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Research Article

Implicit Vector Integral Equations Associated with Discontinuous Operators

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Let $I := [0, 1]$. We consider the vector integral equation $h(u(t)) = f\left(t, \int_I g(t, z), u(z), dz\right)$ for a.e. $t \in I$, where $f : I \times J \to \mathbf{R}$, $g : I \times I \to [0, +\infty[$, and $h : X \to \mathbf{R}$ are given functions and X, J are suitable subsets of $\mathbf{R}^n$. We prove an existence result for solutions $u \in L^s(I, \mathbf{R}^n)$, where the continuity of f with respect to the second variable is not assumed. More precisely, f is assumed to be a.e. equal (with respect to second variable) to a function $f^* : I \times J \to \mathbf{R}$ which is almost everywhere continuous, where the involved null-measure sets should have a suitable geometry. It is easily seen that such a function f can be discontinuous at each point $x \in J$. Our result, based on a very recent selection theorem, extends a previous result, valid for scalar case $n = 1$.

1. Introduction

Let $I := [0, 1]$. Recently, in the paper [1], the following integral equation was studied: given $\lambda > 0$, $f : I \times [0, \lambda] \to \mathbf{R}$, $g : I \times I \to [0, +\infty[$, $h :]0, +\infty[\to \mathbf{R}$, and $s > 1$; find $u \in L^s(I)$ such that

$$h(u(t)) = f\left(t, \int_I g(t, z)\, u(z)\, dz\right) \quad \text{for a.e. } t \in I. \tag{1}$$

In [1], the following existence result was proved, where, unlike other results in the field (see, for instance, the papers [2–5] and references therein, to which we also refer for motivations for studying (1)), the continuity of f with respect to the second variable was not assumed.

Theorem 1 (Theorem 1 of [1]). *Let $\lambda > 0$, $A \subseteq]0, +\infty[$ a closed interval, $h : A \to \mathbf{R}$ a continuous function, $f : I \times [0, \lambda] \to \mathbf{R}$, and $g : I \times I \to [0, +\infty[$ two given functions. Let $s \in]1, +\infty]$, $\phi_0 \in L^j(I)$, with $j > 1$ and $j \geq s'$ (the conjugate exponent of s), $\phi_1 \in L^{s'}(I)$, and $\beta \in L^s(I)$. Assume that*

(i) *there exists a function $f^* : I \times [0, \lambda] \to \mathbf{R}$, two negligible sets $E_1, E_2 \subseteq [0, \lambda]$, with E_2 closed, and a countable dense subset D of $[0, \lambda]$, such that for all $x \in D$ the function $f^*(\cdot, x)$ is measurable, and for a.e. $t \in I$ one has*

$$\begin{gathered} \{x \in [0, \lambda] : f(t, x) \neq f^*(t, x)\} \subseteq E_1, \\ \{x \in [0, \lambda] : f^*(t, \cdot) \text{ is discontinuous at } x\} \subseteq E_2; \end{gathered} \tag{2}$$

(ii) *for all $z \in \operatorname{int}(h(A))$ (the interior of $h(A)$), one has $\operatorname{int} h^{-1}(z) = \emptyset$;*

(iii) *if one puts, for all $t \in I$,*

$$v(t) := \operatorname*{ess\,inf}_{x \in [0,\lambda]} f(t, x), \qquad z(t) := \operatorname*{ess\,sup}_{x \in [0,\lambda]} f(t, x), \tag{3}$$

then for a.e. $t \in I$ one has

$$[v(t), z(t)] \subseteq h(A), \qquad \sup h^{-1}([v(t), z(t)]) \leq \beta(t); \tag{4}$$

(iv) *one has*

$$0 < \|\phi_0\|_{L^{s'}(I)} \le \frac{\lambda}{\|\beta\|_{L^s(I)}}; \tag{5}$$

(v) *for all $t \in I$, the function $g(t,\cdot)$ is measurable;*

(vi) *for a.e. $z \in I$, the function $g(\cdot,z)$ is continuous in I, differentiable in $]0,1[$ and*

$$g(t,z) \le \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t,z) \le \phi_1(z) \quad \forall t \in]0,1[. \tag{6}$$

Then, there exists a solution $u \in L^s(I)$ to (1).

Of course, the main peculiarity of Theorem 1 resides in the kind of discontinuity that is allowed for f. Indeed, it is easy to construct examples of functions f, g, and h satisfying the assumptions of Theorem 1 and such that for all $t \in I$ the function $f(t,\cdot)$ is discontinuous at all points $x \in [0,\lambda]$.

Theorem 1 extends a previous result (Theorem 1 of [6]), valid for the case where f does not depend on t explicitly. At this point, it is natural to ask if Theorem 1 above can be extended to the more general case where the function u takes its values in the space $\mathbf{R}^n$. In this direction, we note that some results exist for the vector explicit equation

$$u(t) = f\left(t, \int_I g(t,z)\,u(z)\,dz\right) \tag{7}$$

(see [7, 8]), while for the implicit equation (1) the problem is still unsolved.

The aim of this note is exactly to provide such an extension. In the following, if $n \in \mathbf{N}$ and $i \in \{1,\dots,n\}$, we will denote by $P_i : \mathbf{R}^n \to \mathbf{R}$ the projection over the ith axis. Moreover, we will denote by m_n the n-dimensional Lebesgue measure over $\mathbf{R}^n$. If $\lambda := (\lambda_1,\dots,\lambda_n) \in \mathbf{R}^n$, with $\lambda_i > 0$ for all $i = 1,\dots,n$, we will put $J_\lambda := \prod_{i=1}^n [0,\lambda_i]$. Finally, if n and λ are as above, we will denote by $\mathcal{F}_{n,\lambda}$ the family of all subsets $F \subseteq J_\lambda$ such that there exist sets $F_1, F_2, \dots, F_n \subseteq \mathbf{R}^n$, with $m_1(P_i(F_i)) = 0$ for all $i = 1\dots,n$, such that $F := \bigcup_{i=1}^n F_i$. The following is our main result (where $\mathbf{R}^+_n$ denotes the positive open orthant of $\mathbf{R}^n$, and $\operatorname{int}_A(B)$ is the interior of B in A).

Theorem 2. *Let $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbf{R}^n_+$, and let $X \subseteq \mathbf{R}^n_+$ be a nonempty, closed, connected, and locally connected subset of $\mathbf{R}^n$, with $\inf P_i(X) > 0$ for all $i = 1,\dots,n$. Let $h : X \to \mathbf{R}$ be a continuous function and $f : I \times J_\lambda \to \mathbf{R}$ and $g : I \times I \to [0,+\infty[$ two given functions. Let $s \in]1,+\infty]$, $\phi_0 \in L^j(I)$, with $j > 1$ and $j \ge s'$, $\phi_1 \in L^{s'}(I)$, and $\beta \in L^s(I,\mathbf{R}^n)$. Finally, let D be a countable dense subset of J_λ.*

Assume that there exists a function $f^* : I \times J_\lambda \to \mathbf{R}$ and two sets $E, F \in \mathcal{F}_{n,\lambda}$, with F closed, such that

(i) *for all $x \in D$ the function $f^*(\cdot,x)$ is measurable;*

(ii) *for a.e. $t \in I$ one has*

$$\begin{gathered}\{x \in J_\lambda : f(t,x) \ne f^*(t,x)\} \subseteq E,\\ \{x \in J_\lambda : f^*(t,\cdot) \text{ is discontinuous at } x\} \subseteq F.\end{gathered} \tag{8}$$

Moreover, assume that

(iii) $\operatorname{int}_X(h^{-1}(t)) = \emptyset$, *for all $t \in \operatorname{int}_{\mathbf{R}}(h(X))$;*

(iv) *if one puts, for all $t \in I$;*

$$v(t) := \operatorname*{ess\,inf}_{x\in J_\lambda} f(t,x), \qquad z(t) := \operatorname*{ess\,sup}_{x\in J_\lambda} f(t,x), \tag{9}$$

then for a.e. $t \in I$ and all $i = 1,\dots,n$ one has

$$\begin{gathered}[v(t), z(t)] \subseteq h(X),\\ \sup P_i\left(h^{-1}([v(t), z(t)])\right) \le \beta_i(t),\end{gathered} \tag{10}$$

(where $\beta_i : I \to \mathbf{R}$ denotes the ith component of the function β);

(v) *one has*

$$0 < \|\phi_0\|_{L^{s'}(I)} \le \min_{1\le i\le n} \frac{\lambda_i}{\|\beta_i\|_{L^s(I)}}; \tag{11}$$

(vi) *for all $t \in I$, the function $g(t,\cdot)$ is measurable;*

(vii) *for a.e. $z \in I$, the function $g(\cdot,z)$ is continuous in I, differentiable in $]0,1[$ and*

$$g(t,z) \le \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t,z) \le \phi_1(z) \quad \forall t \in]0,1[. \tag{12}$$

Then, there exists $u \in L^s(I,\mathbf{R}^n)$ such that

$$h(u(t)) = f\left(t, \int_I g(t,z)\,u(z)\,dz\right) \quad \textit{for a.e. } t \in I. \tag{13}$$

Theorem 2 will be proved as an application of the following selection theorem, recently proved in [9], which we now state for the reader's convenience (in the following, if S is a topological space, we will denote by $\mathcal{B}(S)$ the Borel family of S).

Theorem 3 (Theorem 2.2 of [9]). *Let T and $X_1, X_2, \dots X_k$ be complete separable metric spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^k X_j$ (endowed with the product topology). Let $\mu, \psi_1, \dots, \psi_k$ be positive regular Borel measures over $T, X_1, X_2, \dots X_k$, respectively, with μ finite and $\psi_1,\dots,\psi_k$ σ-finite.*

Let S be a separable metric space, and let $F : T \times X \to 2^S$ be a multifunction with nonempty complete values. Finally, let $E \subseteq X$ be a given set, and, for each $i \in \{1,\dots,k\}$, let $P_{*,i} : X \to X_i$ be the projection over X_i. Assume that

(i) *the multifunction F is $\mathcal{T}_\mu \otimes \mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_k)$-measurable (where $\mathcal{T}_\mu$ denotes the completion of the Borel σ-algebra $\mathcal{B}(T)$ of T with respect to the measure μ);*

(ii) *for a.e. $t \in T$, one has*

$$\begin{gathered}\{x := (x_1,\dots,x_k) \in X : F(t,\cdot)\\ \text{is not lower semicontinuous at } x\} \subseteq E.\end{gathered} \tag{14}$$

Then, there exist sets $Q_1,\dots,Q_k$, with $Q_i \in \mathscr{B}(X_i)$ and $\psi_i(Q_i)=0$ for all $i=1,\dots,k$, and a function $\phi: T\times X \to S$ such that

(a) $\phi(t,x)\in F(t,x)$ *for all* $(t,x)\in T\times X$;

(b) *for all* $x:=(x_1,x_2,\dots,x_k)\in X\setminus[(\bigcup_{i=1}^{k}P_{*,i}^{-1}(Q_i))\cup E]$, *the function* $\phi(\cdot,x)$ *is* $\mathscr{T}_\mu$*-measurable over* T;

(c) *for a.e.* $t\in T$, *one has*

$$\{x:=(x_1,x_2,\dots,x_k)\in X:\phi(t,\cdot)\ \text{is discontinuous at } x\}\subseteq E\cup\left[\bigcup_{i=1}^{k}P_{*,i}^{-1}(Q_i)\right]. \tag{15}$$

The proof of Theorem 2 will be given in Section 2. Further, we will point out some counterexamples to possible improvements of Theorem 2.

2. Proof of Theorem 2

Before giving the proof of Theorem 2, we fix some notations. If $n\in\mathbf{N}$, the space $\mathbf{R}^n$ will be considered with its Euclidean norm $\|\cdot\|_n$. Moreover, if $x\in\mathbf{R}^n$ and $r>0$, we put

$$\begin{aligned}B(x,r)&:=\{v\in\mathbf{R}^n:\|v-x\|_n<r\},\\ \overline{B}(x,r)&:=\{v\in\mathbf{R}^n:\|v-x\|_n\le r\}.\end{aligned}\tag{16}$$

If $p\in[1,+\infty]$, the space $L^p(I,\mathbf{R}^n)$ will be considered with the usual norm

$$\begin{aligned}\|u\|_{L^p(I,\mathbf{R}^n)}&:=\left(\int_I\|u(t)\|_n^p\,dt\right)^{1/p}\quad \text{if } p<+\infty,\\ \|u\|_{L^\infty(I,\mathbf{R}^n)}&:=\operatorname*{ess\,sup}_{t\in I}\|u(t)\|_n\quad \text{if } p=+\infty.\end{aligned}\tag{17}$$

As usual, we put $L^p(I):=L^p(I,\mathbf{R})$. For the basic definitions and facts about multifunctions, we refer to [10].

Proof of Theorem 2. Without loss of generality, we can assume that (8) and (10) hold for all $t\in I$. Moreover, we can suppose that $j<+\infty$. Firstly, we prove that the functions v and z are measurable. Observe that, by assumption (ii), for all $t\in I$, one has

$$v(t)=\inf_{x\in J_\lambda\setminus F}f^*(t,x),\qquad z(t)=\sup_{x\in J_\lambda\setminus F}f^*(t,x).\tag{18}$$

To see this, fix $t\in I$, and let $\psi(t):=\sup_{x\in J_\lambda\setminus F}f^*(t,x)$. Since $m_n(E\cup F)=0$ we get

$$z(t)\le\sup_{x\in J_\lambda\setminus(E\cup F)}f(t,x)=\sup_{x\in J_\lambda\setminus(E\cup F)}f^*(t,x)\le\psi(t).\tag{19}$$

Now, assume that $z(t)<\psi(t)$. Hence, there is $x^*\in J_\lambda\setminus F$ such that $f^*(t,x^*)>z(t)$. Since the function $f^*(t,\cdot)$ is continuous at x^*, there exist $\delta,\varepsilon>0$ such that

$$f^*(t,x)>z(t)+\varepsilon\quad\forall x\in J_\lambda\cap B(x^*,\delta).\tag{20}$$

Since $m_n(J_\lambda\cap B(x^*,\delta))>0$, we get

$$z(t):=\operatorname*{ess\,sup}_{x\in J_\lambda}f(t,x)=\operatorname*{ess\,sup}_{x\in J_\lambda}f^*(t,x)\ge z(t)+\varepsilon,\tag{21}$$

which is absurd. Therefore, the second equality in (18) is proved. The first one can be checked in analogous way.

Since F is closed, it can be easily checked that the set $D\cap(J_\lambda\setminus F)$ is nonempty, countable, and dense in $(J_\lambda\setminus F)$. Consequently, by Lemma at page 198 of [11], the function $f^*|_{I\times(J_\lambda\setminus F)}$ is $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda\setminus F)$-measurable (where $\mathscr{L}(I)$ denotes the family of all Lebesgue-measurable subsets of I). By (18) and Lemma III.39 of [12], it follows that the functions v and z are measurable over I, as claimed.

By assumption (iii) and Theorem 2.4 di [13], there exists a set $Y\subseteq X$ such that $h(Y)=h(X)$ and the function $h|_Y$ is open (it carries open subsets of Y onto open subsets of $h(X)=h(Y)$). Consequently, the multifunction $T:h(X)\to 2^Y$ defined by putting, for each $s\in h(X)$,

$$T(s):=h^{-1}(s)\cap Y,\tag{22}$$

is lower semicontinuous in $h(X)$ with nonempty values. Let $f_0:I\times J_\lambda\to\mathbf{R}$ be defined by putting, for all $(t,x)\in I\times J_\lambda$,

$$f_0(t,x)=\begin{cases}f^*(t,x) & \text{if } x\notin F\\ z(t) & \text{if } x\in F.\end{cases}\tag{23}$$

Clearly, the function f_0 is $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda)$-measurable and, by (18), one has

$$v(t)\le f_0(t,x)\le z(t)\quad\forall(t,x)\in I\times J_\lambda.\tag{24}$$

Moreover, assumption (ii) and the closedness of F imply that for all $t\in I$ one has

$$\{x\in J_\lambda: f_0(t,\cdot)\ \text{is discontinuous at } x\}\subseteq F.\tag{25}$$

Let $G:I\times J_\lambda\to 2^Y$ be the multifunction defined by setting, for each $(t,x)\in I\times J_\lambda$,

$$G(t,x):=T(f_0(t,x))=h^{-1}(f_0(t,x))\cap Y.\tag{26}$$

Observe that G is well-defined since for all $(t,x)\in I\times J_\lambda$ one has

$$f_0(t,x)\in[v(t),z(t)]\subseteq h(X).\tag{27}$$

Moreover, by the lower semicontinuity of T and by (25), for all $t\in I$, we get

$$\{x\in J_\lambda: G(t,\cdot)\ \text{is not lower semicontinuous at } x\}\subseteq F.\tag{28}$$

Let $\Psi:I\times J_\lambda\to 2^{\mathbf{R}^n}$ (more precisely, $\Psi:I\times J_\lambda\to 2^{\overline{Y}}$) be the multifunction defined by putting, for each $(t,x)\in I\times J_\lambda$, $\Psi(t,x):=\overline{G(t,x)}$. By (28), for all $t\in[0,1]$, we get

$$\{x\in J_\lambda:\Psi(t,\cdot)\ \text{is not lower semicontinuous at } x\}\subseteq F.\tag{29}$$

Moreover, the values of Ψ are closed (in $\mathbf{R}^n$) subsets of X.

Since f_0 is $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda)$-measurable and T is lower semicontinuous, by Proposition 13.2.1 of [10] the multifunction G is $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda)$-weakly measurable. That is, for each set $\Omega\subseteq Y$, with Ω open in the relative topology of Y, the set

$$G^{-}(\Omega) := \{(t,x)\in I\times J_\lambda : G(t,x)\cap\Omega\neq\emptyset\} \tag{30}$$

belongs to $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda)$. By Proposition 2.6 and Theorem 3.5 of [14], the multifunction Ψ is $\mathscr{L}(I)\otimes\mathscr{B}(J_\lambda)$-measurable.

Since $F\in\mathscr{F}_{n,\lambda}$, there exist sets $F_1,F_2,\ldots,F_n\subseteq\mathbf{R}^n$ such that $F=\bigcup_{i=1}^n F_i$ and $m_1(P_i(F_i))=0$ for all $i=1\ldots,n$. By Theorem 3, there exist sets $Q_1,Q_2,\ldots,Q_n\subseteq\mathbf{R}$, with $Q_i\in\mathscr{B}([0,\lambda_i])$ and $m_1(Q_i)=0$ for all $i=1,\ldots,n$, and a function $\psi:[0,1]\times J_\lambda\to\mathbf{R}^n$ such that

(a) for all $(t,x)\in I\times J_\lambda$ one has $\psi(t,x)\in\Psi(t,x)$;

(b) for all $x\in J_\lambda\setminus[\bigcup_{i=1}^n(P_i^{-1}(Q_i)\cup F_i)]$, the function $\psi(\cdot,x)$ is $\mathscr{L}(I)$-measurable;

(c) for a.e. $t\in I$, one has

$$\begin{aligned}&\{x\in J_\lambda : \psi(t,\cdot)\ \text{is discontinuous at } x\}\\ &\quad\subseteq\left[\bigcup_{i=1}^n\left(P_i^{-1}(Q_i)\cup F_i\right)\right]\cap J_\lambda.\end{aligned} \tag{31}$$

Since X is closed and h is continuous, for all $(t,x)\in I\times J_\lambda$, the set $h^{-1}(f_0(t,x))$ is closed in $\mathbf{R}^n$. Consequently, for all $(t,x)\in I\times J_\lambda$, we get

$$\begin{aligned}\psi(t,x)\in\Psi(t,x)&=\overline{h^{-1}(f_0(t,x))\cap Y}\subseteq\overline{h^{-1}(f_0(t,x))}\\ &=h^{-1}(f_0(t,x)).\end{aligned} \tag{32}$$

Now, let

$$\alpha := \min_{1\le i\le n}\inf P_i(X)>0. \tag{33}$$

By (27), (32), and assumption (iv) we get

$$\psi(t,x)\in\prod_{i=1}^n[\alpha,\beta_i(t)]\quad\forall(t,x)\in I\times J_\lambda. \tag{34}$$

Let $\psi_1: I\times\mathbf{R}^n\to\mathbf{R}^n$ be defined by putting

$$\psi_1(t,x)=\begin{cases}\psi(t,x) & \text{if } x\in J_\lambda\\ \beta(t) & \text{if } x\in\mathbf{R}^n\setminus J_\lambda.\end{cases} \tag{35}$$

By (34) and (35) we easily get

$$\psi_1(t,x)\in\prod_{i=1}^n[\alpha,\beta_i(t)]\quad\forall(t,x)\in I\times\mathbf{R}^n. \tag{36}$$

Let

$$\Lambda := \left[\bigcup_{i=1}^n\left(P_i^{-1}(Q_i)\cup F_i\right)\right]\cap J_\lambda, \tag{37}$$

and let D_0 be any countable dense subset of $J_\lambda\setminus\Lambda$. Since $m_n(\Lambda)=0$, it is easily seen that D_0 is dense in J_λ. Let D_1 be any countable dense subset of $\mathbf{R}^n\setminus J_\lambda$. Then, the set $D_2 := D_0\cup D_1$ is countable and dense in $\mathbf{R}^n$, and for all $x\in D_2$ the function $\psi_1(\cdot,x)$ is measurable by the above construction. Moreover, taking into account (31), for all $t\in I$, one has

$$\begin{aligned}&\{x\in\mathbf{R}^n : \psi_1(t,\cdot)\ \text{is discontinuous at } x\}\\ &\quad\subseteq\left[\bigcup_{i=1}^n\left(P_i^{-1}(Q_i)\cup P_i^{-1}(\{0,\lambda_i\})\cup F_i\right)\right]\cap J_\lambda.\end{aligned} \tag{38}$$

Let $H:[0,1]\times\mathbf{R}^n\to 2^{\mathbf{R}^n}$ be defined by putting, for all $(t,x)\in[0,1]\times\mathbf{R}^n$,

$$H(t,x)=\bigcap_{m\in\mathbf{N}}\overline{\text{conv}}\left(\overline{\bigcup_{\substack{y\in D_2\\ \|x-y\|_n\le 1/m}}\{\psi_1(t,y)\}}\right), \tag{39}$$

where "$\overline{\text{conv}}$" stands for "closed convex hull." By Proposition 2 of [8], taking into account (36) and (38), we have that

(a) H has nonempty closed convex values;

(b) for all $x\in\mathbf{R}^n$, the multifunction $H(\cdot,x)$ is measurable;

(c) for all $t\in I$, the multifunction $H(t,\cdot)$ has closed graph;

(d) for all $t\in I$, one has

$$\begin{aligned}H(t,x)&=\{\psi_1(t,x)\}\\ &\forall x\in\mathbf{R}^n\setminus\left(\left[\bigcup_{i=1}^n\left(P_i^{-1}(Q_i)\cup P_i^{-1}(\{0,\lambda_i\})\right.\right.\right.\\ &\qquad\left.\left.\left.\cup F_i\right)\right]\cap J_\lambda\right).\end{aligned} \tag{40}$$

Moreover, by (36), we have

$$H(t,x)\subseteq\prod_{i=1}^n[\alpha,\beta_i(t)]\quad\forall(t,x)\in I\times\mathbf{R}^n. \tag{41}$$

Now we want to apply Theorem 1 of [15], choosing $T=[0,1]$, $X=Y=\mathbf{R}^n$, $p=s$, $q=j'$, $V=L^s(I,\mathbf{R}^n)$, $\Psi(u)=u$, $r=\|\beta\|_{L^s(I,\mathbf{R}^n)}$, $\varphi\equiv+\infty$, $F=H$, and

$$\Phi(u)(t)=\int_I g(t,z)\,u(z)\,dz. \tag{42}$$

To this aim, we can argue as in [8]. In particular, observe the following.

(a) $\Phi(L^s(I,\mathbf{R}^n))\subseteq C^0(I,\mathbf{R}^n)$. This follows easily from our assumptions (vi) and (vii) and the classical Lebesgue's dominated convergence theorem.

(b) If $v\in L^s(I,\mathbf{R}^n)$ and $\{v^k\}$ is a sequence in $L^s(I,\mathbf{R}^n)$, weakly convergent to v in $L^{j'}(I,\mathbf{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I,\mathbf{R}^n)$. This follows by Theorem 2 at page 359 of [16], since g is jth power summable in $I\times I$ (note that g is measurable on $I\times I$ by the classical Scorza-Dragoni theorem; see [17] or also [11]).

(c) By (41), the function

$$h_0 : t \in I \longrightarrow \sup_{x\in\mathbf{R}^n} \inf_{y\in H(t,x)} \|y\|_n \tag{43}$$

belongs to $L^s(I)$ and $\| h_0\|_{L^s(I)} \le \|\beta\|_{L^s(I,\mathbf{R}^n)}$.

Therefore, taking into account the above construction, all the assumptions of Theorem 1 of [15] are satisfied. Consequently, there exists a function $u^* \in L^s(I,\mathbf{R}^n)$ and a set $K_1 \subseteq I$, with $m_1(K) = 0$, such that

$$u^*(t) \in H(t, \Phi(u^*)(t)) \subseteq \prod_{i=1}^{n} [\alpha, \beta_i(t)] \quad \forall t \in I \setminus K_1. \tag{44}$$

That is,

$$u^*(t) \in H\left(t, \int_I g(t,z)\, u^*(z)\, dz\right) \subseteq \prod_{i=1}^{n} [\alpha, \beta_i(t)] \quad \forall t \in I \setminus K_1. \tag{45}$$

We now prove that the function u^* satisfies our conclusion. To this aim, observe that, since $E \in \mathcal{F}_{n,\lambda}$, there exist sets $E_1, E_2, \ldots, E_n \subseteq \mathbf{R}^n$, with $m_1(P_i(E_i)) = 0$ for all $i = 1,\ldots,n$, such that $E := \bigcup_{i=1}^n E_i$.

Fix $i \in \{1,\ldots,n\}$. Let $\gamma_i : I \to \mathbf{R}$ be the function

$$\gamma_i(t) := P_i(\Phi(u^*)(t)) = \int_I g(t,z)\, u_i^*(z)\, dz. \tag{46}$$

By (45) we get

$$u_i^*(t) \in [\alpha, \beta_i(t)] \quad \forall t \in I \setminus K_1; \tag{47}$$

hence for all $t \in I$ we get the inequality

$$\begin{aligned} 0 \le \gamma_i(t) &\le \|\phi_0\|_{L^{s'}(I)} \|u^*\|_{L^s(I)} \\ &\le \frac{\lambda_i}{\|\beta_i\|_{L^s(I)}} \cdot \|\beta_i\|_s = \lambda_i, \end{aligned} \tag{48}$$

hence $\gamma_i(I) \subseteq [0, \lambda_i]$. By (vi), (vii), and (45) we have that γ_i is strictly increasing. Moreover, by Lemma 2.2. at page 226 of [18] we get

$$\gamma_i'(t) = \int_I \frac{\partial g}{\partial t}(t,z)\, u_i^*(z)\, dz > 0 \quad \forall t \in \,]0,1[\,. \tag{49}$$

Consequently, by Theorem 2 of [19], the function γ_i^{-1} is absolutely continuous. By Theorem 18.25 of [20], the set

$$W_i := \gamma_i^{-1}\left[(P_i(E_i \cup F_i) \cup Q_i \cup \{0, \lambda_i\}) \cap \gamma_i(I)\right] \tag{50}$$

has null Lebesgue measure. Now, put

$$\Omega := \left(\bigcup_{i=1}^{n} W_i\right) \cup K_1. \tag{51}$$

Of course, $m_1(\Omega) = 0$. Choose any point $t^* \in I \setminus \Omega$. Since $t^* \notin K_1$, by (45) we get

$$u^*(t^*) \in H(t^*, \Phi(u^*)(t^*)). \tag{52}$$

For each $i \in \{1,\ldots,n\}$, since $t^* \notin W_i$, taking into account (48), we have

$$\gamma_i(t^*) \in [0, \lambda_i] \setminus [P_i(E_i \cup F_i) \cup \{0, \lambda_i\} \cup Q_i]. \tag{53}$$

Therefore, $\Phi(u^*)(t^*) \in J_\lambda$ and for all $i \in \{1,\ldots,n\}$ we have

$$\Phi(u^*)(t^*) \notin \left[(E_i \cup F_i) \cup P_i^{-1}(\{0, \lambda_i\}) \cup P_i^{-1}(Q_i)\right]. \tag{54}$$

Consequently, we get

$$\begin{aligned} &\Phi(u^*)(t^*) \\ &\quad \in J_\lambda \setminus \bigcup_{i=1}^{n} \left[(E_i \cup F_i) \cup P_i^{-1}(\{0, \lambda_i\}) \cup P_i^{-1}(Q_i)\right]. \end{aligned} \tag{55}$$

By (40) we get

$$\begin{aligned} u^*(t^*) \in H(t^*, \Phi(u^*)(t^*)) &= \{\psi_1(t^*, \Phi(u^*)(t^*))\} \\ &= \{\psi(t^*, \Phi(u^*)(t^*))\}. \end{aligned} \tag{56}$$

By (32) and (56) we get

$$u^*(t^*) \in h^{-1}(f_0(t^*, \Phi(u^*)(t^*))), \tag{57}$$

hence

$$h(u^*(t^*)) = f_0(t^*, \Phi(u^*)(t^*)). \tag{58}$$

Since $\Phi(u^*)(t^*) \notin E \cup F$, we get

$$\begin{aligned} h(u^*(t^*)) &= f^*(t^*, \Phi(u^*)(t^*)) \\ &= f(t^*, \Phi(u^*)(t^*)) \\ &= f\left(t^*, \int_i g(t^*,z)\, u^*(z)\, dz\right). \end{aligned} \tag{59}$$

This completes the proof. □

Remark 4. Of course, a function f satisfying the assumptions of Theorem 2 can be discontinuous at each point $x \in J_\lambda$. The Example at the end of [8] shows that, in the statement of Theorem 2, none of the sets E and F can be assumed to depend on t. Moreover, the Example at the end of [6] shows that the second inequality in assumption (vii) cannot be weakened by assuming that

$$0 \le \frac{\partial g}{\partial t}(t,z) \le \phi_1(z). \tag{60}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] G. Anello and P. Cubiotti, "Non-autonomous implicit integral equations with discontinuous right-hand side," *Commentationes Mathematicae Universitatis Carolinae*, vol. 45, no. 3, pp. 417–429, 2004.

[2] J. Banaś and Z. Knap, "Integrable solutions of a functional-integral equation," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 2, no. 1, pp. 31–38, 1989.

[3] G. Emmanuele, "About the existence of integrable solutions of a functional-integral equation," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 4, no. 1, pp. 65–69, 1991.

[4] G. Emmanuele, "Integrable solutions of a functional-integral equation," *Journal of Integral Equations and Applications*, vol. 4, no. 1, pp. 89–94, 1992.

[5] M. Fečkan, "Nonnegative solutions of nonlinear integral equations," *Commentationes Mathematicae Universitatis Carolinae*, vol. 36, no. 4, pp. 615–627, 1995.

[6] F. Cammaroto and P. Cubiotti, "Implicit integral equations with discontinuous right-hand side," *Commentationes Mathematicae Universitatis Carolinae*, vol. 38, no. 2, pp. 241–246, 1997.

[7] F. Cammaroto and P. Cubiotti, "Vector integral equations with discontinuous right-hand side," *Commentationes Mathematicae Universitatis Carolinae*, vol. 40, no. 3, pp. 483–490, 1999.

[8] P. Cubiotti, "Non-autonomous vector integral equations with discontinuous right-hand side," *Commentationes Mathematicae Universitatis Carolinae*, vol. 42, no. 2, pp. 319–329, 2001.

[9] P. Cubiotti and J. C. Yao, "Two-point problem for vector differential inclusions with discontinuous right-hand side," *Applicable Analysis*, 2013.

[10] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley & Sons, New York, NY, USA, 1984.

[11] A. Kucia, "Scorza Dragoni type theorems," *Polska Akademia Nauk. Fundamenta Mathematicae*, vol. 138, no. 3, pp. 197–203, 1991.

[12] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, Germany, 1977.

[13] B. Ricceri, "Sur la semi-continuité inférieure de certaines multifonctions," *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, vol. 294, no. 7, pp. 265–267, 1982.

[14] C. J. Himmelberg, "Measurable relations," *Polska Akademia Nauk. Fundamenta Mathematicae*, vol. 87, pp. 53–72, 1975.

[15] O. N. Ricceri and B. Ricceri, "An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and application to a multivalued boundary value problem," *Applicable Analysis*, vol. 38, no. 4, pp. 259–270, 1990.

[16] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, UK, 1964.

[17] G. Scorza Dragoni, "Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 17, pp. 102–106, 1948.

[18] S. Lang, *Real and Functional Analysis*, Springer, New York, NY, USA, 1993.

[19] A. Villani, "On Lusin's condition for the inverse function," *Rendiconti del Circolo Matematico di Palermo II*, vol. 33, no. 3, pp. 331–335, 1984.

[20] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, Berlin, Germany, 1975.