+(\beta_1+\gamma_1\eta_1)\right]\right|$$
$$\left.\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,v(s))|\,d_qs\right\}$$

$$\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs\right.$$
$$+|\gamma_1|\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\times\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$+|\gamma_2|\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\times\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$+|\alpha_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\times\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$\left.+|\beta_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\times\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs\right\}$$
$$\leq \varphi(r)\left\{\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p_1(s)\,d_qs+|\gamma_1|\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p_1(s)\,d_qs+|\gamma_2|\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p_1(s)\,d_qs\right.$$
$$\left.+|\alpha_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p_1(s)\,d_qs+|\beta_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}p_1(s)\,d_qs\right\}$$
$$+\left\{\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}q_1(s)\,d_qs+|\gamma_1|\delta_1\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}q_1(s)\,d_qs+|\gamma_2|\delta_2\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}q_1(s)\,d_qs\right.$$
$$\left.+|\alpha_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}q_1(s)\,d_qs+|\beta_2|\delta_2\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}q_1(s)\,d_qs\right\}. \quad (45)$$

As before, it can be shown that

$$\|T_1v\|\leq\varphi(r)\,\omega_1+\omega_2. \quad (46)$$

Similarly, we have

$$\|T_2u\|\leq\psi(r)\,\varpi_1+\varpi_2. \quad (47)$$

Thus, T maps bounded sets into bounded sets in $X\times X$.

Next, we show that T maps bounded sets into equicontinuous sets of $X\times X$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $(u,v)\in U$, where U is a bounded set of $X\times X$. Then taking into consideration the inequality $(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\leq(t_2-t_1)$, for $0<t_1<t_2$, we obtain

$$\left|(T_1v)(t_2)-(T_1v)(t_1)\right|$$
$$\leq\left|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,v(s))\,d_qs-\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,v(s))\,d_qs\right|$$
$$+\frac{1}{\Delta}\left\{\left|\gamma_1(\alpha_2-\gamma_2)\right|(t_2-t_1)\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f(s,v(s))\right|d_qs\right.$$
$$+\gamma_2\left|\alpha_1-\gamma_1\right|(t_2-t_1)\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f(s,v(s))\right|d_qs$$
$$+|\alpha_2|\left|\alpha_1-\gamma_1\right|(t_2-t_1)\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f(s,v(s))\right|d_qs$$
$$\left.+|\beta_2|\left|\alpha_1-\gamma_1\right|(t_2-t_1)\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left|f(s,v(s))\right|d_qs\right\}$$
$$\leq\left|\int_0^{t_1}\frac{\left[(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right]}{\Gamma_q(\alpha)}\times\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs\right|$$

$$+\frac{1}{\Delta}\left\{\left|\gamma_1(\alpha_2-\gamma_2)\right|(t_2-t_1)\right.$$
$$\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$+\gamma_2\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$+\left|\alpha_2\right|\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs$$
$$+\left|\beta_2\right|\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\left.\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(\|v\|)+q_1(s)\right]d_qs\right\}$$
$$\leq\left|\int_0^{t_1}\frac{(t_2-t_1)}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs\right.$$
$$\left.+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs\right|$$
$$+\frac{1}{\Delta}\left\{\left|\gamma_1(\alpha_2-\gamma_2)\right|(t_2-t_1)\right.$$
$$\times\int_0^{\eta_1}\frac{(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs$$
$$+\gamma_2\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\times\int_0^{\eta_2}\frac{(\eta_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs$$
$$+\left|\alpha_2\right|\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\times\int_0^{1}\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs$$
$$+\left|\beta_2\right|\left|\alpha_1-\gamma_1\right|(t_2-t_1)$$
$$\left.\times\int_0^{1}\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)}\left[p_1(s)\varphi(r)+q_1(s)\right]d_qs\right\}. \tag{48}$$

Clearly, the right-hand side of the above inequality tends to zero independently of $v \in U$ as $t_2 \to t_1$. Thus, it follows by the Arzelá-Ascoli theorem that T_1 is completely continuous. Similarly, T_2 is completely continuous. Therefore, $T : X \times X \to X \times X$ is completely continuous.

Let us set $\Omega = \{(u,v) \in U : \|(u,v)\| < M\}$. Note that the operator $T : \overline{\Omega} \to X \times X$ is continuous and completely continuous. From the choice of Ω, assume that there is $(u,v) \in \partial\Omega$ such that $(u,v) = \lambda T(u,v)$, for some $\lambda \in (0,1)$. By (H3), we obtain

$$\begin{aligned}\|(u,v)\| &= \lambda\|T(u,v)\| < \|T(u,v)\|\\ &= \max\{\|T_1v\|,\|T_2u\|\} < M,\end{aligned} \tag{49}$$

which is a contradiction. In consequence, by the nonlinear alternative of Leray-Schauder type (Lemma 9), we deduce that T has a fixed point $(u,v) \in \overline{\Omega}$ which is a solution of the system (1). The proof is complete. □

In the sequel we present two examples which illustrate Theorems 11 and 12.

4. Examples

Example 1. Consider the following fractional q-difference nonlocal boundary value problem:

$$\begin{gathered}{}^CD_q^{3/2}u(t)=\frac{L_1}{2}\left(v+\tan^{-1}v+\sin t\right),\quad 0\le t\le 1,\\ {}^CD_q^{7/4}v(t)=\frac{L_2}{3}\left(u+\tan^{-1}u+\sin t\right),\quad 0\le t\le 1,\\ u(0)-\frac{1}{2}D_qu(0)=u\left(\frac{1}{3}\right),\\ \frac{1}{4}u(1)+\frac{3}{4}D_qu(1)=u\left(\frac{2}{3}\right),\\ v(0)-\frac{1}{2}D_qv(0)=v\left(\frac{1}{3}\right),\\ \frac{1}{4}v(1)+\frac{3}{4}D_qv(1)=v\left(\frac{2}{3}\right).\end{gathered} \tag{50}$$

In this case, $\alpha = 3/2$, $\beta = 7/4$, $\alpha_1 = 1$, $\beta_1 = 1/2$, $\alpha_2 = 1/4$, $\beta_2 = 3/4$, $\alpha_3 = 1$, $\beta_3 = 1/2$, $\alpha_4 = 1/4$, $\beta_4 = 3/4$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $\eta_3 = 1/3$, $\eta_4 = 2/3$, $\gamma_1 = 1 = \gamma_2$, $\gamma_3 = 1 = \gamma_4$, and L_1, L_2 are constants to be fixed later on. Moreover, $\Delta = 5/8$, $\blacktriangle = 5/8$, $\delta_1 = 26/15$, $\delta_2 = 4/3$, $\delta_3 = 26/15$, and $\delta_4 = 4/3$. Consider

$$\begin{gathered}f(t,v)=\frac{L_1}{2}\left(v+\tan^{-1}v+\sin t\right),\\ g(t,u)=\frac{L_2}{3}\left(u+\tan^{-1}u+\sin t\right),\\ t\in[0,1],\ u,v\in C[0,1].\end{gathered} \tag{51}$$

Clearly, we have

$$\begin{gathered}\left|f(t,v)-f(t,v_1)\right|\le L_1\left|v-v_1\right|,\\ \left|g(t,u)-g(t,u_1)\right|\le L_2\left|u-u_1\right|,\end{gathered}$$

$$\kappa_1 := \frac{L_1}{\Gamma_{1/2}(3/2)}\left(1+\frac{2\sqrt{2}\left(13+10\sqrt{2}+15\sqrt{3}\right)}{15\sqrt{3}\left(2\sqrt{2}-1\right)}\right), \tag{52}$$

$$\kappa_2 := \frac{L_2}{\Gamma_{1/2}(7/4)}\left(1+\frac{2\sqrt{2}\left(13+10\sqrt{2}+15\sqrt{3}\right)}{15\sqrt{3}\left(2\sqrt{2}-1\right)}\right).$$

Choose

$$L_1 := \left[\frac{1}{\Gamma_{1/2}(3/2)} \left(1 + \frac{2\sqrt{2}\left(13 + 10\sqrt{2} + 15\sqrt{3}\right)}{15\sqrt{3}\left(2\sqrt{2} - 1\right)} \right) \right]^{-1},$$

$$L_2 := \left[\frac{1}{\Gamma_{1/2}(7/4)} \left(1 + \frac{2\sqrt{2}\left(13 + 10\sqrt{2} + 15\sqrt{3}\right)}{15\sqrt{3}\left(2\sqrt{2} - 1\right)} \right) \right]^{-1}. \tag{53}$$

Hence all the assumptions of Theorem 11 are satisfied. Therefore, by Theorem 11, the problem (50) has a unique solution.

Example 2. Consider the following fractional boundary value problem:

$$\begin{aligned}
{}^{C}D_q^{3/2} u(t) &= \frac{1}{4}\cos t^2 \sin\left(\frac{|v|}{2}\right) + \frac{e^{-x^2(t^2+1)}}{1 + (t^2+1)} + \frac{1}{3}, \\
&\qquad 0 \le t \le 1, \\
{}^{C}D_q^{3/2} v(t) &= \frac{1}{16\pi}\sin(2\pi u(t)) + \frac{|u(t)|}{2(1 + |u(t)|)} + \frac{1}{2}, \\
&\qquad 0 \le t \le 1, \\
u(0) - \frac{1}{2}D_q u(0) &= u\left(\frac{1}{3}\right), \\
\frac{1}{4}u(1) + \frac{3}{4}D_q u(1) &= u\left(\frac{2}{3}\right), \\
v(0) - \frac{1}{2}D_q v(0) &= v\left(\frac{1}{3}\right), \\
\frac{1}{4}v(1) + \frac{3}{4}D_q v(1) &= v\left(\frac{2}{3}\right).
\end{aligned} \tag{54}$$

In this case, $\alpha = 3/2$, $\beta = 3/2$, $\alpha_1 = 1$, $\beta_1 = 1/2$, $\alpha_2 = 1/4$, $\beta_2 = 3/4$, $\alpha_3 = 1$, $\beta_3 = 1/2$, $\alpha_4 = 1/4$, $\beta_4 = 3/4$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $\eta_3 = 1/3$, $\eta_4 = 2/3$, $\gamma_1 = 1 = \gamma_2$, $\gamma_3 = 1 = \gamma_4$, and L_1, L_2 are constants to be fixed later on. Moreover, $\Delta = 5/8$, $\blacktriangle = 5/8$, $\delta_1 = 26/15$, $\delta_2 = 4/3$, $\delta_3 = 26/15$, and $\delta_4 = 4/3$. Clearly

$$\begin{aligned}
\left|f(t,v)\right| &= \left| \frac{1}{4}\cos t^2 \sin\left(\frac{|v|}{2}\right) + \frac{e^{-x^2(t^2+1)}}{1 + (t^2+1)} + \frac{1}{3} \right| \\
&\le \frac{1}{8}|v| + 1, \\
\left|g(t,u)\right| &= \left| \frac{1}{16\pi}\sin(2\pi u(t)) + \frac{|u(t)|}{2(1 + |u(t)|)} + \frac{1}{2} \right| \\
&\le \frac{1}{8}|u| + 1.
\end{aligned} \tag{55}$$

Clearly $p_1 = 1/8$, $q_1 = 1$, $\varphi(M) = M$, $p_2 = 1/8$, $q_2 = 1$, and $\psi(M) = M$. Consequently, $\omega_1 \approx 0.567129414$, $\omega_2 \approx 4.536963312$, $\varpi_1 \approx 0.567129414$, $\varpi_2 \approx 4.536963312$, and conditions (43) imply that $M > 10.48055997$. Thus, all the assumptions of Theorem 12 are satisfied. Therefore, the conclusion of Theorem 12 applies to problems (54).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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