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Research Article

Caristi Fixed Point Theorem in Metric Spaces with a Graph

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We discuss Caristi's fixed point theorem for mappings defined on a metric space endowed with a graph. This work should be seen as a generalization of the classical Caristi's fixed point theorem. It extends some recent works on the extension of Banach contraction principle to metric spaces with graph.

Dedicated to Rashed Saleh Alfuraidan and Prof. Miodrag Mateljević for his 65th birthday

1. Introduction

This work was motivated by some recent works on the extension of Banach contraction principle to metric spaces with a partial order [1] or a graph [2]. Caristi's fixed point theorem is maybe one of the most beautiful extensions of Banach contraction principle [3, 4]. Recall that this theorem states the fact that any map $T : M \to M$ has a fixed point provided that M is a complete metric space and there exists a lower semicontinuous map $\phi : M \to [0, +\infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad (1)$$

for every $x \in M$. Recall that $x \in M$ is called a fixed point of T if $T(x) = x$. This general fixed point theorem has found many applications in nonlinear analysis. It is shown, for example, that this theorem yields essentially all the known inwardness results [5] of geometric fixed point theory in Banach spaces. Recall that inwardness conditions are the ones which assert that, in some sense, points from the domain are mapped toward the domain. Possibly, the weakest of the inwardness conditions, the Leray-Schauder boundary condition, is the assumption that a map points x of ∂M anywhere except to the outward part of the ray originating at some interior point of M and passing through x.

The proofs given to Caristi's result vary and use different techniques (see [3, 6–8]). It is worth to mention that because of Caristi's result of close connection to the Ekeland's [9] variational principle, many authors refer to it as Caristi-Ekeland fixed point result. For more on Ekeland's variational principle and the equivalence between Caristi-Ekeland fixed point result and the completeness of metric spaces, the reader is advised to read [10].

2. Main Results

Maybe one of the most interesting examples of the use of metric fixed point theorems is the proof of the existence of solutions to differential equations. The general approach is to convert such equations to integral equations which describes exactly a fixed point of a mapping. The metric spaces in which such mapping acts are usually a function space. Putting a norm (in the case of a vector space) or a distance gives us a metric structure rich enough to use the Banach contraction principle or other known fixed point theorems. But one structure naturally enjoyed by such function spaces is rarely used. Indeed we have an order on the functions inherited from the order of $\mathbb{R}$. In the classical use of Banach contraction principle, the focus is on the metric behavior of the mapping. The connection with the natural order is usually ignored.

In [1, 11], the authors gave interesting examples where the order is used combined with the metric conditions.

Example 1 (see [1]). Consider the periodic boundary value problem

$$u'(t) = f(t, u(t)), \quad t \in [0, T], \qquad u(0) = u(T), \tag{2}$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Clearly any solution to this problem must be continuously differentiable on $[0, T]$. So the space to be considered for this problem is $C^1([0, T], \mathbb{R})$. The above problem is equivalent to the integral problem

$$u(t) = \int_0^T G(t, s)\left[f(s, u(s)) + \lambda u(s)\right] ds, \tag{3}$$

where $\lambda > 0$ and the Green function is given by

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} & 0 \le s < t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} & 0 \le t < s \le T. \end{cases} \tag{4}$$

Define the mapping $\mathscr{A} : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ by

$$\mathscr{A}(u)(t) = \int_0^T G(t, s)\left[f(s, u(s)) + \lambda u(s)\right] ds. \tag{5}$$

Note that if $u(t) \in C([0, T], \mathbb{R})$ is a fixed point of $\mathscr{A}$, then $u(t) \in C^1([0, T], \mathbb{R})$ is a solution to the original boundary value problem. Under suitable assumptions, the mapping $\mathscr{A}$ satisfies the following property:

(i) if $u(t) \le v(t)$, then we have $\mathscr{A}(u) \le \mathscr{A}(v)$;

(ii) if $u(t) \le v(t)$, then

$$\|\mathscr{A}(u) - \mathscr{A}(v)\| \le k\|u - v\|, \tag{6}$$

for a constant $k < 1$ independent of u and v.

The contractive condition is only valid for comparable functions. It does not hold on the entire space $C([0, T], \mathbb{R})$. This condition led the authors in [1] to use a weaker version of the Banach contraction principle to prove the existence of the solution to the original boundary value problem, a result which was already known [12] using different techniques.

Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$. We will say that T is monotone increasing if

$$x \preceq y \Longrightarrow T(x) \preceq T(y), \quad \text{for any } x, y \in X. \tag{7}$$

The main result of [1] is the following theorem.

Theorem 2 (see [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and monotone increasing mapping such that there exists $k < 1$ with*

$$d(T(x), T(y)) \le kd(x, y), \quad \textit{whenever } y \preceq x. \tag{8}$$

If there exists $x_0 \in X$, with $x_0 \preceq T(x_0)$, then T has a fixed point.

Clearly, from this theorem, one may see that the contractive nature of the mapping T is restricted to the comparable elements of $(X, \preceq)$ not to the entire set X. The detailed investigation of the example above shows that such mappings may exist which are not contractive on the entire set X. Therefore the classical Banach contraction principle will not work in this situation. The analogue to Caristi's fixed theorem in this setting is the following result.

Theorem 3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and monotone increasing mapping. Assume that there exists a lower semicontinuous function $\phi : X \to [0, +\infty)$ such that*

$$d(x, T(x)) \le \phi(x) - \phi(T(x)), \quad \textit{whenever } T(x) \preceq x. \tag{9}$$

Then T has a fixed point if and only if there exists $x_0 \in X$, with $T(x_0) \preceq x_0$.

Proof. Clearly, if x_0 is a fixed point of T, that is, $T(x_0) = x_0$, then we have $T(x_0) \preceq x_0$. Assume that there exists $x_0 \in X$ such that $T(x_0) \preceq x_0$. Since T is monotone increasing, we have $T^{n+1}(x_0) \preceq T^n(x_0)$, for any $n \ge 1$. Hence

$$d\left(T^n(x_0), T^{n+1}(x_0)\right) \le \phi\left(T^n(x_0)\right) - \phi\left(T^{n+1}(x_0)\right), \quad n = 1, 2, \ldots. \tag{10}$$

Hence $\{\phi(T^n(x_0))\}$ is a decreasing sequence of positive numbers. Let $\phi_0 = \lim_{n\to\infty}\phi(T^n(x_0))$. For any $n, h \ge 1$, we have

$$d\left(T^n(x_0), T^{n+h}(x_0)\right) \le \sum_{k=0}^{h-1} d\left(T^{n+k}(x_0), T^{n+k+1}(x_0)\right) \le \phi\left(T^n(x_0)\right) - \phi\left(T^{n+h}(x_0)\right). \tag{11}$$

Therefore $\{T^n(x_0)\}$ is a Cauchy sequence in X. Since X is complete, there exists $\overline{x} \in X$ such that $\lim_{n\to\infty} T^n(x_0) = \overline{x}$. Since T is continuous, we conclude that $T(\overline{x}) = \overline{x}$; that is, $\overline{x}$ is a fixed point of T. □

The continuity assumption of T may be relaxed if we assume that X satisfies the property (OSC).

Definition 4. Let $(X, \preceq)$ be a partially ordered set. Let d be a distance defined on X. One says that X satisfies the property (OSC) if and only if for any convergent decreasing sequence $\{x_n\}$ in X, that is, $x_{n+1} \preceq x_n$, for any $n \ge 1$, one has $\lim_{m\to\infty} x_m = \inf\{x_n, n \ge 1\}$.

One has the following improvement to Theorem 3.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Assume that X satisfies the property (OSC). Let $T : X \to X$ be a monotone increasing mapping. Assume that* *there exists a lower semicontinuous function* $\phi : X \to [0, +\infty)$ *such that*

$$d(x, T(x)) \leq \phi(x) - \phi(T(x)), \quad \text{whenever } T(x) \preceq x. \tag{12}$$

Then T *has a fixed point if and only if there exists* $x_0 \in X$, *with* $T(x_0) \preceq x_0$.

Proof. We proceed as we did in the proof of Theorem 3. Let $x_0 \in X$ such that $T(x_0) \preceq x_0$. Write $x_n = T^n(x_0)$, $n \geq 1$. Then we have the fact that $\{x_n\}$ is decreasing and $\lim_{n\to\infty} x_n = x_\omega$ exists in X. Since we did not assume T continuous, then x_ω may not be a fixed point of T. The idea is to use transfinite induction to build a transfinite orbit to help catch the fixed point. Note that since X satisfies (OSC), then we have $x_\omega = \inf\{x_n, n \geq 1\}$. Since T is monotone increasing, then we will have $T(x_\omega) \preceq x_\omega$. These basic facts so far will help us seek the transfinite orbit $\{x_\alpha\}_{\alpha\Gamma}$, where Γ is the set of all ordinals. This transfinite orbit must satisfy the following properties:

(1) $T(x_\alpha) = x_{\alpha+1}$, for any $\alpha \in \Gamma$;

(2) $x_\alpha = \inf\{x_\beta, \beta < \alpha\}$, if α is a limit ordinal;

(3) $x_\alpha \preceq x_\beta$, whenever $\beta < \alpha$;

(4) $d(x_\alpha, x_\beta) \leq \phi(x_\beta) - \phi(x_\alpha)$, whenever $\beta < \alpha$.

Clearly the above properties are satisfied for any $\alpha \in \{0, 1, \ldots, \omega\}$. Let α be an ordinal number. Assume that the properties (1)–(4) are satisfied by $\{x_\beta\}_{\beta<\alpha}$. We have two cases as follows.

(i) If $\alpha = \beta + 1$, then set $x_\alpha = T(x_\beta)$.

(ii) Assume that α is a limit ordinal. Set $\phi_0 = \inf\{\phi(x_\beta), \beta < \alpha\}$. Then one can easily find an increasing sequence of ordinals $\{\beta_n\}$, with $\beta_n < \alpha$, such that $\lim_{n\to\infty} \phi(x_{\beta_n}) = \phi_0$. Property (4) will force $\{x_{\beta_n}\}$ to be Cauchy. Since X is complete, then $\lim_{n\to\infty} x_{\beta_n} = \overline{x}$ exists in X. The property (OSC) will then imply $\overline{x} = \inf\{x_{\beta_n}, n \geq 1\}$. Let us show that $\overline{x} = \inf\{x_\beta, \beta < \alpha\}$. Let $\beta < \alpha$. If $\beta_n < \beta$, for all $n \geq 1$, then we have

$$d\left(x_\beta, x_{\beta_n}\right) \leq \phi\left(x_{\beta_n}\right) - \phi\left(x_\beta\right), \quad n = 1, 2, \ldots. \tag{13}$$

But $\phi(x_\beta) \geq \phi_0 = \lim_{n\to\infty} \phi(x_{\beta_n}) \geq \phi(x_\beta)$. Hence $\phi(x_\beta) = \phi_0$ which implies that $\lim_{n\to\infty} x_{\beta_n} = x_\beta$. Hence $x_\beta = \overline{x}$. Assume otherwise that there exists $n_0 \geq 1$ such that $\beta < \beta_{n_0}$. Hence $x_{\beta_{n_0}} \preceq x_\beta$ which implies $\overline{x} \preceq x_\beta$. In any case, we have $\overline{x} \preceq x_\beta$, for any $\beta < \alpha$. Therefore we have

$$\overline{x} = \inf\left\{x_{\beta_n}, n \geq 1\right\} \leq \inf\left\{x_\beta, \beta < \alpha\right\} \leq \inf\left\{x_{\beta_n}, n \geq 1\right\}. \tag{14}$$

Hence $\overline{x} = \inf\{x_\beta, \beta < \alpha\}$. Set $x_\alpha = \overline{x}$. Let us prove that $\{x_\beta, \beta \leq \alpha\}$ satisfies all properties (1)–(4). Clearly (1) and (2) are satisfied. Let us focus on (3) and (4). Let $\beta < \alpha$. We need to show that $x_\alpha \preceq x_\beta$. If α is a limit ordinal, this is obvious. Assume that $\alpha - 1$ exists. We have two cases; if $\alpha - 2$ exists, then we have $x_{\alpha-1} \preceq x_{\alpha-2}$. Since T is monotone increasing, then $T(x_{\alpha-1}) \preceq T(x_{\alpha-2})$); that is, $x_\alpha \preceq x_{\alpha-1}$. Otherwise, if $\alpha - 2$ is an ordinal limit, then $x_{\alpha-2} = \inf\{x_\gamma, \gamma < \alpha - 2\}$. Since T is monotone increasing, then we have

$$x_{\alpha-1} = T\left(x_{\alpha-2}\right) \preceq x_{\gamma+1}, \quad \text{for any } \gamma < \alpha - 2, \tag{15}$$

which implies $x_\alpha \preceq x_{\gamma+2}$, for any $\gamma < \alpha - 2$. Therefore we have $x_\alpha \preceq x_\beta$, which completes the proof of (3). Let us prove (4). Let $\beta < \alpha$. First assume that $\alpha - 1$ exists. Then, in the proof of (3), we saw that $x_\alpha = T(x_{\alpha-1}) \preceq x_{\alpha-1}$. Our assumption on T will then imply

$$d\left(x_{\alpha-1}, x_\alpha\right) \leq \phi\left(x_{\alpha-1}\right) - \phi\left(x_\alpha\right). \tag{16}$$

If $\beta = \alpha - 1$, we are done. Otherwise, if $\beta < \alpha - 1$, then we use the induction assumption to get

$$d\left(x_\beta, x_{\alpha-1}\right) \leq \phi\left(x_\beta\right) - \phi\left(x_{\alpha-1}\right). \tag{17}$$

The triangle inequality will then imply

$$d\left(x_\beta, x_\alpha\right) \leq \phi\left(x_\beta\right) - \phi\left(x_\alpha\right). \tag{18}$$

Next we assume that α is a limit ordinal. Then there exists an increasing sequence of ordinals $\{\beta_n\}$, with $\beta_n < \alpha$, such that $x_\alpha = \lim_{n\to\infty} x_{\beta_n}$. Given $\beta < \alpha$, assume that we have $\beta_n < \beta$, for all $n \geq 1$. In this case, we have seen that $x_\alpha = x_\beta$. Otherwise, let us assume that there exists $n_0 \geq 1$ such that $\beta < \beta_{n_0}$. In this case, from our induction assumption and the triangle inequality, we get

$$d\left(x_\beta, x_{\beta_n}\right) \leq \phi\left(x_\beta\right) - \phi\left(x_{\beta_n}\right), \quad n \geq n_0. \tag{19}$$

Using the lower semicontinuity of ϕ, we conclude that

$$d\left(x_\beta, x_\alpha\right) \leq \phi\left(x_\beta\right) - \phi\left(x_\alpha\right), \tag{20}$$

which completes the proof of (4). By the transfinite induction we conclude that the transfinite orbit $\{x_\alpha\}$ exists which satisfies the properties (1)–(4). Using Proposition A.6 ([13, page 284]), there exists an ordinal β such that $\phi(x_\alpha) = \phi(x_\beta)$, for any $\alpha \geq \beta$. In particular, we have $\phi(x_\alpha) = \phi(x_{\alpha+1})$, for any $\alpha \geq \beta$. Property (4) will then force $x_{\alpha+1} = x_\alpha$; that is, $T(x_\alpha) = x_\alpha$. Therefore T has a fixed point.

One may wonder if Theorem 5 is truly an extension of the main results of [1, 2, 11]. The following example shows that it is the case.

Example 6. Let $X = L^1([0, 1], dx)$ be the classical Banach space with the natural pointwise order generated by $\mathbb{R}$. Let $C = \{f \in X, f(t) \geq 0 \ a.e.\}$ be the positive cone of X. Define $T : C \to C$ by

$$T(f)(t) = \begin{cases} f(t) & \text{if } f(t) > \dfrac{1}{2}, \\ 0 & \text{if } f(t) \leq \dfrac{1}{2}. \end{cases} \tag{21}$$

First note that C is a closed subset of X. Hence C is complete for the norm-1 distance. Also it is easy to check that the property (OSC) holds in this case. Note that, for any

$f \in C$, we have $0 \leq T(f) \leq f$. Also we have $T^2(f) = T(T(f)) = T(f))$; that is, $T(f)$ is a fixed point of T for any $f \in C$. Note that for any $f \in C$ we have

$$d(f, T(f)) = \int_0^1 |f(t) - T(f)(t)|\, dt = \int_0^1 f(t)\, dt - \int_0^1 T(f)(t)\, dt = \|f\| - \|T(f)\|. \tag{22}$$

Therefore all the assumptions of Theorem 5 are satisfied. But T fails to satisfy the assumptions of [1, 2, 11]. Indeed if we take

$$f(t) = \frac{1}{2}, \quad f_n(t) = \frac{1}{2} + \frac{1}{n}, \quad n \geq 1, \tag{23}$$

then we have $T(f) = 0$ and $T(f_n) = f_n$, for any $n \geq 1$. Therefore T is not continuous since $\{f_n\}$ converges uniformly (and in norm-1 as well) to f. Note also that $f \leq f_n$, for $n \geq 1$. So any Lipschitz condition on the partial order of C will not be satisfied by T in this case. □

3. Caristi's Theorem in Metric Spaces with Graph

It seems that the terminology of graph theory instead of partial ordering sets can give more clear pictures and yield to generalize the theorems above. In this section, we give the graph versions of our two main results.

Throughout this section we assume that (X, d) is a metric space and G is a directed graph (digraph) with set of vertices $V(G) = X$ and set of edges $E(G)$ containing all the loops; that is, $(x, x) \in E(G)$ for any $x \in X$. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G), E(G))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [14, 15]. A digraph G is called an oriented graph; if whenever $(u, v) \in E(G)$, then $(v, u) \notin E(G)$.

Let $(X, \preceq)$ be a partially ordered set. We define the oriented graph $G_{\preceq}$ on X as follows. The vertices of $G_{\preceq}$ are the elements of X, and two vertices $x, y \in X$ are connected by a directed edge (arc) if $x \preceq y$. Therefore, $G_{\preceq}$ has no parallel arcs as $x \preceq y$ & $y \preceq x \Rightarrow x = y$.

If x, y are vertices of the digraph G, then a directed path from x to y of length N is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that

$$x_0 = x, \qquad x_N = y, \qquad (x_i, x_{i+1}) \in E(G), \quad i = 0, 1, \ldots, N. \tag{24}$$

A closed directed path of length $N > 1$ from x to y, that is, $x = y$, is called a directed cycle. An acyclic digraph is a digraph that has no directed cycle.

Given an acyclic digraph, G, we can always define a partially order $\preceq_G$ on the set of vertices of G by defining that $x \preceq_G y$ whenever there is a directed path from x to y.

Definition 7. One says that a mapping $T : X \to X$ is G-edge preserving if

$$\forall x, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G). \tag{25}$$

T is said to be a Caristi G-mapping if there exists a lower semicontinuous function $\phi : X \to [0, +\infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \text{whenever } (T(x), x) \in E(G). \tag{26}$$

One can now give the graph theory versions of our two mean Theorems 3 and 5 as follows.

Theorem 8. *Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be continuous, G-edge preserving, and a G-Caristi mapping. Then T has a fixed point if and only if there exists $x_0 \in X$, with $(T(x_0), x_0) \in E(G)$.*

Proof. G has all the loops. In particular, if x_0 is a fixed point of T, that is, $T(x_0) = x_0$, then we have $(T(x_0), x_0) \in E(G)$. Assume that there exists $x_0 \in X$ such that $(T(x_0), x_0) \in E(G)$. Since T is G-edge preserving, we have $(T^{n+1}(x_0), T^n(x_0)) \in E(G)$, for any $n \geq 1$. Hence

$$d\left(T^n(x_0), T^{n+1}(x_0)\right) \leq \phi\left(T^n(x_0)\right) - \phi\left(T^{n+1}(x_0)\right), \quad n = 1, 2, \ldots. \tag{27}$$

Hence $\{\phi(T^n(x_0))\}$ is a decreasing sequence of positive numbers. Let $\phi_0 = \lim_{n \to \infty} \phi(T^n(x_0))$. For any $n, h \geq 1$, we have

$$d\left(T^n(x_0), T^{n+h}(x_0)\right) \leq \sum_{k=0}^{h-1} d\left(T^{n+k}(x_0), T^{n+k+1}(x_0)\right) \leq \phi\left(T^n(x_0)\right) - \phi\left(T^{n+h}(x_0)\right). \tag{28}$$

Therefore $\{T^n(x_0)\}$ is a Cauchy sequence in X. Since X is complete, there exists $\overline{x} \in X$ such that $\lim_{n \to \infty} T^n(x_0) = \overline{x}$. Since T is continuous, we conclude that $T(\overline{x}) = \overline{x}$; that is, $\overline{x}$ is a fixed point of T. □

The following definition is needed to prove the analogue to Theorem 5.

Definition 9. Let G be an acyclic oriented graph on the set X with $E(G)$ containing all loops. One says that G satisfies the property (OSC) if and only if $(X, \preceq_G)$ satisfies (OSC).

The analogue to Theorem 5 may be stated as follows.

Theorem 10. *Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that G satisfies the property (OSC). Let $T : X \to X$ be a G-edge preserving and a Caristi G-mapping. Then T has a fixed point if and only if there exists $x_0 \in X$, with $(T(x_0), x_0) \in E(G)$.*

The proof of Theorem 10 is similar to the proof of Theorem 5. In fact it is easy to check that these two theorems are equivalent.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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