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Research Article

Antiperiodic Solutions to Impulsive Cohen-Grossberg Neural Networks with Delays on Time Scales

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We use the method of coincidence degree and construct suitable Lyapunov functional to investigate the existence and global exponential stability of antiperiodic solutions of impulsive Cohen-Grossberg neural networks with delays on time scales. Our results are new even if the time scale $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. An example is given to illustrate our feasible results.

1. Introduction

It is well known that Cohen-Grossberg neural networks (CGNNs) include many models from different research fields, such as neurobiology, population biology, and evolutionary theory, as well as Hopfield neural networks and other recurrent neural network models. Over the past few years, a large number of scholars have extensively studied the dynamical behaviors, in particular, the existence and stability of the equilibrium point and periodic and almost-periodic solutions of Cohen-Grossberg neural networks. There have been considerable results on CGNNs (e.g., see [1–16]). In contrast, however, very few results are available on the existence and exponential stability of antiperiodic solutions for neural networks, while the existence of antiperiodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [17–22]).

In [17], the authors studied the existence and exponential stability of antiperiodic solutions for the following Cohen-Grossberg neural networks with bounded and unbounded delays:

$$\begin{aligned}\dot{x}_i(t) &= -a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) \\ &\quad - \sum_{j=1}^{n} d_{ij}(t) g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \\ &\quad - \sum_{j=1}^{n} e_{ij}(t) \int_0^{+\infty} K_{ij}(s) h_j\left(x_j(t-s)\right) ds, \\ &\quad - I_i(t)\Bigg], \quad t \neq t_k,\ t \geq 0,\end{aligned} \tag{1}$$

where n corresponds to the number of units in the neural networks, x_i denotes the potential (or voltage) of cell i at time t, a_i represents an amplification function, b_i is an appropriately behaved function, c_{ij}, d_{ij}, and e_{ij} denote the strengths of connectivity between cells i and j at time t, respectively. The activation functions f_j, g_j, and h_j show how the jth neuron reacts to the input, τ_{ij} corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell at time t, K_{ij} is the kernel, and I_i denotes the ith component of an external input source introduced from outside the network to cell i at time $t, i, j = 1, 2, \ldots, n$.

In fact, both continuous and discrete systems are very important in implementation and application. Therefore, the study of dynamic equations on time scales has received much attention (see [18, 19, 23–28]) which displays a combination of characteristics of both continuous-time and discrete-time system. For example, in [23], the authors extended the almost-periodic theory on time scales with the delta derivative to that with the nabla derivative and then derived some sufficient conditions ensuring the existence, uniqueness, and exponential stability of almost-periodic solutions for a class of cellular neural networks with time-varying delays in leakage terms on time scales.

Also, differential equations with impulses provide an adequate mathematical model of many evolutionary processes that suddenly change their states at certain moments. For example, [18] applied the method of coincidence degree to investigate the existence of antiperiodic solutions to the following impulsive shunting inhibitory cellular neural networks on time scales:

$$\begin{aligned} x_{ij}^{\Delta}(t) &= -a_{ij}(t)\,x_{ij}(t) - \sum_{c_{kl}\in N_r(i,j)} c_{ij}^{kl}(t)\int_0^{+\infty} K_{ij}(u)\,x_{ij}(t)\,f_{ij}(x_{kl}(t-u))\,\Delta u \\ &\quad + L_{ij}(t), \quad t\in\mathbb{T}^+,\ t\neq t_h,\ h\in\mathbb{N}, \\ \Delta x_{ij}(t_h) &= x_{ij}(t_h^+) - x_{ij}(t_h^-) = I_{ijk}(x_{ij}(t_h)), \\ &\quad t = t_h, \quad i = 1,2,\ldots,m, \quad j = 1,\ldots,n. \end{aligned} \tag{2}$$

Motivated by the abovementioned works, in this paper, we will apply the method of coincidence degree and construct suitable Lyapunov functional to investigate the existence and global exponential stability of antiperiodic solutions to the following impulsive CGNN model with delays on time scales:

$$\begin{aligned} x_i^{\Delta}(t) &= -\alpha_i(x_i(t))\Bigg[\beta_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}(t)\,f_j(x_j(t)) \\ &\quad - \sum_{j=1}^{n} b_{ij}(t)\,g_j(x_j(t-\tau_{ij}(t))) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t)\int_0^{+\infty} K_{ij}(s)\,h_j(x_j(t-s))\,\Delta s \\ &\quad + J_i(t)\Bigg], \quad t\in\mathbb{T}^+,\ t\neq t_k,\ k\in\mathbb{N}, \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)), \\ &\quad t = t_k, \quad i = 1,2,\ldots,n, \end{aligned} \tag{3}$$

where $\mathbb{T}$ is an $(\omega/2)$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$, $\mathbb{T}^+ = \{t \in \mathbb{T} : t \geq 0\}$, $x_i(t_k^+)$, $x_i(t_k^-)$ represent right and left limit of $x_i(t_k)$ in the sense of time scales, t_k is a sequence of real numbers such that $0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$, and there exists a positive integer p such that $t_{k+p} = t_k + \omega/2$, $I_{i(k+p)}(x_i(t_{k+p})) = -I_{ik}(-x_i(t_k))$, $k \in \mathbb{N}$. Without loss of generality, we also assume that $[0,\omega/2)_{\mathbb{T}} \bigcap \{t_k : k \in \mathbb{N}\} = \{t_1, t_2, \ldots, t_q\}$.

The initial conditions associated with system (3) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0]_{\mathbb{T}},\ i = 1,2,\ldots,n, \tag{4}$$

where $\varphi_i(s)$ is a bounded continuous function on $(-\infty, 0]_{\mathbb{T}}$.

Throughout this paper, we make the following assumptions:

(H_1) J_i, a_{ij}, b_{ij}, c_{ij}, and $\tau_{ij} \in C(\mathbb{T},\mathbb{R})$, $a_{ij}(t+\omega/2) = a_{ij}(t)$, $b_{ij}(t+\omega/2) = b_{ij}(t)$, $c_{ij}(t+\omega/2) = c_{ij}(t)$, $\tau_{ij}(t+\omega/2) = \tau_{ij}(t)$, $J_i(t+\omega/2) = -J_i(t)$, and $|J_i(t)| < \infty$, $i, j = 1,2,\ldots,n$;

(H_2) $\alpha_i \in C(\mathbb{R},\mathbb{R}^+)$, $\alpha_i(-u) = \alpha_i(u)$, and there exist positive constants α_i^m, α_i^M, α_i^L such that $\alpha_i^m \leq \alpha_i(u) \leq \alpha_i^M$ and $|\alpha_i(v_1) - \alpha_i(v_2)| \leq \alpha_i^L|v_1 - v_2|$ for all $u, v_1, v_2 \in \mathbb{R}$, $i = 1,2,\ldots,n$;

(H_3) $\beta_i \in C(\mathbb{R},\mathbb{R})$ is delta differential and $0 < \rho_i \leq \beta_i^{\Delta} \leq \delta_i$, $\beta_i(-u) = -\beta_i(u)$, $\beta_i(0) = 0$, $i = 1,2,\ldots,n$;

(H_4) there exists a positive constant α_i^l such that $|\alpha_i(u)\beta_i(u) - \alpha_i(v)\beta_i(v)| > \alpha_i^l|\beta_i(u) - \beta_i(v)|$ for all $u, v \in \mathbb{R}$;

(H_5) f_j, g_j, and $h_j \in C(\mathbb{R},\mathbb{R})$, $f_j(-u) = -f_j(u)$, $g_j(-u) = -g_j(u)$, $h_j(-u) = -h_j(u)$, $f_j(0) = g_j(0) = h_j(0) = 0$, and there exist positive constants $f_j^M, g_j^M, h_j^M, f_j^L, g_j^L$, and h_j^L such that $|f_j(u)| \leq f_j^M$, $|g_j(u)| \leq g_j^M$, $|h_j(u)| \leq h_j^M$, $|f_j(v_1) - f_j(v_2)| \leq f_j^L|v_1 - v_2|$, $|g_j(v_1) - g_j(v_2)| \leq g_j^L|v_1 - v_2|$, and $|h_j(v_1) - h_j(v_2)| \leq h_j^L|v_1 - v_2|$, for all $u, v_1, v_2 \in \mathbb{R}$, $j = 1,2,\ldots,n$;

(H_6) $\int_0^{+\infty}|K_{ij}(u)|\Delta u < +\infty$, $i, j = 1,2,\ldots,n$;

(H_7) $I_{ik} \in C(\mathbb{R},\mathbb{R})$ and there exist positive constants ρ_{ik} such that $|I_{ik}(u) - I_{ik}(v)| \leq \rho_{ik}|u - v|$ for all $u, v \in \mathbb{R}$, $k \in \mathbb{N}$, and $i = 1,2,\ldots,n$.

For the sake of convenience, we introduce some notations:

$$a_{ij}^M = \max_{t\in[0,\omega]_{\mathbb{T}}} a_{ij}(t), \qquad a_{ij}^m = \min_{t\in[0,\omega]_{\mathbb{T}}} a_{ij}(t),$$

$$J_i^M = \max_{t\in[0,\omega]_{\mathbb{T}}} J_i(t), \qquad \tau = \sup_{t\in[0,\omega]_{\mathbb{T}}} \max_{1\leq i,j\leq n}\{\tau_{ij}(t)\},$$

$$B_i = \sum_{j=1}^{n} a_{ij}^M f_j^M + \sum_{j=1}^{n} b_{ij}^M g_j^M + \sum_{j=1}^{n} c_{ij}^M h_j^M \int_0^{\infty} \left|K_{ij}(s)\right| \Delta s,$$

$$E_i = \rho_i \alpha_i^m \omega \left(1 - \alpha_i^M \delta_i \omega\right) - \left(1 + \rho_i \alpha_i^m \omega\right) \sum_{k=1}^{2q} \rho_{ik},$$

$$D_i = \left(1 + \alpha_i^m \rho_i \omega\right) \left(\alpha_i^M \omega \left(B_i + J_i^M\right) + \sum_{k=1}^{2q} \left|I_{ik}(0)\right| \right),$$

$$R_i = \sum_{j=1}^{n} \left(a_{ij}^M f_j^L + b_{ij}^M g_j^L e_\varepsilon \left(t, t - \tau_{ij}(t)\right) + c_{ij}^M h_j^L \int_0^{+\infty} \left|K_{ij}(s)\right| e_\varepsilon (t, t-s) \Delta s \right), \quad (5)$$

where $i, j = 1, 2, \ldots, n$.

The organization of the rest of this paper is as follows. In Section 2, we introduce some definitions and make some preparations for later sections. In Sections 3 and 4, we establish our main results for the existence and exponential stability of antiperiodic solutions of (3). Finally, we present an example to illustrate the feasibility and effectiveness of our results obtained in previous sections.

2. Preliminaries

In this section, we recall some basic definitions and lemmas which are used in what follows.

Definition 1 (see [24, 26]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t. \quad (6)$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2 (see [24, 26]). A vector function $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$.

Definition 3 (see [24, 26]). For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by Banach space), the (delta) derivative is defined by

$$f^\Delta = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \quad (7)$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^\Delta = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{t \to s} \frac{f(t) - f(s)}{t - s}, \quad (8)$$

provided that this limit exists.

Definition 4 (see [23, 24, 26]). If $F^\Delta(t) = f(t)$, then one defines the delta integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a). \quad (9)$$

Definition 5 (see [23, 24]). If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)_{\mathbb{T}}$, then we define the improper integral by

$$\int_a^{\infty} f(s) \Delta s := \lim_{b \to \infty} \int_a^b f(s) \Delta s, \quad (10)$$

provided that this limit exists, and one says that the improper integral converges in this case. If this limit does not exist, then one says that the improper integral diverges.

Definition 6 (see [24, 26]). A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathscr{R}$, while the set $\mathscr{R}^+$ is given by $\{f \in \mathscr{R} : 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathscr{R}$. The exponential function is defined by

$$e_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right\} \quad \text{for } s, t \in \mathbb{T} \quad (11)$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \quad (12)$$

Definition 7 (see [18, 24]). For each $t \in \mathbb{R}$, let N be a neighborhood of t. Then one defined the generalized derivative (or Dini derivative) $D^+u^\Delta(t)$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N(\varepsilon) \subset N$ of ϵ such that

$$\frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} < D^+u^\Delta(t) + \epsilon \quad (13)$$

for each $s \in N(\epsilon)$, $s > t$.

In case t is right-scattered and $u(t)$ is continuous at t, this reduces to

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}, \quad (14)$$

where the upper right Dini derivative is defined as

$$D^+u(t) = \lim_{h \to 0^+} \sup \frac{u(t+h) - u(t)}{h}. \quad (15)$$

Definition 8 (see [18]). One says that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is $(\omega/2)$-antiperiodic if there exists a natural number n such that $\omega/2 = np$, $f(t + \omega/2) = -f(t)$ for all $t \in \mathbb{T}$ and $\omega/2$ is the smallest number such that $f(t + \omega/2) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, one says that f is $(\omega/2)$-antiperiodic if $\omega/2$ is the smallest positive number such that $f(t + \omega/2) = -f(t)$ for all $t \in \mathbb{T}$.

Lemma 9 (see [23, 24, 26]). *Assume that $p, q \in \mathscr{R}$. Then*

(a) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(b) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(c) $1/e_p(t,s) = e_{\ominus p}(t,s)$, *where* $\ominus p(t) := -p(t)/(1 + \mu(t)p(t))$;

(d) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(e) $e_p^{\Delta}(\cdot, s) = p e_p(\cdot, s)$.

Lemma 10 (see [28]). *If $p \in \mathscr{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t (p(u))\,\Delta u\right). \tag{16}$$

Lemma 11 (see [23, 24, 26]). *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in T^k$. Then*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)\,g(t) + f(\sigma(t))\,g^{\Delta}(t) \\ &= f(t)\,g^{\Delta}(t) + f^{\Delta}(t)\,g(\sigma(t)).\end{aligned} \tag{17}$$

Lemma 12 (see [18, 24, 27]). *Let $t_1, t_2 \in [0,\omega]_{\mathbb{T}}$. If $x : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$\begin{aligned}x(t) &\leq x(t_1) + \int_0^{\omega} \left|x^{\Delta}(s)\right| \Delta s, \\ x(t) &\geq x(t_2) - \int_0^{\omega} \left|x^{\Delta}(s)\right| \Delta s.\end{aligned} \tag{18}$$

Definition 13 (see [18]). The antiperiodic solution $x^*(t) = (x_1^*(t), \dots, x_n^*(t))$ of system (3) with initial value $\varphi^*(t) = (\varphi_1^*(t), \dots, \varphi_n^*(t))$ is said to be globally exponentially stable if there exist positive constants λ and $M = M(\lambda) \geq 1$, for any solution $x(t) = (x_1(t), \dots, x_n(t))^T$ of system (3) with the initial value $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T$, such that

$$\begin{gathered}\left|x_i(t) - x_i^*(t)\right| \leq M(\lambda)\, e_{\ominus\lambda}(t,\alpha) \left\|\varphi - \varphi^*\right\|_{\infty}, \\ \forall t \in (0,\infty)_{\mathbb{T}}, \quad i = 1, 2, \dots, n,\end{gathered} \tag{19}$$

where

$$\begin{gathered}\left\|\varphi - \varphi^*\right\|_{\infty} = \sup_{-\infty < s < 0} \max_{1 \leq i \leq n} \left|\varphi_i(s) - \varphi_i^*(s)\right|, \\ \alpha \in (-\infty, 0]_{\mathbb{T}}.\end{gathered} \tag{20}$$

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 14 (see [18, 29]). *Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces and let $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \emptyset$ and $N : \overline{\Omega} \to \mathbb{Y}$ is L-compact. Further, one also assumes that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ *for all* $x \in D(L) \cap \partial\Omega$, $\lambda \in (0,1]$.

Then $Lx = Nx$ has at least one solution on $D(L) \cap \overline{\Omega}$.

Lemma 15 (mean value theorem, [6, 30]). *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ which is Δ-differentiable on $[a,b)_{\mathbb{T}}$, and then there exist $\xi, \tau \in [a,b)_{\mathbb{T}}$ such that*

$$f^{\Delta}(\xi)(b-a) \leq f(b) - f(a) \leq f^{\Delta}(\tau)(b-a). \tag{21}$$

3. Existence of Antiperiodic Solutions

In this section, by using fixed point theorem of coincidence degree, we will study the existence of at least one antiperiodic solution for system (3).

Theorem 16. *Assume that (H_1)–(H_7) hold. Suppose further that $E_i > 0$, $i = 1, 2, \dots, n$. Then system (3) has at least one $(\omega/2)$-antiperiodic solution.*

Proof. Let $C^k([0,\omega; t_1, t_2, \dots, t_q, t_{q+1}, \dots, t_{2q}]_{\mathbb{T}}, \mathbb{R}^n) = \{x : [0,\omega]_{\mathbb{T}} \to \mathbb{R}^n \mid x^{(k)}(t)$ be a piecewise continuous map with first-class discontinuous points in $[0,\omega]_{\mathbb{T}} \cap \{t_k : k \in \mathbb{N}\}$ and at each discontinuous point it is continuous on the left$\}$, $k = 0, 1$. Let

$$\begin{gathered}\mathbb{X} = \left\{x \in C\left(\left[0,\omega; t_1, t_2, \dots, t_q, t_{q+1}, \dots, t_{2q}\right]_{\mathbb{T}}, \mathbb{R}^n\right)\right. \\ \left. : x\left(t + \frac{\omega}{2}\right) = -x(t)\ \ \forall t \in \left[0, \frac{\omega}{2}\right]_{\mathbb{T}}\right\}, \\ \mathbb{Y} = \mathbb{X} \times \mathbb{R}^{n \times q}\end{gathered} \tag{22}$$

be two Banach spaces equipped with the norms

$$\|x\|_{\mathbb{X}} = \sum_{i=1}^{n} |x_i|_0, \qquad \|y\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} + \|z\| \quad \forall x \in \mathbb{X},\ z \in \mathbb{R}^{n \times q}, \tag{23}$$

in which $|x_i|_0 = \max_{t \in [0,\omega]_{\mathbb{T}}} |x_i(t)|$, $i = 1, 2, \dots, n$ and $\|\cdot\|$ is any norm of $\mathbb{R}^{n \times q}$. Set

$$\begin{gathered}L : \operatorname{Dom} L \cap \mathbb{X} \longrightarrow \mathbb{Y}, \\ x \longrightarrow \left(x^{\Delta}, \Delta x(t_1), \Delta x(t_2), \dots, \Delta x\left(t_q\right)\right),\end{gathered} \tag{24}$$

where

$$\begin{gathered}\operatorname{Dom} L = \left\{x \in C^1\left(\left[0,\omega; t_1, t_2, \dots, t_{2q}\right]_{\mathbb{T}}, \mathbb{R}^n\right)\right. \\ \left. : x\left(t + \frac{\omega}{2}\right) = -x(t)\ \ \forall t \in \left[0, \frac{\omega}{2}\right]_{\mathbb{T}}\right\},\end{gathered} \tag{25}$$

and $N : \mathbb{X} \to \mathbb{Y}$ and

$$Nx = \left(\begin{pmatrix} A_1(t) \\ A_2(t) \\ \vdots \\ A_n(t) \end{pmatrix}, \begin{pmatrix} I_{11}(x_1(t_1)) \\ I_{21}(x_2(t_1)) \\ \vdots \\ I_{n1}(x_n(t_1)) \end{pmatrix}, \begin{pmatrix} I_{12}(x_1(t_2)) \\ I_{22}(x_2(t_2)) \\ \vdots \\ I_{n2}(x_n(t_2)) \end{pmatrix}, \dots, \begin{pmatrix} I_{1q}(x_1(t_q)) \\ I_{2q}(x_2(t_q)) \\ \vdots \\ I_{nq}(x_n(t_q)) \end{pmatrix} \right), \tag{26}$$

where

$$\begin{aligned} A_i(t) = {} & -\alpha_i(x_i(t)) \times \left[\beta_i(x_i(t)) - \sum_{j=1}^n a_{ij}(t) f_j(x_j(t)) \right. \\ & - \sum_{j=1}^n b_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) \\ & - \sum_{j=1}^n c_{ij}(t) \int_0^{+\infty} K_{ij}(s) h_j(x_j(t-s)) \Delta s \\ & \left. + J_i(t) \right], \end{aligned} \tag{27}$$

for $i = 1, 2, \dots, n$. It is easy to see that

$$\begin{aligned} \operatorname{Ker} L &= \{0\}, \\ \operatorname{Im} L &= \left\{ y = (g, c_1, \dots, c_q) \in \mathbb{Y} : \int_0^\omega g(s) \Delta s = 0 \right\} \equiv \mathbb{Y}. \end{aligned} \tag{28}$$

Thus, $\dim \operatorname{Ker} L = 0 = \operatorname{codim} \operatorname{Im} L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P : \mathbb{X} \to \operatorname{Ker} L$ and the averaging projector $Q : \mathbb{Y} \to \mathbb{Y}$ by

$$\begin{aligned} Px &= \int_0^\omega x(s) \Delta s = 0, \\ Qy &= Q(g, c_1, \dots, c_q) = \left(\frac{1}{\omega} \int_0^\omega g(s) \Delta s, 0, \dots, 0 \right). \end{aligned} \tag{29}$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Denoting by $L_p^{-1} : \operatorname{Im} L \to \operatorname{Dom}(L) \cap \operatorname{Ker} P$ the inverse of $L|_{\operatorname{Dom}(L) \cap \operatorname{Ker} P}$, we have

$$L_p^{-1} y = \int_0^t g(s) \Delta s + \sum_{t > t_k} c_k - \frac{1}{2} \int_0^{\omega/2} g(s) \Delta s - \frac{1}{2} \sum_{k=1}^q c_k, \tag{30}$$

in which $c_{q+i} = -c_i$ for all $1 \le i \le q$.

Similar to [24], it is not difficult to show that $QN(\overline{\Omega})$, $L_p^{-1}(I - Q)(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 14, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$\begin{aligned} x_i^\Delta(t) &= \frac{1}{1+\lambda} G_i(t, x) - \frac{\lambda}{1+\lambda} G_i(t, -x), \\ & \qquad t \in \mathbb{T}^+, \quad t \ne t_k, \quad k \in \mathbb{N}, \\ \Delta x_i(t_k) &= \frac{1}{1+\lambda} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} I_{ik}(-x_i(t_k)), \\ & \qquad i = 1, 2, \dots, n, \end{aligned} \tag{31}$$

where

$$\begin{aligned} G_i(t, x) = {} & -\alpha_i(x_i(t)) \times \left[\beta_i(x_i(t)) - \sum_{j=1}^n a_{ij}(t) f_j(x_j(t)) \right. \\ & - \sum_{j=1}^n b_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) \\ & - \sum_{j=1}^n c_{ij}(t) \int_0^{+\infty} K_{ij}(s) h_j(x_j(t-s)) \Delta s \\ & \left. + J_i(t) \right], \\ G_i(t, -x) = {} & -\alpha_i(-x_i(t)) \times \left[\beta_i(-x_i(t)) - \sum_{j=1}^n a_{ij}(t) f_j(-x_j(t)) \right. \\ & - \sum_{j=1}^n b_{ij}(t) g_j(-x_j(t - \tau_{ij}(t))) \\ & - \sum_{j=1}^n c_{ij}(t) \int_0^{+\infty} K_{ij}(s) h_j(-x_j(t-s)) \Delta s \\ & \left. + J_i(t) \right], \end{aligned} \tag{32}$$

for $i = 1, 2, \dots, n$.

Set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$. Then, by (31), (H_2), (H_3), (H_5), (H_6), and Lemma 15, we obtain that

$$\begin{aligned}
&\int_0^\omega \left|x_i^\Delta(t)\right| \Delta t \\
&= \sum_{k=1}^{2q+1} \int_{t_{k-1}^+}^{t_k} \left|x_i^\Delta(t)\right| \Delta t + \sum_{k=1}^{2q} \left|\Delta x_i(t_k)\right| \\
&\le \int_0^\omega \left|\frac{1}{1+\lambda} G_i(t,x) - \frac{\lambda}{1+\lambda} G_i(t,-x)\right| \Delta t \\
&\quad + \sum_{k=1}^{2q} \left|\frac{1}{1+\lambda} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} I_{ik}(-x_i(t_k))\right| \\
&\le \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right) \int_0^\omega \max\left\{\left|G_i(t,x)\right|, \left|G_i(t,-x)\right|\right\} \Delta t \\
&\quad + \sum_{k=1}^{2q} \frac{1}{1+\lambda} \left|I_{ik}(x_i(t_k)) - I_{ik}(0)\right| \\
&\quad + \sum_{k=1}^{2q} \frac{\lambda}{1+\lambda} \left|I_{ik}(-x_i(t_k)) - I_{ik}(0)\right| + \sum_{k=1}^{2q} \left|I_{ik}(0)\right| \\
&\le \alpha_i^M \left(\int_0^\omega \left|\beta_i(x_i(t))\right| \Delta t + \sum_{j=1}^n a_{ij}^M \int_0^\omega \left|f_j(x_j(t))\right| \Delta t \right. \\
&\qquad + \sum_{j=1}^n b_{ij}^M \int_0^\omega \left|g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)\right| \Delta t \\
&\qquad + \sum_{j=1}^n c_{ij}^M \int_0^\infty \left|K_{ij}(s)\right| \Delta s \\
&\qquad\qquad \left. \times \int_0^\omega \left|h_j\left(x_j(t-s)\right)\right| \Delta t + J_i^M \omega \right) \\
&\quad + \sum_{k=1}^{2q} \rho_{ik} \left|x_i\right|_0 + \sum_{k=1}^{2q} \left|I_{ik}(0)\right| \\
&\le \alpha_i^M \omega \left(\delta_i \left|x_i\right|_0 + \sum_{j=1}^n a_{ij}^M f_j^M + \sum_{j=1}^n b_{ij}^M g_j^M \right. \\
&\qquad \left. + \sum_{j=1}^n c_{ij}^M h_j^M \int_0^\infty \left|K_{ij}(s)\right| \Delta s + J_i^M \right) \\
&\quad + \sum_{k=1}^{2q} \rho_{ik} \left|x_i\right|_0 + \sum_{k=1}^{2q} \left|I_{ik}(0)\right| \\
&= \alpha_i^M \omega \left(\delta_i \left|x_i\right|_0 + B_i + J_i^M\right) + \sum_{k=1}^{2q} \rho_{ik} \left|x_i\right|_0 + \sum_{k=1}^{2q} \left|I_{ik}(0)\right|, \\
&\qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{33}$$

Integrating (31) from 0 to ω, we have by (33)

$$\begin{aligned}
&\left|\int_0^\omega \alpha_i(x_i(t)) \beta_i(x_i(t)) \Delta t\right| \\
&= \left|\frac{1}{1+\lambda} \int_0^\omega \alpha_i(x_i(t)) \sum_{j=1}^n a_{ij}(t) f_j\left(x_j(t)\right) \Delta t \right. \\
&\quad - \frac{\lambda}{1+\lambda} \int_0^\omega \alpha_i(-x_i(t)) \sum_{j=1}^n a_{ij}(t) f_j\left(-x_j(t)\right) \Delta t \\
&\quad + \frac{1}{1+\lambda} \int_0^\omega \alpha_i(x_i(t)) \sum_{j=1}^n b_{ij}(t) g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \Delta t \\
&\quad - \frac{\lambda}{1+\lambda} \int_0^\omega \alpha_i(-x_i(t)) \sum_{j=1}^n b_{ij}(t) g_j\left(-x_j\left(t-\tau_{ij}(t)\right)\right) \Delta t \\
&\quad + \frac{\lambda}{1+\lambda} \sum_{k=1}^{2q} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} \sum_{k=1}^{2q} I_{ik}(-x_i(t_k)) \\
&\quad + \frac{1}{1+\lambda} \int_0^\omega \alpha_i(x_i(t)) \\
&\quad \times \sum_{j=1}^n c_{ij}(t) \int_0^\infty K_{ij}(s) h_j\left(x_j(t-s)\right) \Delta s \Delta t \\
&\quad - \frac{\lambda}{1+\lambda} \int_0^\omega \alpha_i(-x_i(t)) \\
&\quad \times \sum_{j=1}^n c_{ij}(t) \int_0^\infty K_{ij}(s) h_j\left(-x_j(t-s)\right) \Delta s \Delta t \\
&\quad - \frac{1}{1+\lambda} \int_0^\omega \alpha_i(x_i(t)) J_i(t) \Delta t \\
&\quad \left. + \frac{\lambda}{1+\lambda} \int_0^\omega \alpha_i(x_i(t)) J_i(t) \Delta t \right| \\
&\le \alpha_i^M \omega \left(\sum_{j=1}^n a_{ij}^M f_j^M + \sum_{j=1}^n b_{ij}^M g_j^M \right. \\
&\qquad \left. + \sum_{j=1}^n c_{ij}^M h_j^M \int_0^\infty \left|K_{ij}(s)\right| \Delta s + J_i^M \right) \\
&\quad + \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right) \\
&\quad \times \max\left\{\left|\sum_{k=1}^{2q} I_{ik}(x_i(t_k))\right|, \left|\sum_{k=1}^{2q} I_{ik}(-x_i(t_k))\right|\right\} \\
&\le \alpha_i^M \omega \left(\sum_{j=1}^n a_{ij}^M f_j^M + \sum_{j=1}^n b_{ij}^M g_j^M \right.
\end{aligned}$$

$$+\sum_{j=1}^{n}c_{ij}^{M}h_{j}^{M}\int_{0}^{\infty}\left|K_{ij}(s)\right|\Delta s+J_{i}^{M}\Bigg)$$
$$+\max\left\{\left|\sum_{k=1}^{2q}I_{ik}(x_i(t_k))-I_{ik}(0)\right|,\right.$$
$$\left.\left|\sum_{k=1}^{2q}I_{ik}(-x_i(t_k))-I_{ik}(0)\right|\right\}+\sum_{k=1}^{2q}I_{ik}(0)$$
$$\leq \alpha_i^M\omega\left(B_i+J_i^M\right)+\sum_{k=1}^{2q}\rho_{ik}\left|x_{i0}\right|+\sum_{k=1}^{2q}I_{ik}(0),$$
$$i=1,2,\ldots,n. \tag{34}$$

In view of (34), (H_3), and Lemma 15, we get

$$\left|\int_0^\omega \alpha_i(x_i(t))x_i(t)\Delta t\right|\leq\frac{1}{\rho_i}\left|\int_0^\omega\alpha_i(x_i(t))\beta_i(x_i(t))\Delta t\right|$$
$$\leq\frac{1}{\rho_i}\alpha_i^M\omega\left(B_i+J_i^M\right)$$
$$+\frac{1}{\rho_i}\left(\sum_{k=1}^{2q}\rho_{ik}\left|x_{i0}\right|+\sum_{k=1}^{2q}I_{ik}(0)\right), \tag{35}$$

for $i=1,2,\ldots,n$. In addition, from Lemma 12, for any $\xi_i,\eta_i\in[0,\omega]_{\mathbb{T}}$, we have

$$\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t\leq\int_0^\omega\alpha_i(x_i(t))x_i(\xi_i)\Delta t+\int_0^\omega\alpha_i(x_i(t))\left(\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\right)\Delta t, \tag{36}$$

$$\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t\geq\int_0^\omega\alpha_i(x_i(t))x_i(\eta_i)\Delta t-\int_0^\omega\alpha_i(x_i(t))\left(\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\right)\Delta t, \tag{37}$$

where $i=1,2,\ldots,n$. Dividing by $\int_0^\omega\alpha_i(x_i(t))\Delta t$ on the two sides of (36) and (37), respectively, we obtain that

$$x_i(\xi_i)\geq\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t-\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t, \tag{38}$$

$$x_i(\eta_i)\leq\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t+\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t, \tag{39}$$

where $i=1,2,\ldots,n$.

Let $\overline{t}_i,\underline{t}_i\in[0,\omega]_{\mathbb{T}}$ such that $x_i(\overline{t}_i)=\max_{t\in[0,\omega]_{\mathbb{T}}}x_i(t)$, $x_i(\underline{t}_i)=\min_{t\in[0,\omega]_{\mathbb{T}}}x_i(t)$, by the arbitrariness of ξ_i,η_i; we get from (33)–(39) that

$$\begin{aligned}
x_i(\underline{t}_i)&\geq\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t-\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\\
&\geq-\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\left|\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t\right|-\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\\
&\geq-\frac{1}{\rho_i\alpha_i^m}\alpha_i^M\left(B_i+J_i^M\right)-\alpha_i^M\omega\left(\delta_i|x_i|_0+B_i+J_i^M\right)\\
&\quad-\left(\frac{1}{\rho_i\alpha_i^m\omega}+1\right)\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0+\sum_{k=1}^{2q}\left|I_{ik}(0)\right|\right)\\
&=-\left(\frac{\alpha_i^M}{\rho_i\alpha_i^m}+\alpha_i^M\omega\right)\left(B_i+J_i^M\right)-\alpha_i^M\delta_i\omega|x_i|_0\\
&\quad-\left(\frac{1}{\rho_i\alpha_i^m\omega}+1\right)\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0+\sum_{k=1}^{2q}\left|I_{ik}(0)\right|\right),\\
&\qquad i=1,2,\ldots,n,\\
x_i(\overline{t}_i)&\leq\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t+\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\\
&\leq\frac{1}{\int_0^\omega\alpha_i(x_i(t))\Delta t}\left|\int_0^\omega\alpha_i(x_i(t))x_i(t)\Delta t\right|+\int_0^\omega\left|x_i^\Delta(t)\right|\Delta t\\
&\leq\frac{1}{\rho_i\alpha_i^m}\alpha_i^M\left(B_i+J_i^M\right)+\alpha_i^M\omega\left(\delta_i|x_i|_0+B_i+J_i^M\right)\\
&\quad+\left(\frac{1}{\rho_i\alpha_i^m\omega}+1\right)\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0+\sum_{k=1}^{2q}\left|I_{ik}(0)\right|\right)\\
&=\left(\frac{\alpha_i^M}{\rho_i\alpha_i^m}+\alpha_i^M\omega\right)\left(B_i+J_i^M\right)+\alpha_i^M\delta_i\omega|x_i|_0\\
&\quad+\left(\frac{1}{\rho_i\alpha_i^m\omega}+1\right)\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0+\sum_{k=1}^{2q}\left|I_{ik}(0)\right|\right),\\
&\qquad i=1,2,\ldots,n.
\end{aligned} \tag{40}$$

Thus, we have from (40) that

$$\begin{aligned}|x_i|_0 &= \max_{t\in[0,\omega]_{\mathbb{T}}} |x_i(t)| \\ &\le \left(\frac{\alpha_i^M}{\rho_i\alpha_i^m} + \alpha_i^M\omega\right)\left(B_i + J_i^M\right) + \alpha_i^M\delta_i\omega|x_i|_0 \\ &\quad + \left(\frac{1}{\rho_i\alpha_i^m\omega} + 1\right)\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0 + \sum_{k=1}^{2q}|I_{ik}(0)|\right), \\ &\qquad i = 1, 2, \ldots, n.\end{aligned} \tag{41}$$

From (41), we have

$$\begin{aligned}\rho_i\alpha_i^m\omega|x_i|_0 &\le \alpha_i^M\omega\left(1 + \rho_i\alpha_i^m\omega\right)\left(B_i + J_i^M\right) \\ &\quad + \rho_i\alpha_i^M\alpha_i^m\delta_i\omega^2|x_i|_0 + \left(1 + \rho_i\alpha_i^m\omega\right) \\ &\quad \times\left(\sum_{k=1}^{2q}\rho_{ik}|x_i|_0 + \sum_{k=1}^{2q}|I_{ik}(0)|\right), \\ &\qquad i = 1, 2, \ldots, n.\end{aligned} \tag{42}$$

Then, by the assumption of Theorem 16 and (42), we have

$$|x_i|_0 \le \frac{D_i}{E_i} := M_i, \quad i = 1, 2, \ldots, n. \tag{43}$$

Let

$$M = \sum_{i=1}^{n} M_i + 1. \tag{44}$$

Clearly, M is independent of λ. Then take

$$\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M\}. \tag{45}$$

It is clear that Ω satisfies all the requirements in Lemma 14 and condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 14 that system (3) has at least one $(\omega/2)$-antiperiodic solution. This completes the proof. □

4. Global Exponential Stability of Antiperiodic Solutions

In this section, we will construct some suitable Lyapunov functions to study the global exponential stability of antiperiodic solutions of system (3).

Theorem 17. *Assume that (H_1)–(H_7) hold. Suppose further the following.*

(H_8) *The impulsive operators $I_{ik}(x_i(t))$ satisfy*

$$\begin{gathered}I_{ik}(x_i(t)) = -\gamma_{ik}x_i(t_k), \\ 0 \le \gamma_{ik} \le 2, \quad i = 1, 2, \ldots, n, \quad k \in \mathbb{N}.\end{gathered} \tag{46}$$

(H_9) *For $t \in (0, \infty)_{\mathbb{T}}$, there exist constants $\epsilon > 0$ and $\eta > 0$ such that*

$$\begin{gathered}l_i = \epsilon - \alpha_i^l\rho_i + (1 + \mu(t)\epsilon)\left(\alpha_i^L B_i + \alpha_i^M R_i\right) < -\eta < 0, \\ i = 1, 2, \ldots, n.\end{gathered} \tag{47}$$

Then the $(\omega/2)$-antiperiodic solution of system (3) is globally exponentially stable.

Proof. According to Theorem 16 and its proof, we know that system (3) has an $(\omega/2)$-antiperiodic solution $x^* = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ with the initial value $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$ and $|x_i^*|_0 \le M_i$, and suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of system (3) with the initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$. Set $y_i(t) = x_i(t) - x_i^*(t)$. Then it follows from system (3) and (H_8) that

$$\begin{aligned}y_i^\Delta(t) &= -\left(\alpha_i(x_i(t))\beta_i(x_i(t)) - \alpha_i(x_i^*(t))\beta_i(x_i^*(t))\right) \\ &\quad + \Bigg(\alpha_i(x_i(t))\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t)) \\ &\qquad -\alpha_i(x_i^*(t))\sum_{j=1}^{n}a_{ij}(t)f_j(x_j^*(t))\Bigg) \\ &\quad + \Bigg(\alpha_i(x_i(t))\sum_{j=1}^{n}b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &\qquad -\alpha_i(x_i^*(t))\sum_{j=1}^{n}b_{ij}(t)g_j(x_j^*(t-\tau_{ij}(t)))\Bigg) \\ &\quad + \Bigg(\alpha_i(x_i(t))\sum_{j=1}^{n}c_{ij}(t)\int_0^{+\infty}K_{ij}(s)h_j(x_j(t-s))\Delta s \\ &\qquad - \alpha_i(x_i^*(t)) \\ &\qquad \times\sum_{j=1}^{n}c_{ij}(t)\int_0^{+\infty}K_{ij}(s)h_j(x_j^*(t-s))\Delta s\Bigg), \\ \Delta y_i(t_k) &= -\gamma_{ik}y_i(t_k), \quad i = 1, 2, \ldots, n.\end{aligned} \tag{48}$$

In view of the above system and (H_2)–(H_6), for $t \in \mathbb{T}^+$, $t \ne t_k$, $k \in \mathbb{N}$, $i = 1, 2, \ldots, n$, similar to [6], we have

$$\begin{aligned}&D^+\left|y_i^\Delta(t)\right| \\ &\quad\le -\alpha_i^l\left|\beta_i(x_i(t)) - \beta_i(x_i^*(t))\right| \\ &\qquad + \alpha_i^L\sum_{j=1}^{n}\left(a_{ij}^M f_j^M + b_{ij}^M g_j^M + c_{ij}^M h_j^M\int_0^{+\infty}\left|K_{ij}(s)\right|\Delta s\right) \\ &\qquad \times|y_i(t)| + \alpha_i^M\sum_{j=1}^{n}a_{ij}^M f_j^L\left|y_j(t)\right| \\ &\qquad + \alpha_i^M\sum_{j=1}^{n}b_{ij}^M g_j^L\left|y_j(t-\tau_{ij}(t))\right| \\ &\qquad + \alpha_i^M\sum_{j=1}^{n}c_{ij}^M h_j^L\int_0^{+\infty}\left|K_{ij}(s)\right|\left|y_j(t-s)\right|\Delta s\end{aligned}$$

$$
\begin{aligned}
&\le -\alpha_i^l \rho_i |y_i(t)| + \alpha_i^L B_i |y_i(t)| + \alpha_i^M \sum_{j=1}^n a_{ij}^M f_j^L |y_j(t)| \\
&\quad + \alpha_i^M \sum_{j=1}^n b_{ij}^M g_j^L |y_j(t-\tau_{ij}(t))| \\
&\quad + \alpha_i^M \sum_{j=1}^n c_{ij}^M h_j^L \int_0^{+\infty} |K_{ij}(s)| |y_j(t-s)| \Delta s.
\end{aligned} \tag{49}
$$

For any $a \in (-\infty, 0]_{\mathbb{T}}$, we consider the following Lyapunov function:

$$
V_i(t) = |y_i(t)| e_\varepsilon(t, a), \quad i = 1, 2, \ldots n. \tag{50}
$$

For $t \in \mathbb{T}^+, t \neq t_k, k \in \mathbb{N}$, calculating the upper right derivative of $V_i(t)$ by (48)–(50), we have

$$
\begin{aligned}
&D^+ |V_i^\Delta(t)| \\
&\quad = D^+ |y_i^\Delta(t)| e_\varepsilon(\sigma(t), a) + |y_i(t)| e_\varepsilon^\Delta(t, a) \\
&\quad \le \epsilon |y_i(t)| e_\varepsilon(t, a) + e_\varepsilon(\sigma(t), a) \\
&\qquad \times \Bigg(-\alpha_i^l \rho_i |y_i(t)| + \alpha_i^L B_i |y_i(t)| + \alpha_i^M \sum_{j=1}^n a_{ij}^M f_j^L |y_j(t)| \\
&\qquad\quad + \alpha_i^M \sum_{j=1}^n b_{ij}^M g_j^L |y_j(t-\tau_{ij}(t))| + \alpha_i^M \\
&\qquad\quad \times \sum_{j=1}^n c_{ij}^M h_j^L \int_0^{+\infty} |K_{ij}(s)| |y_j(t-s)| \Delta s e_\varepsilon(\sigma(t), a) \Bigg) \\
&\quad = \epsilon V_i(t) + (1 + \epsilon \mu(t)) \\
&\qquad \times \Bigg(\left(-\alpha_i^l \rho_i + \alpha_i^L B_i\right) V_i(t) + \alpha_i^M \sum_{j=1}^n a_{ij}^M f_j^L V_j(t) \\
&\qquad\quad + \alpha_i^M \sum_{j=1}^n b_{ij}^M g_j^L V_j(t-\tau_{ij}(t)) e_\varepsilon(t, t-\tau_{ij}(t)) + \alpha_i^M \\
&\qquad\quad \times \sum_{j=1}^n c_{ij}^M h_j^L \int_0^{+\infty} |K_{ij}(s)| e_\varepsilon(t, t-s) V_j(t-s) \Delta s \Bigg),
\end{aligned} \tag{51}
$$

for $i = 1, 2, \ldots, n$. In addition, for $t \in \mathbb{T}^+$, $t = t_k$, $k \in \mathbb{N}$, we have from (H_8) that

$$
\begin{aligned}
x_i(t_k^+) - x_i^*(t_k^+) &= |1 - \gamma_{ik}| |x_i(t_k) - x_i^*(t_k)| \\
&\le |x_i(t_k) - x_i^*(t_k)|, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{52}
$$

Now, let $M > 1$ denote an arbitrary real number and set

$$
\|\varphi - \varphi^*\|_\infty = \sup_{-\infty < s < 0} \max_{1 \le i \le n} |\varphi_i(s) - \varphi_i^*(s)| > 0. \tag{53}
$$

It follows from (50) and Definition 13 that

$$
\begin{aligned}
V_i(t) &= |y_i(t)| e_\varepsilon(t, a) < M \|\varphi - \varphi^*\|_\infty, \\
&\forall t \in (-\infty, 0]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{54}
$$

We claim that

$$
\begin{aligned}
V_i(t) &= |y_i(t)| e_\varepsilon(t, a) < M \|\varphi - \varphi^*\|_\infty, \\
&\forall t \in (0, \infty)_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{55}
$$

If it is not true, in view of the arbitrariness of M, there exist $i \in \{1, 2, \ldots, n\}$ and $0 < t_i < t_1$ such that

$$
V_i(t_i) \ge M \|\varphi - \varphi^*\|_\infty, \tag{56}
$$

and for $j \neq i$, $j = 1, 2, \ldots, n$, we have

$$
V_j(t) < M \|\varphi - \varphi^*\|_\infty, \quad \forall t \in (-\infty, t_i)_{\mathbb{T}}. \tag{57}
$$

Let $r_i = V_i(t_i)/M\|\varphi - \varphi^*\|_\infty$. Then it follows from (56) and (57) that

$$
\begin{aligned}
r_i \ge 1, \qquad & V_i(t_i) = r_i M \|\varphi - \varphi^*\|_\infty, \\
& V_j(t) < r_i M \|\varphi - \varphi^*\|_\infty.
\end{aligned} \tag{58}
$$

Together with (48), (51), (58), and Lemma 10, we obtain

$$
\begin{aligned}
0 &\le D^+ |V_i^\Delta(t_i)| \\
&= D^+ |y_i^\Delta(t_i)| e_\varepsilon(\sigma(t_i), a) + |y_i(t_i)| e_\varepsilon^\Delta(t_i, a) \\
&\le \epsilon V_i(t_i) + (1 + \epsilon \mu(t_i)) \\
&\quad \times \Bigg(\left(-\alpha_i^l \rho_i + \alpha_i^L B_i\right) V_i(t_i) + \alpha_i^M \sum_{j=1}^n a_{ij}^M f_j^L V_j(t_i) \\
&\qquad + \alpha_i^M \sum_{j=1}^n b_{ij}^M g_j^L V_j(t_i - \tau_{ij}(t_i)) e_\varepsilon(t_i, t_i - \tau_{ij}(t_i)) \\
&\qquad + \alpha_i^M \sum_{j=1}^n c_{ij}^M h_j^L \\
&\qquad \times \int_0^{+\infty} |K_{ij}(s)| e_\varepsilon(t_i, t_i - s) V_j(t_i - s) \Delta s \Bigg) \\
&\le \left\{ \epsilon - \alpha_i^l \rho_i + (1 + \epsilon \mu(t_i)) \left(\alpha_i^L B_i + \alpha_i^M R_i\right) \right\} \\
&\quad \times r_i M \|\varphi - \varphi^*\|_\infty.
\end{aligned} \tag{59}
$$

Thus,

$$
0 \le \epsilon - \alpha_i^l \rho_i + (1 + \epsilon \mu(t_i)) \left(\alpha_i^L B_i + \alpha_i^M R_i\right), \tag{60}
$$

which contradicts (H_9). Hence, (55) holds. It follows that

$$
\begin{aligned}
|x_i(t) - x_i^*(t)| &< M e_{\ominus \epsilon} \|\varphi - \varphi^*\|_\infty, \\
&\forall t \in (-\infty, t_1)_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{61}
$$

When $t = t_1$, the second expression of (48) implies

$$\begin{aligned} \left|y_i\left(t_1^+\right)\right| &= \left|y_i\left(t_1^-\right) - \gamma_{i1} y_i\left(t_1\right)\right| \\ &\leq \left|1-\gamma_{i1}\right| \left|y_i\left(t_1\right)\right| \leq \lim_{t \to t_1^-} \left|y_i(t)\right| \\ &< M e_{\ominus\epsilon} \left\|\varphi - \varphi^*\right\|_\infty; \end{aligned} \tag{62}$$

that is,

$$\left|x_i\left(t_1^+\right) - x_i^*\left(t_1^+\right)\right| < M e_{\ominus\epsilon} \left\|\varphi - \varphi^*\right\|_\infty, \tag{63}$$

where $i = 1, 2, \ldots, n$. Similar to the step of (51)–(63), we can also prove that

$$\begin{gathered} \left|x_i(t) - x_i^*(t)\right| < M e_{\ominus\epsilon} \left\|\varphi - \varphi^*\right\|_\infty, \\ \forall t \in \left[t_1, t_2\right)_{\mathbb{T}}, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{64}$$

When $t = t_2$, again, from the second expression of (48), we have

$$\left|x_i\left(t_2^+\right) - x_i^*\left(t_2^+\right)\right| < M e_{\ominus\epsilon} \left\|\varphi - \varphi^*\right\|_\infty, \quad i = 1, 2, \ldots, n. \tag{65}$$

By repeating the same procedure, we obtain

$$\begin{gathered} \left|x_i(t) - x_i^*(t)\right| < M e_{\ominus\epsilon} \left\|\varphi - \varphi^*\right\|_\infty, \\ \forall t \in (0, \infty)_{\mathbb{T}}, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{66}$$

In view of Definition 13, the $(\omega/2)$-antiperiodic solution $x^*(t)$ of system (3) is globally exponentially stable. This completes the proof. □

5. Example

In this section, we give an example to illustrate that our results are feasible.

When $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\mu(t) = 1$, we consider the following Cohen-Grossberg neural networks system with impulses:

$$\begin{aligned} x_i^\Delta(t) = &-\alpha_i\left(x_i(t)\right) \\ &\times \Bigg[\beta_i\left(x_i(t)\right) - \sum_{j=1}^n a_{ij}(t) f_j\left(x_j(t)\right) \\ &\quad - \sum_{j=1}^n b_{ij}(t) g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - \sum_{j=1}^n c_{ij}(t) \\ &\quad \times \int_0^{+\infty} K_{ij}(s) h_j\left(x_j(t-s)\right) \Delta s + J_i(t)\Bigg], \\ &\qquad t \in \mathbb{T}^+, \quad t \neq t_k, \quad k \in \mathbb{N}, \\ \Delta x_i\left(t_k\right) = &\, x_i\left(t_k^+\right) - x_i\left(t_k^-\right) = \frac{1}{800} x_i\left(t_k\right), \\ &\qquad t = t_k, \quad i, k = 1, 2, \end{aligned} \tag{67}$$

where

$$\begin{gathered} \alpha_1\left(x_1(t)\right) = \frac{1}{200} + \frac{1}{300}\cos\left(x_1(t)\right), \\ \alpha_2\left(x_2(t)\right) = \frac{5}{800} + \frac{1}{800}\cos\left(x_2(t)\right), \\ \beta_1\left(x_1(t)\right) = 8x_1(t) + 2\sin\left(x_1(t)\right), \\ \beta_2\left(x_2(t)\right) = 9x_2(t) + 4\sin\left(x_2(t)\right), \\ f_j(u) = \frac{1}{200}\sin u, \qquad g_j(u) = \frac{1}{300}\sin u, \\ h_j(u) = \frac{1}{400}\sin u, \qquad J_i(t) = \sin\left(\frac{\pi}{2}t\right), \\ K_{ij}(t) = e^{-300t}, \qquad \tau_{ij}(t) = 2\left|\cos\left(\frac{\pi}{2}t\right)\right|, \quad i, j = 1, 2, \\ \left(a_{ij}(t)\right)_{2\times2} = \begin{pmatrix} \frac{1}{300} + \frac{1}{200}\left|\sin\left(\frac{\pi}{2}t\right)\right| & \frac{1}{300} - \frac{1}{400}\left|\cos\left(\frac{\pi}{2}t\right)\right| \\ \frac{1}{300}\left|\cos\left(\frac{\pi}{2}t\right)\right| & \frac{1}{400}\left|\sin\left(\frac{\pi}{2}t\right)\right| \end{pmatrix}, \\ \left(b_{ij}(t)\right)_{2\times2} = \begin{pmatrix} \frac{1}{400}\left|\cos\left(\frac{\pi}{2}t\right)\right| & \frac{1}{500}\left|\sin\left(\frac{\pi}{2}t\right)\right| \\ \frac{1}{600}\left|\sin\left(\frac{\pi}{2}t\right)\right| & \frac{1}{700}\left|\sin\left(\frac{\pi}{2}t\right)\right| \end{pmatrix}, \\ \left(c_{ij}(t)\right)_{2\times2} = \begin{pmatrix} \frac{1}{800}\left|\sin\left(\frac{\pi}{2}t\right)\right| & \frac{1}{900}\left|\cos\left(\frac{\pi}{2}t\right)\right| \\ \frac{1}{100}\left|\cos\left(\frac{\pi}{2}t\right)\right| & \frac{1}{200}\left|\cos\left(\frac{\pi}{2}t\right)\right| \end{pmatrix}. \end{gathered} \tag{68}$$

By calculation, we have

$$\begin{gathered} \alpha_1^M = \frac{5}{600}, \qquad \alpha_1^m = \alpha_1^l = \frac{1}{600}, \qquad \alpha_2^M = \frac{3}{400}, \\ \alpha_2^m = \alpha_2^l = \frac{1}{200}, \qquad \alpha_1^L = \frac{1}{300}, \qquad \alpha_2^L = \frac{1}{800}, \\ \rho_1 = 6, \qquad \delta_1 = 10, \qquad \rho_2 = 5, \qquad \delta_2 = 13, \\ f_j^M = f_j^L = \frac{1}{200}, \qquad g_j^M = g_j^L = \frac{1}{300}, \\ h_j^M = h_j^L = \frac{1}{400}, \qquad J_i^M = 1, \qquad \rho_{ik} = \frac{1}{800}, \\ \omega = 4, \\ i, j, k = 1, 2, \\ \left(a_{ij}^M\right)_{2\times2} = \begin{pmatrix} \frac{5}{600} & \frac{1}{300} \\ \frac{1}{300} & \frac{1}{400} \end{pmatrix}, \end{gathered}$$

$$\left(b_{ij}^{M}\right)_{2\times 2}=\begin{pmatrix}\frac{1}{400} & \frac{1}{500}\\ \frac{1}{600} & \frac{1}{700}\end{pmatrix},$$
$$\left(c_{ij}^{M}\right)_{2\times 2}=\begin{pmatrix}\frac{1}{800} & \frac{1}{900}\\ \frac{1}{100} & \frac{1}{200}\end{pmatrix}. \quad (69)$$

Therefore, $E_1 = 0.02407$, $E_2 = 0.05825$. Take $\epsilon = 10^{-6}$ and $\eta = 10^{-3}$, and then

$$l_1 \approx -0.009998, \qquad l_2 \approx -0.024999. \quad (70)$$

It is easy to see that (H_1)–(H_9) hold. According to Theorems 16 and 17, system (67) has a 2-antiperiodic solution which is globally exponentially stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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