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Research Article

Multiple Solutions to Elliptic Equations on $\mathbb{R}^N$ with Combined Nonlinearities

Miaomiao Yang[1] and Anran Li[2]

[1] *School of Science, Qilu University of Technology, Jinan, Shandong 250353, China*
[2] *School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China*

Correspondence should be addressed to Anran Li; anran0200@163.com

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In this paper, we are concerned with the multiplicity of nontrivial radial solutions for the following elliptic equations $(P)_\lambda$: $-\Delta u + V(x)u = \lambda Q(x)|u|^{q-2}u + Q(x)f(u), x \in \mathbb{R}^N$; $u(x) \to 0$, as $|x| \to +\infty$, where $1 < q < 2, 0 < \lambda \in \mathbb{R}, N \geq 3, V$, and Q are radial positive functions, which can be vanishing or coercive at infinity, and f is asymptotically linear at infinity.

1. Introduction and Main Results

In this paper, we deal with the multiplicity of nontrivial radial solutions for the following elliptic equations:

$$-\Delta u + V(x)u = \lambda Q(x)|u|^{q-2}u + Q(x)f(u) \quad x \in \mathbb{R}^N,$$
$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty, \tag{P$_\lambda$}$$

where $1 < q < 2$, $0 < \lambda \in \mathbb{R}$, $N \geq 3$, V, and Q are radial positive functions, which can be vanishing or coercive at infinity.

When Ω is a smooth bounded domain in $\mathbb{R}^N$, the problem

$$-\Delta u = \pm\lambda|u|^{q-2}u + f(x,u) \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \Omega, \tag{P$'_\pm$)$_\lambda$}$$

where $1 < q < 2$, $0 < \lambda \in \mathbb{R}$, and $N \geq 3$, has been widely studied in the literature and plays a central role in modern mathematical sciences, in the theory of heat conduction in electrically conduction materials and in the study of non-Newtonian fluids. However, it is not possible to give here a complete bibliography. Here we just list some representative results. In the case where f is superlinear near infinity, problem $(P'_+)_\lambda$ is the famous concave-convex problem; after the celebrated work [1, 2], this kind of problem has drawn much attention. In the case where f is linear in u, the authors in [3] have proved that there exist at least two nonnegative solutions for a more general question:

$$-\Delta u = h(x)u^q + f(x,u),$$
$$0 \leq u \in H_0^1(\Omega), \quad 0 < q < 1, \tag{1}$$

where $h(x) \in L^\infty(\Omega)$ satisfies some additional conditions. For problem $(P'_-)_\lambda$, in the special case $f(u) = au + |u|^p$, where $2 < p < 2^*$, one nonnegative solution for any $a \in \mathbb{R}$ and $\lambda > 0$ was found in [4] via Mountain Pass Theorem. In the last years, several papers have also been devoted to the study of nonlinearities with indefinite sign, for example, [5, 6] and the references therein.

When $\Omega = \mathbb{R}^N$, there are a large number of papers devoted to the following equation:

$$-\Delta u + V(x)u = f(x,u) \quad \text{with } u \in W^{1,2}(\mathbb{R}^N). \tag{R}$$

So far, in almost all the results concerning (R), the nonlinear function f is assumed to be globally superlinear, that is, $\lim_{|u|\to 0}(f(x,u)/u) = 0$ and there exists $\theta > 2$ such that $0 < \theta F(x,u) \leq uf(x,u)$ for all $(x,u) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$, where $F(x,u) = \int_0^u f(x,t)dt$. The case in which $V(x) \to +\infty$, $|x| \to \infty$, and f is globally superlinear was first studied by Rabinowitz in [7]. The assumptions in [7] ensure that the associated functional of the equation satisfies the Palais-Smale condition; this fact was observed in [8, 9] where the results in [7] were generalized. For a radially symmetric Schrödinger equation with an asymptotically linear term, one radial solution has been obtained in [10, 11] by Stuart and Zhou and their results were generalized to more general situations in [12–15].

Since the class Sobolev embedding is $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $p \in (2, 2N/(N-2))$, we cannot study the sublinear problems in $W^{1,2}(\mathbb{R}^N)$ via variation method. In order to overcome this obstacle, a regular way is to add some restrictions on potentials V and Q. For example, in [16], the authors obtained the existence of infinitely many nodal solutions for problem (R), where $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \geq 1, \int_{\mathbb{R}^N}(1/V(x))dx < +\infty$, and the nonlinearity f is symmetric in the sense of being odd in u and may involve a combination of concave and convex terms. There are also some other results about concave and convex problem on $\mathbb{R}^N$, such as [17–19] and the references therein. However, as we have known, there are few results about problem $(\mathrm{P})_\lambda$ with both sublinear terms and asymptotically linear terms.

Recently, in [20], the authors established a weighted Sobolev type embedding of radially symmetric functions which provides a basic tool to study quasilinear elliptic equations with sublinear nonlinearities. Motivated by the works of [20], we consider $(\mathrm{P})_\lambda$ with more general potentials and combined nonlinearities. In our paper, we assume the following.

(V) $V(x) \in C(\mathbb{R}^N, (0, +\infty))$ is radially symmetric and there exists $a_1 \in \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} \frac{V(x)}{|x|^{a_1}} > 0. \tag{2}$$

(Q) $Q(x) \in C(\mathbb{R}^N, (0, +\infty))$ is radially symmetric and there exists $a_2 \in \mathbb{R}$ such that

$$\limsup_{|x|\to\infty} \frac{Q(x)}{|x|^{a_2}} < \infty. \tag{3}$$

It is clear that the indexes a_1 and a_2 describe the behavior of V and Q near infinity. On a_1, a_2, we assume the following:

(A_1) $a_2 \geq ((2(N-1)+a_1)/2)-N$, $((N-2)/2)-N \leq a_2 \leq -2$;

(A_2) $a_2 < ((2(N-1)+a_1)/4)-N$, $((N-2)/2)-N \leq a_2 \leq -2$;

(A_3) $a_1 \leq -2$, $((N-2)/2)-N < a_2 < ((2(N-1)+a_1)/2)-N$;

(A_4) $a_2 \leq ((N-2)/2)-N$, $((2(N-1)+a_1)/4)-N \leq a_2 < ((2(N-1)+a_1)/2)-N$;

(A_5) $a_1 \geq -2$, $((2(N-1)+a_1)/4)-N \leq a_2 < ((2(N-1)+a_1)/2)-N$.

According to the indexes a_1, a_2, we define the bottom index 2_*:

$$2_* = \begin{cases} \dfrac{2(a_2+N)}{N-2}, & \text{if } (a_1,a_2) \in A_i,\ i=1,2,3; \\ \dfrac{4(a_2+N)}{2(N-1)+a_1}, & \text{if } (a_1,a_2) \in A_i,\ i=4,5. \end{cases} \tag{4}$$

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \left\{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial}\right\}. \tag{5}$$

Denote by $D_r^{1,2}(\mathbb{R}^N)$ the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}. \tag{6}$$

Define

$$W_r^{1,2}(\mathbb{R}^N; V) := \left\{u \in D_r^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2 dx < \infty\right\}, \tag{7}$$

which is a Hilbert space [21, 22] equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\ dx\right)^{1/2}. \tag{8}$$

Let

$$\begin{aligned} &L^p(\mathbb{R}^N; Q) \\ &\quad := \Big\{u : \mathbb{R}^N \longmapsto \mathbb{R} \mid u \text{ be Lebesgue measurabe}, \\ &\qquad \int_{\mathbb{R}^N} Q(x)|u|^p dx < \infty\Big\}, \end{aligned} \tag{9}$$

which is a Banach space equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^N;Q)} = \left(\int_{\mathbb{R}^N} Q(x)|u|^p dx\right)^{1/p}. \tag{10}$$

Following Theorem 1.2 in [20], under the assumptions (V), (Q), and (A_i), $i = 1, \ldots, 5$, it holds that the embedding $W_r^{1,2}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q)$ is compact for $p \in (2_*, 2N/(N-2))$. We remark that the index $2_* < 2$ by (A_i), $i = 1, \ldots, 5$, so it is possible to study $(\mathrm{P})_\lambda$ with sublinear nonlinearities. We make the following assumptions on f:

(f_1) $f(u) \in C(\mathbb{R}, \mathbb{R})$;

(f_2) $\lim_{|u|\to\infty}(2F(u)/|u|^2) = b$.

Since under the assumptions (V), (Q), and (A_i), $i = 1, \ldots, 5$, it holds that the embedding $W_r^{1,2}(\mathbb{R}^N; V) \hookrightarrow L^2(\mathbb{R}^N; Q)$ is compact, the eigenvalue problem

$$\begin{gathered} -\Delta u + V(x)u = \mu Q(x)u \quad x \in \mathbb{R}^N, \\ u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow +\infty \end{gathered} \tag{P$_\mu$}$$

has the eigenvalue sequence

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \longrightarrow +\infty. \tag{11}$$

Similar to the eigenvalue problem on bounded domain, $\mu_1 > 0$ is simple and isolated and has an associated eigenfunction ϕ_1 which is positive in $\mathbb{R}^N$.

Our main results are the following.

Theorem 1. *Under the assumptions (V), (Q), and (A_i), $i = 1,\ldots,5$, if f satisfies (f_1), (f_2) with $\mu_1 < b < +\infty$, moreover*

(f_3) $F(u) = \int_0^u f(s)ds \geq 0$, $u \in \mathbb{R}$;

(f_4) *there exist $C' \in (0,\mu_1)$ and $r_0 > 0$ small, such that $|f(u)| \leq C'|u|$, $|u| \leq r_0$,*

then there exists $\lambda_1 > 0$ such that, for any $\lambda \in (0,\lambda_1)$, $(P)_\lambda$ has at least four nontrivial solutions.

Theorem 2. *Under the assumptions (V), (Q), and (A_i), $i = 1,\ldots,5$, if f satisfies (f_1), (f_2) with $\mu_{k+1} < b < +\infty$ for some $k \in N$, moreover,*

(f_5) *b is not an eigenvalue, $F(u) \geq (\mu_m/2)u^2$, $u \in \mathbb{R}$, $\limsup_{u\to 0}(2F(u)/u^2) < \mu_{m+1}$, for some $m \in N, m \leq k$,*

then there exists $\lambda_2 > 0$ such that, for $0 < \lambda < \lambda_2$, $(P)_\lambda$ has at least one nontrivial solution.

Remark 3. In Theorem 1, f may be assumed as superlinear near zero; we can get four nontrivial solutions by Mountain Pass Theorem and Ekeland's variational principle and truncation technique. In Theorem 2, under the assumptions on f near zero, the functional associated to problem $(\mathrm{P})_\lambda$ enjoys linking structure, and $(\mathrm{P})_\lambda$ has a linking solution.

Remark 4. In Theorem 1, b may be an eigenvalue of problem $(\mathrm{P})_\mu$; then problem $(\mathrm{P})_\lambda$ may be resonant near infinity.

Remark 5. As we have known, there are few results about problem on $\mathbb{R}^n$ with both sublinear and asymptotically linear nonlinearities at the same time.

The paper is organized as follows. In Section 2, we give some preliminary results. The proof of our main results will be given in Section 3.

2. Preliminary

In this section we give some preliminaries that will be used to prove the main results of the paper. We begin with a special case of results on Sobolev type embedding which is due to [20].

Lemma 6 (see [20]). *Let (V), (Q), and (A_i), $i = 1,\ldots,5$, be satisfied; the space $W_r^{1,2}(\mathbb{R}^N;V)$ is compactly embedded in $L^p(\mathbb{R}^N;Q)$, for any p such that $2_* < p < 2N/(N-2)$.*

For $W_r^{1,2}(\mathbb{R}^N;V)$, we denote

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u|^q dx - \int_{\mathbb{R}^N} Q(x)F(u)dx,$$
$$I_\lambda^\pm(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)\left|u^\pm\right|^q dx - \int_{\mathbb{R}^N} Q(x)F(u^\pm)dx, \tag{12}$$

where $u^+ = \max\{u,0\}$, $u^- = \min\{u,0\}$; then, under conditions (f_1) and (f_2), I_λ and $I_\lambda^\pm \in C^1(W_r^{1,2}(\mathbb{R}^N;V),\mathbb{R})$.

Recall that a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence for the functional I, if

$$I(u_n) \longrightarrow c, \quad I'(u_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{13}$$

A sequence $\{u_n\}$ is a $(\mathrm{C})_c$ sequence for the functional I, if

$$I(u_n) \longrightarrow c, \quad (1+\|u_n\|)I'(u_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{14}$$

Definition 7. Assume X is a Banach space, $I \in C^1(X,\mathbb{R})$; one says that I satisfies the $(\mathrm{PS})_c$ condition, if every $(\mathrm{PS})_c$ sequence $\{u_n\}$ has a convergent subsequence. I satisfies (PS) condition if I satisfies $(\mathrm{PS})_c$ at any $c \in \mathbb{R}$.

Definition 8. Assume X is a Banach space, $I \in C^1(X,\mathbb{R})$; one says that I satisfies the $(\mathrm{C})_c$ condition, if every $(\mathrm{C})_c$ sequence $\{u_n\}$ has a convergent subsequence. I satisfies (C) condition if I satisfies $(\mathrm{C})_c$ at any $c \in \mathbb{R}$.

Lemma 9 (Ekeland's variational principle, [23]). *Let V be a complete metric space and let $I : V \to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous, bounded from below. For any $\varepsilon > 0$, there is some point $v \in V$ with*

$$I(v) \leq \inf_V I + \varepsilon, \qquad I(w) \geq I(v) - \varepsilon d(v,w) \quad \forall w \in V. \tag{15}$$

Lemma 10 (Mountain Pass Theorem, Ambrosetti-Rabinowitz, 1973, [24]). *Let X be a Banach space, $I \in C^1(X,\mathbb{R})$. Let $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} I(u) > I(0) \geq I(e). \tag{16}$$

If I satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$
$$\Gamma := \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}, \tag{17}$$

then c is a critical value of I.

Lemma 11 (Linking Theorem, Rabinowitz, 1978, [24]). *Let $X = Y \bigoplus Z$ be a Banach space with* $\dim Y < \infty$. *Let $R > r > 0$ and $z \in Z$ be such that $\|z\| = r$. Define*

$M := \{u = y + tz \mid \|u\| \leq R, t \geq 0, y \in Y\}$,

$M_0 := \{u = y + tz \mid y \in Y, \|u\| = R$ *and* $t \geq 0$ *or* $\|u\| \leq R$ *and* $t = 0\}$,

$N := \{u \in Z \mid \|u\| = r\}$.

Let $I \in C^1(X,\mathbb{R})$ be such that

$$d := \inf_N I > a := \max_{M_0} I. \tag{18}$$

If I satisfies the $(PS)_c$ condition with

$$c =: \inf_{\gamma\in\Gamma}\max_{u\in M} I(\gamma(u)),$$
$$\Gamma := \{\gamma \in C(M,X) : \gamma|_{M_0} = id\}, \tag{19}$$

then c is a critical value of I.

It is well known that the above two minimax theorems are still valid under $(\mathrm{C})_c$ condition. In our paper, we denote $X := W_r^{1,2}(\mathbb{R}^N; V)$; C is denoted to be various positive constants.

Lemma 12. *Under the assumptions (V), (Q), and (A_i), $i = 1, \ldots, 5$, if f satisfies (f_1), (f_4), and (f_2) with $\mu_1 < b < +\infty$, then there exists $\lambda^* > 0$ such that, for $0 < \lambda < \lambda^*$, one has the following.*

(i) *There exist $\rho_\lambda^\pm, \beta_\lambda^\pm > 0$, such that*

$$I_\lambda^\pm(u) \geq \beta_\lambda^\pm > 0 \quad \forall u \in X \text{ with } \|u\| = \rho_\lambda^\pm. \tag{20}$$

(ii) *There exists $e_\lambda^\pm \in X$ with $\|e_\lambda^\pm\| > \rho_\lambda^\pm$ such that $I_\lambda^\pm(e_\lambda^\pm) < 0$.*

Proof. We only prove the above results for I_λ^+.

(i) By (f_1), (f_2), and (f_4), there exists $C > 0$ and $p \in (2, 2N/(N-2))$, such that $|F(u^+)| \leq (C'/2)|u^+|^2 + (C/2)|u^+|^p$. Then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^+|^q dx - \int_{\mathbb{R}^N} Q(x) F(u^+)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C'}{2}\int_{\mathbb{R}^N} Q(x)|u|^2 dx - \frac{C}{2}\int_{\mathbb{R}^N} Q(x)|u|^p dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u|^q dx \\ &\geq \left(\frac{\mu_1 - C'}{2\mu_1} - \frac{C_1}{2}\|u\|^{p-2} - \frac{C_2\lambda}{q}\|u\|^{q-2}\right)\|u\|^2. \end{aligned} \tag{21}$$

Set

$$g(t) = \frac{C_1}{2}t^{p-2} + \frac{C_2\lambda}{q}t^{q-2} \quad \text{for } t > 0, \tag{22}$$

where $q \in (2_*, 2)$ and $p \in (2, 2N/(N-2))$. By $g'(t_0) = 0$, we have

$$t_0 = \left(\frac{2C_1(2-q)}{qC_2(p-2)}\lambda\right)^{1/(p-q)}. \tag{23}$$

Then there exists $C_0 > 0$ such that $g(t_0) = C_0\lambda^{(p-2)/(p-q)}$. Thus, there exists $\lambda_* > 0$ such that, for $\lambda \in (0, \lambda_*)$, $(\mu_1 - C')/2\mu_1 > C_0\lambda^{(p-2)/(p-q)}$. Furthermore, set $\rho_\lambda^+ = t_0$; we have

$$\begin{aligned} I_\lambda^+(u) &\geq \left(\frac{\mu_1 - C'}{2\mu_1} - C_0\lambda^{(p-2)/(p-q)}\right)\left(\frac{2C_1(2-q)}{qC_2(p-2)}\lambda\right)^{2/(p-q)} \\ &= \beta_\lambda^+ > 0 \quad \forall u \in X \text{ with } \|u\| = \rho_\lambda^+. \end{aligned} \tag{24}$$

(ii) Let $\phi_1 > 0$ be a μ_1-eigenfunction; for $t > 0$ we have

$$\begin{aligned} I_\lambda^+(t\phi_1) &= \frac{t^2}{2}\|\phi_1\|^2 - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} Q(x)|\phi_1|^q dx \\ &\quad - \int_{\mathbb{R}^N} Q(x) F(t\phi_1)\,dx \\ &= \frac{t^2\|\phi_1\|^2}{2} \\ &\quad \times \left(1 - \frac{2\lambda t^{q-2}}{q\|\phi_1\|^2}\int_{\mathbb{R}^N} Q(x)|\phi_1|^q dx \right. \\ &\quad \left. - \frac{2}{t^2\|\phi_1\|^2}\int_{\mathbb{R}^N} Q(x) F(t\phi_1)\,dx\right). \end{aligned} \tag{25}$$

By (f_2), $\mu_1 < b < +\infty$ and $q < 2$; then there exists $T_{0,\lambda} > 0$ large enough such that

$$\begin{aligned} 1 &- \frac{2\lambda T_{0,\lambda}^{q-2}}{q\|\phi_1\|^2}\int_{\mathbb{R}^N} Q(x)|\phi_1|^q dx \\ &- \frac{2}{T_{0,\lambda}^2\|\phi_1\|^2}\int_{\mathbb{R}^N} Q(x) F(T_{0,\lambda}\phi_1)\,dx < 0. \end{aligned} \tag{26}$$

So, we can choose $e_\lambda = T_{0,\lambda}\phi_1$; then (ii) is proved. □

Lemma 13. *Under the assumptions (V), (Q), and (A_i), $i = 1, \ldots, 5$, if f satisfies (f_1)–(f_3) with $\mu_{k+1} < b < +\infty$ and (f_5), then*

(i) *for any given λ and $u \in X_m := \bigoplus_{j=1}^m \ker(-\Delta + V - \mu_j Q)$, we have*

$$I_\lambda(u) \leq 0; \tag{27}$$

(ii) *there exists λ^{**} satisfying the fact that for $\lambda \in (0, \lambda^{**})$ there exist two positive constants $d(\lambda)$ and $r(\lambda)$ such that for all $u \in N := \{u \in X_m^\perp, \|u\| = r(\lambda)\}$, one has*

$$I_\lambda(u) \geq d(\lambda) > 0; \tag{28}$$

(iii) *there exists $R > 0$ such that, for any given λ and $u \in X_{m+1}$, and $\|u\| \geq R$, we have $I_\lambda(u) \leq 0$.*

Proof. (i) Let $u \in X_m$; by (f_5), $F(u) \geq (1/2)\mu_m u^2$, $u \in \mathbb{R}$, then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q|u|^q dx - \int_{\mathbb{R}^N} QF(u)\,dx \\ &\leq \frac{1}{2}\|u\|^2 - \frac{\mu_m}{2}\int_{\mathbb{R}^N} Q|u|^2 dx \leq 0. \end{aligned} \tag{29}$$

(ii) Let $u \in X_m^\perp$; by (f_1) and (f_2) with $\mu_{k+1} < b < +\infty$, and $\limsup_{u\to 0}(F(u)/u^2) < (1/2)\mu_{m+1}$, we have that there exist $\varepsilon_0 > 0$, $C > 0$, $p > 2$, such that $F(u) \leq (1/2)(\mu_{m+1} - \varepsilon_0)u^2 + C|u|^p$. Then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q|u|^q dx - \int_{\mathbb{R}^N} QF(u)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\mu_{m+1} - \varepsilon_0)\int_{\mathbb{R}^N} Q|u|^2 dx \\ &\quad - C\int_{\mathbb{R}^N} Q|u|^q dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q|u|^q dx \\ &\geq \frac{1}{2}\left(1 - \frac{\mu_{m+1} - \varepsilon_0}{\mu_{m+1}}\right)\|u\|^2 - C\|u\|^p - C\lambda\|u\|^q. \end{aligned} \tag{30}$$

The rest of the proof is similar to the proof of (i) of Lemma 12.

(iii) For any $u \in X_{m+1}$, set $f(u) = bu + g(u)$; by (f_2), we have $G(u)/u^2 \to 0$, as $|u| \to \infty$, where $G(u) = \int_0^u g(s)ds$. Then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q|u|^q dx - \frac{b}{2}\int_{\mathbb{R}^N} Q|u|^2 dx \\ &\quad - \int_{\mathbb{R}^N} QG(u)\,dx. \end{aligned} \tag{31}$$

Since $b > \mu_{m+1}$, for every $z \in \operatorname{span}\{\phi_{m+1}\}$, $t \in \mathbb{R}$, $w \in X_m$,

$$t^2\|z\|^2 + \|w\|^2 - b\int_{\mathbb{R}^N} Q(tz + w)^2 dx < 0, \qquad \text{for } tz + w \neq 0. \tag{32}$$

Arguing by contradiction, we find a sequence $\{u_n\}$, satisfying $\|u_n\| \to \infty$, $u_n = t_n z_0 + w_n$, where $z_0 \in \operatorname{span}\{\phi_{m+1}\}$, $t_n \in \mathbb{R}$, $w_n \in X_m$, such that

$$\begin{aligned} I_\lambda(u_n) &= \frac{1}{2}t_n^2\|z_0\|^2 + \frac{1}{2}\|w_n\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} Q|u_n|^q dx \\ &\quad - \int_{\mathbb{R}^N} QF(u_n)\,dx \geq 0. \end{aligned} \tag{33}$$

Dividing $\|u_n\|^2$ in both sides of the above equality, there holds

$$\begin{aligned} \frac{I_\lambda(u_n)}{\|u_n\|^2} &= \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q|u_n|^q dx \\ &\quad - \int_{\mathbb{R}^N} Q\frac{F(u_n)}{\|u_n\|^2}dx \geq 0, \end{aligned} \tag{34}$$

where $\tau_n := t_n/\|u_n\|$, $v_n := w_n/\|u_n\|$. Since $\tau_n^2\|z_0\|^2 + \|v_n\|^2 = 1$, after passing to a subsequence $\tau_n \to \tau$, in $\mathbb{R}$, $v_n \to v$ in X_m. Let $u' = \tau z_0 + v$; by (32), there exists a bounded domain $\Omega \subset \mathbb{R}^N$, such that

$$\tau^2\|z_0\|^2 + \|v\|^2 - b\int_\Omega Q(\tau z_0 + v)^2 dx < 0. \tag{35}$$

As $F(u) = (1/2)bu^2 + G(u)$, it follows from (34) that

$$\begin{aligned} 0 &\leq \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \int_\Omega Q\frac{F(u_n)}{\|u_n\|^2}dx \\ &\quad - \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q|u_n|^q dx \\ &= \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \frac{1}{2}b\int_\Omega Q(\tau_n z_0 + v_n)^2 dx \\ &\quad - \int_\Omega Q\frac{G(u_n)}{\|u_n\|^2}dx - \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q|u_n|^q dx \\ &\leq \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \frac{1}{2}b\int_\Omega Q(\tau_n z_0 + v_n)^2 dx \\ &\quad - \int_\Omega Q\frac{G(u_n)}{\|u_n\|^2}dx. \end{aligned} \tag{36}$$

Clearly, $|G(u)| \leq c_0 u^2$, for some $c_0 > 0$ and $G(u)/u^2 \to 0$, as $|u| \to \infty$. Since $\tau_n \to \tau$, in $\mathbb{R}$, $v_n \to v$ in X_m, then $\tau_n z_0 + v_n \to u' = \tau z_0 + v$, in $L^2(\mathbb{R}^N; Q)$. It is easy to see from the Lebesgue dominated converge theorem that

$$\int_\Omega Q\frac{G(u_n)}{\|u_n\|^2}dx = \int_\Omega Q\frac{G(u_n)}{u_n^2}\left(\tau_n^2 + v_n^2\right)dx \longrightarrow 0. \tag{37}$$

Hence $0 \leq (1/2)\tau^2\|z_0\|^2 + (1/2)\|v\|^2 - (1/2)b\int_\Omega Q|tz_0 + v|^2 dx < 0$; this is impossible. □

3. Proof of Main Results

Proof of Theorem 1. Firstly, we will prove that, for any fixed λ, the functionals $I_\lambda^\pm$ have a local minimizer, respectively; then problem $(\mathrm{P})_\lambda$ has two nontrivial solutions: one is nonnegative; the other one is nonpositive.

Similar to [25], for $\rho_\lambda^+ > 0$ given by Lemma 12(i), define

$$\overline{B}_{\rho_\lambda^+} = \{u \in X \mid \|u\| \leq \rho_\lambda^+\}, \qquad \partial B_{\rho_\lambda^+} = \{u \in X \mid \|u\| = \rho_\lambda^+\} \tag{38}$$

and $\overline{B}_{\rho_\lambda^+}$ is a complete metric space with the distance

$$\operatorname{dist}(u, v) = \|u - v\| \quad \text{for } u, v \in \overline{B}_{\rho_\lambda^+}. \tag{39}$$

By Lemma 12, we have that

$$I_\lambda^+(u) \geq \beta_\lambda^+ > 0, \quad u \in \partial B_{\rho_\lambda^+}. \tag{40}$$

Clearly, $I_\lambda^+ \in C^1(\overline{B}_{\rho_\lambda^+}, \mathbb{R})$; hence I_λ^+ is lower semicontinuous and bounded from below on $\overline{B}_{\rho_\lambda^+}$. Let

$$c_\lambda^1 = \inf_{u \in \overline{B}_{\rho_\lambda^+}} I_\lambda^+(u). \tag{41}$$

By the definition of I_λ^+, we can easily claim that $c_\lambda^1 < 0$. Indeed, since $q < 2$ if $t > 0$ is small enough,

$$\begin{aligned} I_\lambda^+(t\phi_1) &= \frac{t^2}{2}\|\phi_1\|^2 - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} Q(x)|\phi_1|^q dx \\ &\quad - \int_{\mathbb{R}^N} Q(x) F(t\phi_1)\, dx \\ &\le t^2\|\phi_1\|^2 - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} Q(x)|\phi_1|^q dx \quad (\text{by } (f_3)) \\ &< 0. \end{aligned} \tag{42}$$

By Lemma 9, for any $n > 0$, there exists a u_n such that

$$\begin{gathered} c_\lambda^1 \le I_\lambda^+(u_n) \le c_\lambda^1 + \frac{1}{n}, \\ I_\lambda^+(w) \ge I_\lambda^+(u_n) - \frac{1}{n}\|u_n - w\|, \quad \text{for } w \in \overline{B}_{\rho_\lambda^+}. \end{gathered} \tag{43}$$

Then, $\|u_n\| < \rho_\lambda^+$ for $n \ge 1$ large enough. Otherwise, if $\|u_n\| = \rho_\lambda^+$ for infinitely many n, without loss of generality, we may assume that $\|u_n\| = \rho_\lambda^+$ for all $n \in \mathbb{N}$, and it follows from (40) that

$$I_\lambda^+(u_n) \ge \beta_\lambda^+ > 0. \tag{44}$$

Let $n \to \infty$ and combine (43); we can get that $0 < \beta_\lambda^+ \le c_\lambda^1 < 0$. This is a contradiction.

We prove now that $I_\lambda^{+\prime}(u_n) \to 0$, as $n \to \infty$. In fact, for any $u \in X$ with $\|u\| = 1$, let $w_n = u_n + tu$ and, for any fixed $n \ge 1$, we have $\|w_n\| \le \|u_n\| + t < \rho_\lambda^+$ if $t > 0$ is small enough. So it follows from (43) that

$$I_\lambda^+(u_n + tu) \ge I_\lambda^+(u_n) - \frac{t}{n}\|u\|. \tag{45}$$

That is,

$$\frac{I_\lambda^+(u_n + tu) - I_\lambda^+(u_n)}{t} \ge -\frac{1}{n}\|u\| = -\frac{1}{n}. \tag{46}$$

Let $t \to 0$; we see that $\langle I_\lambda^{+\prime}(u_n), u\rangle \ge -1/n$, and this gives

$$\left|\left\langle I_\lambda^{+\prime}(u_n), u\right\rangle\right| < \frac{1}{n} \quad \text{for any } u \in X \text{ with } \|u\| = 1. \tag{47}$$

So, $I_\lambda^{+\prime}(u_n) \to 0$, as $n \to \infty$, and, by (43), $I_\lambda^+(u_n) \to c_\lambda^1 < 0$, as $n \to \infty$. Then, for any given λ, $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_\lambda^1}$ sequence of I_λ^+. By the compactness of Sobolev embedding Lemma 6 and a standard procedure, we see that there exists $u_\lambda^1 \in X$ such that $I_\lambda^{+\prime}(u_\lambda^1) = 0$. Since

$$I_\lambda^{+\prime}\left(u_\lambda^1\right) u_\lambda^{1-} = 0, \tag{48}$$

this implies that $u_\lambda^1 \ge 0$. That is, u_λ^1 is a nontrivial solution of problem $(\mathrm{P})_\lambda$. For the case I_λ^-, by the same argument, we can get that problem $(\mathrm{P})_\lambda$ has another nontrivial solution, which is nonpositive.

Secondly, we will prove that there exists $\lambda_1 > 0$ such that, for $\lambda \in (0, \lambda_1)$, problem $(\mathrm{P})_\lambda$ enjoys two mountain pass solutions.

Define

$$c_\lambda^{2\pm} = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I_\lambda^\pm(\gamma(t)), \tag{49}$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e_\lambda^\pm\}$.

By Lemmas 10 and 12, we only need to prove that $I_\lambda^\pm$ satisfies $(\mathrm{C})_{c_\lambda^{2\pm}}$ condition.

Lemma 14. *Under the assumptions (V), (Q), and (A_i), $i = 1, \dots, 5$, if f satisfies (f_1), (f_2) with $\mu_1 < b$, (f_3), and (f_4) then, for any fixed $\lambda > 0$, the functional $I_\lambda^\pm$ satisfies the $(\mathrm{C})_{c_\lambda^{2\pm}}$ condition.*

Proof. Here, we only prove the case for I_λ^+.

For every $(\mathrm{C})_{c_\lambda^{2+}}$ sequence $\{u_n\}$,

$$I_\lambda^+(u_n) \longrightarrow c_\lambda^{2+}, \quad \text{as } n \longrightarrow +\infty, \tag{50}$$

$$(1 + \|u_n\|) I_\lambda^{+\prime}(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{51}$$

We claim that the sequence $\{u_n\}$ is bounded in X. Seeking a contradiction, we suppose that $\|u_n\| \to \infty$. Let $z_n = u_n/\|u_n\|$; up to a subsequence, we get that

$$\begin{aligned} &z_n \rightharpoonup z \text{ in } X, \\ &z_n \to z \text{ in } L^s(\mathbb{R}^N; Q),\ 2_* < s < 2N/(N-2), \\ &z_n(x) \to z(x) \text{ a.e. } x \in \mathbb{R}^N. \end{aligned}$$

We claim that $z \ne 0$. Otherwise, $z = 0$, since by (51)

$$\begin{aligned} o(1) &= \left\langle I_\lambda^{+\prime}(u_n), u_n \right\rangle \\ &= \|u_n\|^2 - \lambda \int_{\mathbb{R}^N} Q u_n^+(x)^q dx - \int_{\mathbb{R}^N} Q f(u_n^+(x))\, u_n^+(x)\, dx. \end{aligned} \tag{52}$$

Dividing $\|u_n\|^2$ in both sides of (52), we get that

$$o(1) = 1 - \int_{\mathbb{R}^N} Q \frac{f(u_n^+) u_n^+}{\|u_n\|^2} dx. \tag{53}$$

(f_1), (f_2), and (f_4) with $\mu_1 < b < +\infty$ imply that there exists $C > 0$, such that

$$\left|f(u_n^+)\right| \le C u_n^+. \tag{54}$$

Combining (53) and (54), we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} Q \frac{f(u_n^+) u_n^+}{\|u_n\|^2} dx + o(1) \\ &\le C \int_{\mathbb{R}^N} Q(x) \left|z_n^+\right|^2 dx + o(1). \end{aligned} \tag{55}$$

Letting $n \to \infty$, we get a contradiction. Thus, $z \ne 0$ in X.

Set

$$P_n(x) = \begin{cases} \dfrac{f(u_n(x))}{u_n(x)}, & \text{for } x \in \mathbb{R}^N,\ u_n(x) > 0, \\ 0, & \text{for } x \in \mathbb{R}^N,\ u_n(x) \le 0. \end{cases} \tag{56}$$

From $I_\lambda^{+\prime}(u_n) = o(1)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u_n \nabla \phi\, dx + \int_{\mathbb{R}^N} V(x) u_n \phi\, dx - \lambda \int_{\mathbb{R}^N} Q(x) (u_n^+)^{q-1} \phi\, dx \\ &\quad - \int_{\mathbb{R}^N} Q(x) f(u_n^+) \phi\, dx = o(1), \end{aligned} \tag{57}$$

for all $\phi \in C_{0,r}^\infty(\mathbb{R}^N)$. Dividing $\|u_n\|$ in both sides of the above equality, there holds

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla z_n \nabla \phi\, dx + \int_{\mathbb{R}^N} V(x) z_n \phi\, dx - \int_{\mathbb{R}^N} Q(x) P_n(x) z_n^+ \phi\, dx \\ &\quad = o(1). \end{aligned} \tag{58}$$

By (54), $|P_n(x)| \le C$ for $x \in \mathbb{R}^N$. Then we have

$$\begin{aligned} &\left| \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\} \bigcap \operatorname{supp} \phi} Q P_n z_n^+ \phi\, dx \right| \\ &\quad \le C \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\} \bigcap \operatorname{supp} \phi} Q z_n^+ |\phi|\, dx \\ &\quad = o(1) + C \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\} \bigcap \operatorname{supp} \phi} Q z^+ |\phi|\, dx = o(1). \end{aligned} \tag{59}$$

On the other hand, since $z_n^+(x) \to z^+(x)$ for a.e. $x \in \mathbb{R}^N$, we have $\lim_{n \to \infty} u_n^+(x) = +\infty$ for a.e. $x \in \{x \in \mathbb{R}^N \mid z^+(x) > 0\}$, which implies that $\lim_{n \to \infty} P_n(x) = b$, for a.e. $x \in \{x \in \mathbb{R}^N \mid z^+(x) > 0\}$. Besides $|P_n(x)| \le C$, for a.e. $x \in \mathbb{R}^N$. Using the Lebesgue's Dominated Convergence theorem, we obtain that

$$\begin{aligned} &\left| \int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q (P_n(x) - b) z_n^+ \phi\, dx \right| \\ &\quad \le \int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q |P_n(x) - b| z_n^+ |\phi|\, dx \\ &\quad \le \left(\int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q |P_n(x) - b|^2 |\phi|\, dx \right)^{1/2} \\ &\qquad \times \left(\int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q (z_n^+)^2 |\phi|\, dx \right)^{1/2} \\ &\quad \le C \left(\int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q |P_n(x) - b|^2 |\phi|\, dx \right)^{1/2} \\ &\quad = o(1). \end{aligned} \tag{60}$$

By (59) and (60),

$$\begin{aligned} &\int_{\mathbb{R}^N} Q P_n(x) z_n^+ \phi\, dx \\ &\quad = \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\} \bigcap \operatorname{supp} \phi} Q P_n(x) z_n^+ \phi\, dx \\ &\qquad + \int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q P_n(x) z_n^+ \phi\, dx \\ &\quad = o(1) + \int_{\{x \in \mathbb{R}^N | z^+(x) > 0\} \bigcap \operatorname{supp} \phi} Q P_n(x) z_n^+ \phi\, dx \\ &\quad = o(1) + b \int_{\mathbb{R}^N} Q z^+ \phi\, dx. \end{aligned} \tag{61}$$

Combining (58) and (61), letting $n \to \infty$, there holds

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi + V z \phi)\, dx = b \int_{\mathbb{R}^N} Q z^+ \phi\, dx. \tag{62}$$

We claim that $\operatorname{meas}\{x \in \mathbb{R}^N, z^+(x) \ne 0\} > 0$. Otherwise $z^+ = 0$; taking $\phi = z$ in (62), we have $z = 0$, which is impossible. Taking $\phi = z^-$ in (62), then we can get $z \ge 0$. Moreover, by the Hopf' Lemma, we also can get $z > 0$ in $\mathbb{R}^N$. Taking $\phi = \phi_1$ in (62), we obtain

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi_1 + V z \phi_1)\, dx = b \int_{\mathbb{R}^N} Q z^+ \phi_1 dx. \tag{63}$$

Since $\phi_1 > 0$ is the eigenfunction associated to μ_1, and $z \ge 0$, we have

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi_1 + V z \phi_1)\, dx = \mu_1 \int_{\mathbb{R}^N} Q z \phi_1 dx. \tag{64}$$

This is impossible, since $b > \mu_1$. Then $\{u_n\}$ is bounded in X. Since the embedding from X into $L^s(\mathbb{R}^N; Q), s \in (2_*, 2N/(N-2))$ is compact, there exists u_λ^2, such that $u_n \to u_\lambda^2$ strongly in X, and $I_\lambda^+(u_\lambda^2) = c_\lambda^{2+} \ge \beta_1^+ > 0$, $I_\lambda^{+\prime}(u_\lambda^2) = 0$.

Finally, since $I_\lambda^{+\prime}(u_\lambda^2) u_\lambda^{2-} = 0$, then

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\nabla u_\lambda^2 \nabla u_\lambda^{2-} + V(x) u_\lambda^2 u_\lambda^{2-} \right) dx \\ &\quad = \lambda \int_{\mathbb{R}^N} Q(x) \left(u_\lambda^{2+}\right)^{q-1} u_\lambda^{2-} + \int_{\mathbb{R}^N} Q(x) f\left(u_\lambda^{2+}\right) u_\lambda^{2-} dx \\ &\quad = 0. \end{aligned} \tag{65}$$

We have $u_\lambda^{2-} = 0$, i.e., $u_\lambda^2 \ge 0$. Thus, u is a nonnegative solution for problem $(\mathrm{P})_\lambda$. Similarly, for

$$I_\lambda^-(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\mathbb{R}^N} Q |u^-|^q dx - \int_{\mathbb{R}^N} Q F(u^-)\, dx, \tag{66}$$

we can also get a nonpositive solution for problem $(\mathrm{P})_\lambda$.

Thus, problem $(\mathrm{P})_\lambda$ has at least four nontrivial solutions. The proof of Theorem 1 is complete. □

Proof of Theorem 2. In order to prove Theorem 2, we firstly verify that the functional I_λ enjoys the linking structure. This can be easily got form Lemma 13. In fact, for $Y = X_m, Z = X_m^{\perp}$, $z \in \text{span}\{\phi_{m+1}\}$ with $\|z\| = r(\lambda)$,

$M := \{u = y + tz \mid \|u\| \leq R, t \geq 0, y \in Y\}$,

$M_0 := \{u = y + tz \mid y \in Y, \|u\| = R$ and $t \geq 0$ or $\|u\| \leq R$ and $t = 0\}$,

$N := \{u \in Z \mid \|u\| = r(\lambda)\}$.

Lemma 13 implies that there exists $\lambda^{**} > 0$, such that, for $0 < \lambda < \lambda^{**}$,

$$\inf_{u\in N} I_\lambda(u) > \sup_{u\in M_0} I_\lambda(u). \tag{67}$$

Define

$$\begin{aligned} c_\lambda &=: \inf_{\gamma\in\Gamma}\max_{u\in M} I_\lambda(\gamma(u)), \\ \Gamma &:= \left\{\gamma \in C(M, X) : \gamma|_{M_0} = id\right\}. \end{aligned} \tag{68}$$

Next, we prove that the functional I_λ satisfies the $(\mathrm{C})_{c_\lambda}$ condition.

Lemma 15. *Under the assumptions (V), (Q), and (A_i), $i = 1, \ldots, 5$, if f satisfies the assumptions of Theorem 2, then, for any given $\lambda > 0$, the functional I_λ satisfies the $(\mathrm{C})_{c_\lambda}$ condition.*

Proof. For every $(\mathrm{C})_{c_\lambda}$ sequence $\{u_n\}$,

$$\begin{aligned} I_\lambda(u_n) &\longrightarrow c_\lambda, \quad \text{as } n \longrightarrow +\infty, \\ (1 + \|u_n\|) I'_\lambda(u_n) &\longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \end{aligned} \tag{69}$$

Here we just prove that $\{u_n\}$ is bounded. Seeking a contradiction we suppose that $\|u_n\| \to \infty$. Letting $w_n = u_n/\|u_n\|$, up to a subsequence, we get that

$w_n \rightharpoonup w$ in X,

$w_n \to w$ in $L^s(\mathbb{R}^N; Q)$, $2_* < s < 2N/(N-2)$,

$w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$.

Now, we consider the two possible cases.

Case 1 ($w = 0$ in X). From $o(1) = \langle I'_\lambda(u_n), u_n\rangle$, we have

$$o(1) = \|u_n\|^2 - \lambda \int_{\mathbb{R}^N} Q|u_n|^q dx - \int_{\mathbb{R}^N} Qf(u_n)u_n dx. \tag{70}$$

Dividing $\|u_n\|^2$ in both sides of the above equality, we get that

$$o(1) = 1 - \int_{\mathbb{R}^N} Q\frac{f(u_n)u_n}{\|u_n\|^2}dx. \tag{71}$$

Since (f_1), (f_5), and (f_2) with $\mu_{k+1} < b < +\infty$ imply that

$$|f(u_n)u_n| \leq C|u_n|^2, \quad \text{for some } C > 0, \tag{72}$$

combing (71) and (72), we have

$$1 = \int_{\mathbb{R}^N} Q\frac{f(u_n)u_n}{\|u_n\|^2}dx + o(1) \leq C\int_{\mathbb{R}^N} Q|w_n|^2 dx + o(1). \tag{73}$$

Let $n \to \infty$; we get a contradiction.

Case 2 ($w \neq 0$ in X). From $I_\lambda^{+\prime}(u_n) = o(1)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u_n \nabla\phi\, dx + \int_{\mathbb{R}^N} V(x)u_n\phi\, dx \\ &\quad - \lambda\int_{\mathbb{R}^N} Q(x)|u_n|^{q-2}u_n\phi\, dx - \int_{\mathbb{R}^N} Q(x)f(u_n)\phi\, dx \\ &\quad = o(1), \end{aligned} \tag{74}$$

for all $\phi \in C_{0,r}^\infty(\mathbb{R}^N)$. Dividing $\|u_n\|$ in both sides of the above equality, there holds

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla w_n \nabla\phi\, dx + \int_{\mathbb{R}^N} V(x)w_n\phi\, dx \\ &\quad - \int_{\mathbb{R}^N} Q(x)\frac{f(u_n)}{u_n}w_n\phi\, dx = o(1). \end{aligned} \tag{75}$$

By (72), $|f(u_n)/u_n| \leq C$ for $x \in \mathbb{R}^N$. Then we have

$$\begin{aligned} &\left|\int_{\{x\in\mathbb{R}^N|w(x)=0\}\cap\operatorname{supp}\phi} Q\frac{f(u_n)}{u_n}w_n\phi\, dx\right| \\ &\quad \leq C\int_{\{x\in\mathbb{R}^N|w(x)=0\}\cap\operatorname{supp}\phi} Qw_n|\phi|\, dx \\ &\quad = o(1) + C\int_{\{x\in\mathbb{R}^N|w(x)=0\}\cap\operatorname{supp}\phi} Qw|\phi|\, dx = o(1). \end{aligned} \tag{76}$$

On the other hand, since $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^N$, we have $\lim_{n\to\infty}|w_n(x)| = +\infty$ for a.e. $x \in \{x \in \mathbb{R}^N \mid w(x) \neq 0\}$, which implies that $\lim_{n\to\infty}(f(u_n)/u_n) = b$, for a.e. $x \in \{x \in \mathbb{R}^N \mid w(x) \neq 0\}$. Besides $|f(u_n)/u_n| \leq C$, for a.e. $x \in \mathbb{R}^N$. Using the Lebesgue's Dominated Convergence theorem, we obtain that

$$\begin{aligned} &\left|\int_{\{x\in\mathbb{R}^N|w(x)\neq0\}\cap\operatorname{supp}\phi} Q\left(\frac{f(u_n)}{u_n} - b\right)w_n\phi\, dx\right| \\ &\quad \leq \int_{\{x\in\mathbb{R}^N|w(x)\neq0\}\cap\operatorname{supp}\phi} Q\left|\frac{f(u_n)}{u_n} - b\right|w_n|\phi|\, dx \end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_{\{x\in\mathbb{R}^N | w(x) \neq 0\} \bigcap \operatorname{supp}\phi} Q \left| \frac{f(u_n)}{u_n} - b \right|^2 |\phi|\, dx \right)^{1/2} \\
&\quad \times \left(\int_{\{x\in\mathbb{R}^N | w(x) \neq 0\} \bigcap \operatorname{supp}\phi} Q |w_n|^2 |\phi|\, dx \right)^{1/2} \\
&\leq C \left(\int_{\{x\in\mathbb{R}^N | w(x) \neq 0\} \bigcap \operatorname{supp}\phi} Q \left| \frac{f(u_n)}{u_n} - b \right|^2 |\phi|\, dx \right)^{1/2} \\
&= o(1).
\end{aligned} \tag{77}$$

By (76) and (77),

$$\begin{aligned}
&\int_{\mathbb{R}^N} Q \frac{f(u_n)}{u_n} w_n \phi\, dx \\
&\quad = \int_{\{x\in\mathbb{R}^N | w(x)=0\}} Q \frac{f(u_n)}{u_n} w_n \phi\, dx \\
&\quad\quad + \int_{\{x\in\mathbb{R}^N | w(x) \neq 0\}} Q \frac{f(u_n)}{u_n} w_n \phi\, dx \\
&\quad = o(1) + \int_{\{x\in\mathbb{R}^N | w(x) \neq 0\}} Q \frac{f(u_n)}{u_n} w_n \phi\, dx \\
&\quad = o(1) + b \int_{\mathbb{R}^N} Q w \phi\, dx.
\end{aligned} \tag{78}$$

Combining (75) and (78) and letting $n \to \infty$, there holds

$$\int_{\mathbb{R}^N} (\nabla w \nabla \phi + V w \phi)\, dx = b \int_{\mathbb{R}^N} Q w \phi\, dx. \tag{79}$$

It implies that b is an eigenvalue which contradicts (f_5). Thus, $\{u_n\}$ is bounded in X. Since the embedding from X into $L^s(\mathbb{R}^N; Q)$, $s \in (2_*, 2N/(N-2))$ is compact, there exists u_λ, such that $u_n \to u_\lambda$ strongly in X, and $I_\lambda(u_\lambda) = c_\lambda$, $I'_\lambda(u_\lambda) = 0$. □

Remark 16. The sublinear term $|u|^{q-2}u$ can be relaxed to more general type, and the function Q before the sublinear term and asymptotically linear term can also be different.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The authors declare that the study was realized in collaboration with the same responsibility.

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