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*Research Article*

# Numerical Treatment of the Modified Time Fractional Fokker-Planck Equation

**Yuxin Zhang**

*School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China*

Correspondence should be addressed to Yuxin Zhang; zhangyuxin2006@163.com

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A numerical method for the modified time fractional Fokker-Planck equation is proposed. Stability and convergence of the method are rigorously discussed by means of the Fourier method. We prove that the difference scheme is unconditionally stable, and convergence order is $O(\tau + h^4)$, where $\tau$ and $h$ are the temporal and spatial step sizes, respectively. Finally, numerical results are given to confirm the theoretical analysis.

## 1. Introduction

Fractional differential equations have attracted considerable attention due to their frequent appearance in various applications in fluid mechanics, biology, physics, and engineering [1, 2]. Usually, fractional differential equations do not have analytic solutions and can only be solved by some semianalytical and numeric techniques. Recently, several semianalytical methods, such as variational iteration method, homotopy perturbation method, Adomian decomposition method, homotopy analysis method, and collocation method, have been used to solve fractional differential equations [3–7]. Meanwhile, some effective numerical techniques are developed; see [8–17].

In the present paper, we are motivated to study the following modified time fractional Fokker-Planck equation [18]:

$$\frac{\partial u(x,t)}{\partial t} = \left[\kappa_\alpha \frac{\partial^2}{\partial x^2} - \nu_\alpha \frac{\partial}{\partial x}\right] {}_{\mathrm{RL}}D_{0,t}^{1-\alpha} u(x,t) + f(x,t), \quad 0 < x < L,\ 0 \le t \le T, \tag{1}$$

$$\begin{aligned} u(x,0) &= \phi(x), \quad 0 < x < L, \\ u(0,t) &= \varphi_1(t), \quad 0 \le t \le T, \\ u(L,t) &= \varphi_2(t), \quad 0 \le t \le T, \end{aligned} \tag{2}$$

where $\kappa_\alpha > 0$ is a fractional diffusion coefficient, and $\nu_\alpha > 0$ is a fractional friction coefficient. ${}_{\mathrm{RL}}D_{0,t}^{1-\alpha}u(x,t)$ denotes the temporal Riemann-Liouville derivative operator defined as [2]

$${}_{\mathrm{RL}}D_{0,t}^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,s)}{(t-s)^{1-\alpha}} ds, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function.

The outline of the paper is as follows. In Section 2, an effective numerical method for solving the modified time fractional Fokker-Planck equation is proposed. The solvability, stability, and convergence of the numerical method are discussed in Sections 3 and 4, respectively. In Section 5, we give some numerical results demonstrating the convergence orders of the numerical method. Also a conclusion is given in Section 6.

## 2. The Construction of Numerical Method

Let $x_j = jh\, (j = 0, 1, \ldots, M)$ and $t_k = k\tau\, (k = 0, 1, \ldots, N)$, where $h = L/M$ and $\tau = T/N$ are the uniform spatial and temporal mesh sizes, respectively, and $M$, $N$ are two positive integers.

**Lemma 1.** *Suppose that*

$$\left[\kappa_\alpha \frac{\partial^2}{\partial x^2} - \nu_\alpha \frac{\partial}{\partial x}\right] u\left(x_j, t\right) = g\left(x_j, t\right); \quad (4)$$

*then a fourth-order difference scheme for the above equation is given as follows:*

$$\mathscr{A} u\left(x_j, t\right) = \mathscr{L} g\left(x_j, t\right) + \mathscr{O}\left(h^4\right), \quad (5)$$

*where $\mathscr{A}$ and $\mathscr{L}$ are two difference operators and are defined by*

$$\mathscr{A} = \kappa_\alpha \cosh\left(\frac{\sqrt{6}\nu_\alpha h}{6\kappa_\alpha}\right)\delta_x^2 - \nu_\alpha \mu_x \delta_x,$$
$$\mathscr{L} = I + \frac{h^2}{12}\left(\delta_x^2 - \frac{\nu_\alpha}{\kappa_\alpha}\mu_x\delta_x\right), \quad (6)$$

*in which $I$ is an unit operator and $\mu_x\delta_x$ and $\delta_x^2$ are average central and second central difference operators with respect to $x$ and are defined by*

$$\mu_x\delta_x u\left(x_j, t\right) = \frac{u\left(x_{j+1}, t\right) - u\left(x_{j-1}, t\right)}{2h},$$
$$\delta_x^2 u\left(x_j, t\right) = \frac{u\left(x_{j+1}, t\right) - 2u\left(x_j, t\right) + u\left(x_{j-1}, t\right)}{h^2}. \quad (7)$$

*Proof.* In view of Taylor expansion, we can obtain

$$\begin{aligned} E_j(t) &= \mathscr{A} u\left(x_j, t\right) - \mathscr{L} g\left(x_j, t\right) \\ &= \kappa_\alpha \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{\sqrt{6}\nu_\alpha h}{6\kappa_\alpha}\right)^{2k} \\ &\quad \times \left(\frac{\partial^2 u\left(x_j, t\right)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u\left(x_j, t\right)}{\partial x^4} + O\left(h^4\right)\right) \\ &\quad - \nu_\alpha\left(\frac{\partial u\left(x_j, t\right)}{\partial x} + \frac{h^2}{6}\frac{\partial^3 u\left(x_j, t\right)}{\partial x^3} + O\left(h^4\right)\right) \\ &\quad - \left(g\left(x_j, t\right) + \frac{h^2}{12}\frac{\partial^2 g\left(x_j, t\right)}{\partial x^2} + O\left(h^4\right)\right) \\ &\quad + \frac{\nu_\alpha h^2}{12\kappa_\alpha}\left(\frac{\partial g\left(x_j, t\right)}{\partial x} + O\left(h^2\right)\right) \\ &= \left[\left(\kappa_\alpha \frac{\partial^2 u\left(x_j, t\right)}{\partial x^2} - \nu_\alpha \frac{\partial u\left(x_j, t\right)}{\partial x}\right) - g\left(x_j, t\right)\right] \\ &\quad - \frac{\nu_\alpha h^2}{12\kappa_\alpha}\left[\left(\kappa_\alpha \frac{\partial^3 u\left(x_j, t\right)}{\partial x^3} - \nu_\alpha \frac{\partial^2 u\left(x_j, t\right)}{\partial x^2}\right) - \frac{\partial g\left(x_j, t\right)}{\partial x}\right] \\ &\quad + \frac{h^2}{12}\left[\left(\kappa_\alpha \frac{\partial^4 u\left(x_j, t\right)}{\partial x^4} - \nu_\alpha \frac{\partial^3 u\left(x_j, t\right)}{\partial x^3}\right) - \frac{\partial^2 g\left(x_j, t\right)}{\partial x^2}\right] + O\left(h^4\right). \end{aligned} \quad (8)$$

Noting (4), we easily obtain

$$E_j(t) = O\left(h^4\right). \quad (9)$$

This completes the proof. □

Combing (1), (4), and (5), we obtain

$$\begin{aligned} &\mathscr{L} \cdot \frac{\partial u\left(x_j, t_k\right)}{\partial t} \\ &\quad = \mathscr{A} \cdot {}_{\mathrm{RL}}D_{0,t}^{1-\alpha} u\left(x_j, t_k\right) + \mathscr{L} \cdot f\left(x_j, t_k\right) + \mathscr{O}\left(h^4\right). \end{aligned} \quad (10)$$

Using the relation of the Riemann-Liouville fractional derivative ${}_{\mathrm{RL}}D_{0,t}^{1-\alpha}u(x,t)$ and Grünwald-Letnikov fractional derivative ${}_{\mathrm{GL}}D_{0,t}^{1-\gamma}u(x,t)$, we can approximate the Riemann-Liouville fractional derivative ${}_{\mathrm{RL}}D_{0,t}^{1-\alpha}u(x,t)$ by [2]

$${}_{\mathrm{RL}}D_{0,t}^{1-\alpha} u\left(x_j, t_k\right) = \frac{1}{\tau^{1-\alpha}} \sum_{m=0}^{k} \varpi_m^{(1-\alpha)} u\left(x_j, t_{k-m}\right) + O(\tau), \quad (11)$$

where $\varpi_m^{(1-\alpha)} = (-1)^m \binom{1-\alpha}{m}$.

For first-order derivative $\partial u(x_j, t_k)/\partial t$, we apply the following backward difference scheme:

$$\frac{\partial u\left(x_j, t_k\right)}{\partial t} = \frac{u\left(x_j, t_k\right) - u\left(x_j, t_{k-1}\right)}{\tau} + O(\tau). \quad (12)$$

Let $u_i^k$ be the numerical approximation of $u(x_i, t_k)$; substituting (11) and (12) into (10) and omitting the error term $\mathscr{O}(\tau + h^4)$, we can obtain the following difference scheme for solving (1):

$$\mathscr{L} \cdot \left(u_j^k - u_j^{k-1}\right) = \tau^\alpha \mathscr{A} \cdot \sum_{m=0}^{k} \varpi_m^{(1-\alpha)} u_j^{k-m} + \tau \mathscr{L} \cdot f_j^k. \quad (13)$$

The discrete form of above system is

$$\begin{aligned}
&\left(\frac{1}{12}-c_1-c_2+c_3\right)u_{j+1}^{k}+\left(\frac{5}{6}+2c_2\right)u_j^k \\
&\quad+\left(\frac{1}{12}+c_1-c_2-c_3\right)u_{j-1}^{k} \\
&\quad=\left(\frac{1}{12}-c_1+c_2\varpi_1^{(1-\alpha)}-c_3\varpi_1^{(1-\alpha)}\right)u_{j+1}^{k-1} \\
&\quad+\left(\frac{5}{6}-2c_2\varpi_1^{(1-\alpha)}\right)u_j^{k-1} \\
&\quad+\left(\frac{1}{12}+c_1+c_2\varpi_1^{(1-\alpha)}+c_3\varpi_1^{(1-\alpha)}\right)u_{j-1}^{k-1} \\
&\quad-c_3\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(u_{j+1}^{k-m}-u_{j-1}^{k-m}\right) \\
&\quad+c_2\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(u_{j+1}^{k-m}-2u_j^{k-m}+u_{j-1}^{k-m}\right) \\
&\quad+\tau\left(\frac{1}{12}-c_1\right)f_{j+1}^{k}+\frac{5}{6}\tau f_j^k+\tau\left(\frac{1}{12}+c_1\right)f_{j-1}^{k},
\end{aligned} \tag{14}$$

where $c_1=\nu_\alpha h/24\kappa_\alpha$, $c_2=\tau^\alpha\kappa_\alpha\theta/h^2$, $c_3=\tau^\alpha\nu_\alpha/2h$, and $\theta=\cosh(\sqrt{6}\nu_\alpha h/6\kappa_\alpha)$.

The initial and boundary conditions can be discretized as

$$\begin{gathered}
u_j^0=\phi(jh),\quad j=0,1,\ldots,M, \\
u_0^k=\varphi_1(k\tau),\quad u_M^k=\varphi_2(k\tau),\quad k=0,1,\ldots,N.
\end{gathered} \tag{15}$$

Obviously, the local truncation error of difference scheme (14) is $R_j^k=O(\tau+h^4)$.

## 3. The Solvability of the Difference Scheme

Firstly, let us denote

$$\begin{gathered}
\mathbf{U}^0=\left(\phi(x_1),\phi(x_2),\ldots,\phi(x_{M-1})\right)^T, \\
\mathbf{U}^k=\left(u_1^k,u_2^k,\ldots,u_{M-1}^k\right)^T,\qquad \mathbf{F}^k=\left(f_1^k,f_2^k,\ldots,f_{M-1}^k\right)^T, \\
k=0,1,\ldots,N.
\end{gathered} \tag{16}$$

Then we can give the compact form of difference scheme (14) as follows:

$$\mathbf{A}\mathbf{U}^k=\mathbf{B}\mathbf{U}^{k-1}+\sum_{m=2}^{k}\mathbf{C}_m\mathbf{U}^{k-m}+\mathbf{D}\mathbf{F}^k+\mathbf{G},\quad k=1,\ldots,N, \tag{17}$$

where

$$\begin{gathered}
\mathbf{A}=\begin{pmatrix} s_2 & s_3 & & & \\ s_1 & s_2 & s_3 & & \\ & \ddots & \ddots & \ddots & \\ & & s_1 & s_2 & s_3 \\ & & & s_1 & s_2 \end{pmatrix}, \\
\mathbf{B}=\begin{pmatrix} p_2 & p_3 & & & \\ p_1 & p_2 & p_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_1 & p_2 & p_3 \\ & & & p_1 & p_2 \end{pmatrix}, \\
\mathbf{C}_m=\begin{pmatrix} q_2 & q_3 & & & \\ q_1 & q_2 & q_3 & & \\ & \ddots & \ddots & \ddots & \\ & & q_1 & q_2 & q_3 \\ & & & q_1 & q_2 \end{pmatrix}, \\
\mathbf{D}=\begin{pmatrix} r_2 & r_3 & & & \\ r_1 & r_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & r_1 & r_2 & r_3 \\ & & & r_1 & r_2 \end{pmatrix}, \\
G=\Big(g_1,\underbrace{0,\ldots,0}_{M-3},g_{M-1}\Big)^T, \\
s_1=\frac{1}{12}+c_1-c_2-c_3,\qquad s_2=\frac{5}{6}+2c_2, \\
s_3=\frac{1}{12}-c_1-c_2+c_3, \\
p_1=\frac{1}{12}+c_1+c_2\varpi_1^{(1-\alpha)}+c_3\varpi_1^{(1-\alpha)}, \\
p_2=\frac{5}{6}-2c_2\varpi_1^{(1-\alpha)}, \\
p_3=\frac{1}{12}-c_1+c_2\varpi_1^{(1-\alpha)}-c_3\varpi_1^{(1-\alpha)}, \\
q_1=(c_2+c_3)\,\varpi_m^{(1-\alpha)}, \\
q_2=-2c_2\varpi_m^{(1-\alpha)},\qquad q_3=(c_2-c_3)\,\varpi_m^{(1-\alpha)}, \\
r_1=\tau\left(\frac{1}{12}+c_1\right),\qquad r_2=\frac{5}{6}\tau,\qquad r_3=\tau\left(\frac{1}{12}-c_1\right),
\end{gathered} \tag{18}$$

$$\begin{aligned}
g_1=&-s_1\varphi_1(t_k)+p_1\varphi_1(t_{k-1})+(c_2+c_3) \\
&\times\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\varphi_1(t_{k-m})+r_1f(x_0,t_k), \\
g_{M-1}=&-s_3\varphi_2(t_k)+p_3\varphi_2(t_{k-1})+(c_2+c_3) \\
&\times\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\varphi_2(t_{k-m})+r_3f(x_M,t_k).
\end{aligned} \tag{19}$$

**Theorem 2.** *Difference equation* (17) *is uniquely solvable.*

*Proof.* It is well known that the eigenvalues of the matrix $\mathbf{A}$ are

$$\lambda_k = s_2 + 2\sqrt{s_1 s_3}\cos\left(\frac{k\pi}{M}\right), = \frac{5}{6} + \frac{2\tau^\alpha \kappa_\alpha \theta}{h^2} + 2\sqrt{\left(\frac{1}{12} - \frac{\tau^\alpha \kappa_\alpha \theta}{h^2}\right)^2 - \left(\frac{\nu_\alpha h}{24\kappa_\alpha} - \frac{\tau^\alpha \nu_\alpha}{2h}\right)^2}\cos\left(\frac{k\pi}{M}\right), \quad (20)$$

where $k = 1, 2, \ldots, M-1$.

Note that $\theta \geq 1$ and $\kappa_\alpha > 0$; if

$$\left(\frac{1}{12} - \frac{\tau^\alpha \kappa_\alpha \theta}{h^2}\right)^2 - \left(\frac{\nu_\alpha h}{24\kappa_\alpha} - \frac{\tau^\alpha \nu_\alpha}{2h}\right)^2 < 0, \quad (21)$$

then we easily know that $\lambda_k \neq 0$.

If

$$\left(\frac{1}{12} - \frac{\tau^\alpha \kappa_\alpha \theta}{h^2}\right)^2 - \left(\frac{\nu_\alpha h}{24\kappa_\alpha} - \frac{\tau^\alpha \nu_\alpha}{2h}\right)^2 \geq 0, \quad (22)$$

then

$$\lambda_k \geq \frac{5}{6} + \frac{2\tau^\alpha \kappa_\alpha \theta}{h^2} - 2\left|\frac{1}{12} - \frac{\tau^\alpha \kappa_\alpha \theta}{h^2}\right| > 0. \quad (23)$$

At the moment, we obtain $\det(\mathbf{A}) \neq 0$; that is to say, the matrix $\mathbf{A}$ is invertible. Hence, difference equation (17) has a unique solution. □

## 4. Stability and Convergence Analysis

In this section, we analyze the stability and convergence of difference scheme (17) by the Fourier method [8]. Firstly, we give the stability analysis.

**Lemma 3.** *The coefficients* $\varpi_m^{(1-\alpha)} (m = 0, 1, \ldots)$ *satisfy [8] as follows:*

(1) $\varpi_0^{(1-\alpha)} = 1$, $\varpi_1^{(1-\alpha)} = \alpha - 1$, $\varpi_m^{(1-\gamma)} < 0$, $j = 1, 2, \ldots$;

(2) $\sum_{m=0}^{\infty} \varpi_m^{(1-\gamma)} = 0$.

Let $U_j^k$ be the approximate solution of (14) and define

$$\rho_j^k = u_j^k - U_j^k, \quad j = 1, 2, \ldots, M-1, \ k = 0, 1, \ldots, N, \qquad \rho^k = \left(\rho_1^k, \rho_2^k, \ldots, \rho_{M-1}^k\right)^T, \quad k = 0, 1, \ldots, N, \quad (24)$$

respectively.

So, we can easily obtain the following roundoff error equation:

$$\begin{aligned} &\left(\frac{1}{12} - c_1 - c_2 + c_3\right)\rho_{j+1}^k + \left(\frac{5}{6} + 2c_2\right)\rho_j^k + \left(\frac{1}{12} + c_1 - c_2 - c_3\right)\rho_{j-1}^k \\ &= \left(\frac{1}{12} - c_1 + c_2\varpi_1^{(1-\alpha)} - c_3\varpi_1^{(1-\alpha)}\right)\rho_{j+1}^{k-1} + \left(\frac{5}{6} - 2c_2\varpi_1^{(1-\alpha)}\right)\rho_j^{k-1} \\ &\quad + \left(\frac{1}{12} + c_1 + c_2\varpi_1^{(1-\alpha)} + c_3\varpi_1^{(1-\alpha)}\right)\rho_{j-1}^{k-1} \\ &\quad - c_3\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(\rho_{j+1}^{k-m} - \rho_{j-1}^{k-m}\right) \\ &\quad + c_2\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(\rho_{j+1}^{k-m} - 2\rho_j^{k-m} + \rho_{j-1}^{k-m}\right), \\ &\qquad j = 1, 2, \ldots, M-1, \\ &\rho_0^k = \rho_M^k = 0, \quad k = 0, 1, \ldots, N. \end{aligned} \quad (25)$$

Now, we define the grid functions

$$\rho^k(x) = \begin{cases} \rho_j^k, & \text{when } x_j - \dfrac{h}{2} < x \leq x_j + \dfrac{h}{2}, \\ & \quad j = 1, 2, \ldots, M-1, \\ 0, & \text{when } 0 \leq x \leq \dfrac{h}{2} \text{ or } L - \dfrac{h}{2} < x \leq L; \end{cases} \quad (26)$$

then $\rho^k(x)$ can be expanded in a Fourier series:

$$\rho^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l)\exp\left(\frac{2\pi l x}{L} i\right), \quad (27)$$

where

$$\xi_k(l) = \frac{1}{L}\int_0^L \rho^k(x)\exp\left(-\frac{2\pi l x}{L} i\right) dx. \quad (28)$$

We introduce the following norm:

$$\left\|\rho^k\right\|_2 = \left(\sum_{j=1}^{M-1} h\left|\rho_j^k\right|^2\right)^{1/2} = \left[\int_0^L \left|\rho^k(x)\right|^2 dx\right]^{1/2}, \quad (29)$$

and according to the Parseval equality

$$\int_0^L \left|\rho^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} \left|\xi_k(l)\right|^2, \quad (30)$$

we obtain

$$\left\|\rho^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} \left|\xi_k(l)\right|^2. \quad (31)$$

Through the above analysis, we can suppose that the solution of (25) has the following form:

$$\rho_j^k = \xi_k \exp(i\beta j h), \tag{32}$$

where $\beta = 2\pi l/L$.

Substituting the above expression into (25) one gets

$$\begin{aligned}&\left[1-\left(\frac{1}{3}-4c_2\right)\sin^2\left(\frac{1}{2}\beta h\right)-i\cdot 2(c_1-c_3)\sin(\beta h)\right]\xi_k\\&=\left[1-\left(\frac{1}{3}+4c_2\varpi_1^{(1-\alpha)}\right)\sin^2\left(\frac{1}{2}\beta h\right)\right.\\&\qquad\left.-i\cdot 2\left(c_1+c_3\varpi_1^{(1-\alpha)}\right)\sin(\beta h)\right]\xi_{k-1}\\&\quad-\left[4c_2\sin^2\left(\frac{1}{2}\beta h\right)+i\cdot 2c_3\sin(\beta h)\right]\\&\quad\times\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\xi_{k-m},\quad k=0,1,\ldots,N.\end{aligned}\tag{33}$$

**Lemma 4.** *The following relation holds:*

$$\left|\frac{Q}{S}\right|\le 1, \tag{34}$$

*where*

$$\begin{aligned}Q&=\left[1-\left(\frac{1}{3}+4c_2\varpi_1^{(1-\alpha)}\right)\sin^2\left(\frac{1}{2}\beta h\right)\right.\\&\qquad\left.-i\cdot 2\left(c_1+c_3\varpi_1^{(1-\alpha)}\right)\sin(\beta h)\right],\\S&=\left[1-\left(\frac{1}{3}-4c_2\right)\sin^2\left(\frac{1}{2}\beta h\right)\right.\\&\qquad\left.-i\cdot 2(c_1-c_3)\sin(\beta h)\right].\end{aligned}\tag{35}$$

*Proof.* Because of

$$\begin{aligned}8c_2-32c_1c_3&=\frac{8\tau^\alpha\kappa_\alpha}{h^2}\cosh\left(\frac{\sqrt{6}\nu_\alpha h}{6\kappa_\alpha}\right)-\frac{2\tau^\alpha\nu_\alpha^2}{3\kappa_\alpha}\\&=\frac{8\tau^\alpha\kappa_\alpha}{h^2}\left[1+\sum_{k=2}^{\infty}\frac{1}{(2k)!}\left(\frac{\sqrt{6}\nu_\alpha h}{6\kappa_\alpha}\right)^{2k}\right]>0,\end{aligned}\tag{36}$$

we obtain

$$\begin{aligned}&\left\{8\alpha c_2\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right]-32\alpha c_1c_3\cos^2\left(\frac{1}{2}\beta h\right)\right\}\\&\quad+16c_2^2\left(1-\varpi_1^{(1-\alpha)^2}\right)\sin^2\left(\frac{1}{2}\beta h\right)\\&\quad+16c_3^2\left(1-\varpi_1^{(1-\alpha)^2}\right)\cos^2\left(\frac{1}{2}\beta h\right)>0.\end{aligned}\tag{37}$$

Furthermore, we can rewrite the above inequality as

$$\begin{aligned}&\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)-4c_2\varpi_1^{(1-\alpha)}\sin^2\left(\frac{1}{2}\beta h\right)\right]^2\\&\quad+4\left(c_1+c_3\varpi_1^{(1-\alpha)}\right)^2\sin^2(\beta h)\\&\le\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)+4c_2\sin^2\left(\frac{1}{2}\beta h\right)\right]^2\\&\quad+4(c_1-c_3)^2\sin^2(\beta h);\end{aligned}\tag{38}$$

that is,

$$\begin{aligned}&\left|\left(1-\left(\frac{1}{3}+4c_2\varpi_1^{(1-\alpha)}\right)\sin^2\left(\frac{1}{2}\beta h\right)-i\cdot 2\left(c_1+c_3\varpi_1^{(1-\alpha)}\right)\right.\right.\\&\quad\times\sin(\beta h)\Bigg)\times\left(1-\left(\frac{1}{3}-4c_2\right)\sin^2\left(\frac{1}{2}\beta h\right)\right.\\&\qquad\left.\left.-i\cdot 2(c_1-c_3)\sin(\beta h)\right)^{-1}\right|\le 1.\end{aligned}\tag{39}$$

This completes the proof of Lemma 4. □

**Lemma 5.** *Supposing that $\xi_k$ $(k=1,\ldots,N)$ is the solution of* (33), *then we have*

$$|\xi_k|\le\exp(M(k-1)\tau)|\xi_0|,\quad k=1,\ldots,N. \tag{40}$$

*Proof.* For $k=0$, from (33), we get

$$|\xi_1|=\left|\frac{Q}{S}\right||\xi_0|. \tag{41}$$

In the light of Lemma 4, it is clear that

$$|\xi_1|\le|\xi_0|=\exp(M\cdot 0\tau)|\xi_0|. \tag{42}$$

Now, we suppose that

$$|\xi_\ell|\le\exp(M(\ell-1)\tau)|\xi_0|,\quad(\ell=1,2,\ldots,k-1). \tag{43}$$

For $k>0$, from (33) with Lemmas 3 and 4, we have

$$\begin{aligned}|\xi_k|&=\frac{1}{|S|}\left|Q\xi_{k-1}-\left[4c_2\sin^2\left(\frac{1}{2}\beta h\right)+i\cdot 2c_3\sin(\beta h)\right]\right.\\&\qquad\left.\times\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\xi_{k-m}\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|S|}\left|Q\xi_{k-1} - \left[4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right]\right.\\
&\qquad\times\sum_{m=2}^{k-1}\varpi_m^{(1-\alpha)}\xi_{k-m}\\
&\qquad\left.-\left[4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right]\xi_0\right|\\
&\le \frac{1}{|S|}\left\{|Q|\left|\xi_{k-1}\right| - \left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\right.\\
&\qquad\times\sum_{m=2}^{k-1}\varpi_m^{(1-\alpha)}\left|\xi_{k-m}\right|\\
&\qquad\left.-\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\varpi_k^{(1-\alpha)}\left|\xi_0\right|\right\}\\
&\le \frac{1}{|S|}\left\{|Q|\exp(M(k-2)\tau)\left|\xi_0\right|\right.\\
&\qquad-\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\\
&\qquad\times\sum_{m=2}^{k-1}\varpi_m^{(1-\alpha)}\exp(M(k-m-1)\tau)\left|\xi_0\right|\\
&\qquad\left.-\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\varpi_k^{(1-\alpha)}\left|\xi_0\right|\right\}\\
&\le \frac{1}{|S|}\left\{|Q| - \left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\sum_{m=1}^{\infty}\varpi_m^{(1-\alpha)}\right\}\\
&\quad\times\exp(M(k-2)\tau)\left|\xi_0\right|\\
&= \frac{1}{|S|}\left\{|Q| + \alpha\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\right\}\\
&\quad\times\exp(M(k-2)\tau)\left|\xi_0\right|\\
&\le (1+M\tau)\exp(M(k-2)\tau)\left|\xi_0\right|\\
&\le \exp(M\tau)\exp(M(k-2)\tau)\left|\xi_0\right|\\
&= \exp(M(k-1)\tau)\left|\xi_0\right| = K\left|\xi_0\right|.
\end{aligned}
\tag{44}
$$

This finishes the proof of Lemma 5. □

**Lemma 6.** *Difference scheme* (14) *is unconditionally stable.*

*Proof.* According to Lemma 5, we obtain

$$
\begin{aligned}
\left\|\rho^k\right\|_2 &= \left(\sum_{i=1}^{M-1}h\left|\rho_i^k\right|^2\right)^{1/2}\\
&= \left(\sum_{i=1}^{M-1}h\left|\xi_k\exp(i\beta jh)\right|^2\right)^{1/2} = \left(\sum_{i=1}^{M-1}h\left|\xi_k\right|^2\right)^{1/2}\\
&\le K\left(\sum_{i=1}^{M-1}h\left|\xi_0\right|^2\right)^{1/2} = K\left(\sum_{i=1}^{M-1}h\left|\xi_0\exp(i\beta jh)\right|^2\right)^{1/2}\\
&= K\left(\sum_{i=1}^{M-1}h\left|\rho_i^0\right|^2\right)^{1/2} = \left\|\rho^0\right\|_2,\quad k=1,2,\ldots,N,
\end{aligned}
\tag{45}
$$

which means that difference scheme (14) is unconditionally stable.

Next, we give the convergence analysis. Suppose

$$
\mathscr{E}_i^k = u(x_i,t_k) - u_i^k,\quad i=1,\ldots,M-1,\ k=1,\ldots,N,
\tag{46}
$$

and denote

$$
\mathscr{E}^k = \left(\mathscr{E}_1^k,\mathscr{E}_2^k,\ldots,\mathscr{E}_{M-1}^k\right)^T,\qquad R^k = \left(R_1^k,R_2^k,\ldots,R_{M-1}^k\right)^T,\quad k=1,\ldots,N.
\tag{47}
$$

From (14), we obtain

$$
\begin{aligned}
&\left(\frac{1}{12}-c_1-c_2+c_3\right)\mathscr{E}_{j+1}^k + \left(\frac{5}{6}+2c_2\right)\mathscr{E}_j^k\\
&\quad+\left(\frac{1}{12}+c_1-c_2-c_3\right)\mathscr{E}_{j-1}^k\\
&\quad=\left(\frac{1}{12}-c_1+c_2\varpi_1^{(1-\alpha)}-c_3\varpi_1^{(1-\alpha)}\right)\mathscr{E}_{j+1}^{k-1}\\
&\quad+\left(\frac{5}{6}-2c_2\varpi_1^{(1-\alpha)}\right)\mathscr{E}_j^{k-1}\\
&\quad+\left(\frac{1}{12}+c_1+c_2\varpi_1^{(1-\alpha)}+c_3\varpi_1^{(1-\alpha)}\right)\mathscr{E}_{j-1}^{k-1}\\
&\quad-c_3\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(\mathscr{E}_{j+1}^{k-m}-\mathscr{E}_{j-1}^{k-m}\right)\\
&\quad+c_2\sum_{m=2}^{k}\varpi_m^{(1-\alpha)}\left(\mathscr{E}_{j+1}^{k-m}-2\mathscr{E}_j^{k-m}+\mathscr{E}_{j-1}^{k-m}\right)\\
&\quad+\tau\left(\frac{1}{12}-c_1\right)R_{j+1}^k+\frac{5}{6}\tau R_j^k\\
&\quad+\tau\left(\frac{1}{12}+c_1\right)R_{j-1}^k,\\
&\qquad j=1,2,\ldots,M-1,\ k=0,1,\ldots,N.
\end{aligned}
\tag{48}
$$

Similar to the stability analysis method, we define the grid functions

$$
\mathscr{E}^k(x) = \begin{cases}\mathscr{E}_j^k, & \text{when } x_j-\dfrac{h}{2}<x\le x_j+\dfrac{h}{2},\\ & \qquad j=1,2,\ldots,M-1,\\ 0, & \text{when } 0\le x\le\dfrac{h}{2} \text{ or } L-\dfrac{h}{2}<x\le L,\end{cases}
$$

$$R^k(x) = \begin{cases} R_j^k, & \text{when } x_j - \frac{h}{2} < x \le x_j + \frac{h}{2}, \\ & \qquad j = 1, 2, \dots, M-1, \\ 0, & \text{when } 0 \le x \le \frac{h}{2} \text{ or } L - \frac{h}{2} < x \le L; \end{cases} \tag{49}$$

then $\mathscr{E}^k(x)$ and $R^k(x)$ can be expanded to the following Fourier series, respectively:

$$\begin{aligned} \mathscr{E}^k(x) &= \sum_{l=-\infty}^{\infty} \zeta_k(l) \exp\left(\frac{2\pi l x}{L} i\right), \\ R^k(x) &= \sum_{l=-\infty}^{\infty} \eta_k(l) \exp\left(\frac{2\pi l x}{L} i\right), \end{aligned} \tag{50}$$

where

$$\begin{aligned} \zeta_k(l) &= \frac{1}{L} \int_0^L \mathscr{E}^k(x) \exp\left(-\frac{2\pi l x}{L} i\right) dx, \\ \eta_k(l) &= \frac{1}{L} \int_0^L R^k(x) \exp\left(-\frac{2\pi l x}{L} i\right) dx. \end{aligned} \tag{51}$$

The same as before, we also have

$$\left\|\mathscr{E}^k\right\|_2 = \left(\sum_{j=1}^{M-1} h\left|\mathscr{E}_j^k\right|^2\right)^{1/2} = \left(\sum_{l=-\infty}^{\infty} \left|\zeta_k(l)\right|^2\right)^{1/2}, \tag{52}$$

$$\left\|R^k\right\|_2 = \left(\sum_{j=1}^{M-1} h\left|R_j^k\right|^2\right)^{1/2} = \left(\sum_{l=-\infty}^{\infty} \left|\eta_k(l)\right|^2\right)^{1/2}. \tag{53}$$

Based on the above analysis, we can assume that $\mathscr{E}_i^k$ and $R_i^k$ have the following forms:

$$\begin{aligned} \mathscr{E}_j^k &= \zeta_k \exp(i\beta j h), \\ R_j^k &= \eta_k \exp(i\beta j h), \end{aligned} \tag{54}$$

respectively. Substituting the above two expressions into (48) yields

$$\begin{aligned} S\zeta_k &= Q\zeta_{k-1} - \left[4c_2 \sin^2\left(\frac{1}{2}\beta h\right) + i \cdot 2c_3 \sin(\beta h)\right] \\ &\quad \times \sum_{m=2}^{k} \varpi_m^{(1-\alpha)} \zeta_{k-m} \\ &\quad + \tau\left\{\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i \cdot 2c_1 \sin(\beta h)\right\} \eta_k, \\ &\qquad k = 1, 2, \dots, N. \end{aligned} \tag{55}$$

□

**Lemma 7.** *Let $\zeta_k$ $(k = 0, 1, \dots, N)$ be the solution of* (55)*; then there exists a positive constant $C_3$, so that*

$$\left|\zeta_k\right| \le C_3 k\tau \exp(M(k-1)\tau) \left|\eta_1\right|, \quad k = 0, 1, \dots, N. \tag{56}$$

*Proof.* From $\mathscr{E}^0 = 0$, we have

$$\zeta_0 = \zeta_0(l) = 0. \tag{57}$$

In view of the convergence of the series of (53), there is a positive constant $C_1$, such that

$$\left|\eta_k\right| = \left|\eta_k(l)\right| \le C_1 \left|\eta_1\right| = C_1 \left|\eta_1(l)\right|. \tag{58}$$

For $k = 1$, from (55), we have

$$\begin{aligned} \zeta_1 &= \frac{Q}{S}\zeta_0 + \frac{\tau\left\{\left[1 - (1/3)\sin^2((1/2)\beta h)\right] - i \cdot 2c_1 \sin(\beta h)\right\}}{S} \eta_1 \\ &= \frac{\tau\left\{\left[1 - (1/3)\sin^2((1/2)\beta h)\right] - i \cdot 2c_1 \sin(\beta h)\right\}}{S} \eta_1. \end{aligned} \tag{59}$$

Noticing (58), then

$$\left|\zeta_1\right| \le \tau\left|\eta_1\right| \le C_1 \tau \exp(M \cdot 0\tau)\left|\eta_1\right|. \tag{60}$$

Now, we suppose that

$$\left|\zeta_\ell\right| \le C_1 \ell\tau \exp(M(\ell-1)\tau)\left|\eta_1\right|, \quad \ell = 1, \dots, N-1. \tag{61}$$

Then when $k > 1$, we obtain

$$\begin{aligned} \left|\zeta_k\right| &= \frac{1}{|S|}\left| Q\zeta_{k-1} - \left[4c_2 \sin^2\left(\frac{1}{2}\beta h\right) + i \cdot 2c_3 \sin(\beta h)\right] \right. \\ &\quad \times \sum_{m=2}^{k} \varpi_m^{(1-\alpha)} \zeta_{k-m} \\ &\quad \left. + \tau\left\{\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i \cdot 2c_1 \sin(\beta h)\right\} \eta_k \right| \\ &\le \frac{1}{|S|}\left\{ |Q|\left|\zeta_{k-1}\right| - \left|4c_2 \sin^2\left(\frac{1}{2}\beta h\right) + i \cdot 2c_3 \sin(\beta h)\right| \right. \\ &\quad \times \sum_{m=2}^{k} \varpi_m^{(1-\alpha)} \left|\zeta_{k-m}\right| \\ &\quad \left. + \tau\left|\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i \cdot 2c_1 \sin(\beta h)\right| \left|\eta_k\right| \right\} \\ &\le \frac{1}{|S|}\left\{ |Q|(k-1)\exp(M(k-2)\tau) \right. \\ &\quad - \left|4c_2 \sin^2\left(\frac{1}{2}\beta h\right) + i \cdot 2c_3 \sin(\beta h)\right| (k-m) \\ &\quad \times \sum_{m=2}^{k} \varpi_m^{(1-\alpha)} \exp(M(k-m-1)\tau) \\ &\quad + \left.\left|\left[1 - \frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i \cdot 2c_1 \sin(\beta h)\right|\right\} C_1 \tau \left|\eta_1\right| \end{aligned}$$

Table 1: The maximum error, temporal, and spatial convergence orders by difference scheme (14) for different $\alpha$.

| $\alpha$ | | The maximum error | Temporal convergence order | Spatial convergence order |
|---|---|---|---|---|
| 0.1 | $h = 1/4, \tau = 1/4$ | 0.1650 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0103 | 1.0004 | 4.0017 |
| 0.2 | $h = 1/4, \tau = 1/4$ | 0.1526 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0095 | 1.0014 | 4.0057 |
| 0.3 | $h = 1/4, \tau = 1/4$ | 0.1400 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0087 | 1.0021 | 4.0083 |
| 0.4 | $h = 1/4, \tau = 1/4$ | 0.1274 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0080 | 0.9983 | 3.9932 |
| 0.5 | $h = 1/4, \tau = 1/4$ | 0.1150 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0072 | 0.9994 | 3.9975 |
| 0.6 | $h = 1/4, \tau = 1/4$ | 0.1027 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0065 | 0.9955 | 3.9819 |
| 0.7 | $h = 1/4, \tau = 1/4$ | 0.0907 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0058 | 0.9917 | 3.9670 |
| 0.8 | $h = 1/4, \tau = 1/4$ | 0.0789 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0051 | 0.9879 | 3.9515 |
| 0.9 | $h = 1/4, \tau = 1/4$ | 0.0676 | — | — |
| | $h = 1/8, \tau = 1/64$ | 0.0045 | 0.9773 | 3.9090 |

$$\begin{aligned}
&\leq \frac{1}{|S|}\Big\{|Q|(k-1)\exp(M(k-2)\tau) \\
&\quad + \alpha\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right| \\
&\quad \times (k-1)\exp(M(k-1)\tau) \\
&\quad + \left|\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i\cdot 2c_1\sin(\beta h)\right|\Big\} C_1\tau|\eta_1| \\
&\leq \frac{1}{|S|}\Big\{(k-1)\left[|Q| + \alpha\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\right] \\
&\quad + \left|\left[1-\frac{1}{3}\sin^2\left(\frac{1}{2}\beta h\right)\right] - i\cdot 2c_1\sin(\beta h)\right|\Big\} \\
&\quad \times C_1\tau\exp(M(k-2)\tau)|\eta_1| \\
&\leq \Big\{(k-1)\left[|Q| + \alpha\left|4c_2\sin^2\left(\frac{1}{2}\beta h\right) + i\cdot 2c_3\sin(\beta h)\right|\right] \\
&\quad + \frac{\left|\left[1-(1/3)\sin^2((1/2)\beta h)\right] - i\cdot 2c_1\sin(\beta h)\right|}{|S|}\Big\} \\
&\quad \times C_1\tau\exp(M(k-2)\tau)|\eta_1| \\
&\leq \{k(1+M\tau)\}C_1\tau\exp(M(k-2)\tau)|\eta_1| \\
&\leq C_1 k\tau\exp(M\tau)\exp(M(k-2)\tau)|\eta_1| \\
&= C_1 k\tau\exp(M(k-1)\tau)|\eta_1|.
\end{aligned} \tag{62}$$

This completes the proof. □

**Theorem 8.** *Difference scheme* (14) *is convergent, and the convergence order is* $O(\tau + h^4)$.

*Proof.* Firstly, we know that there are exist positive constants $C_2$, such that

$$\left|R_j^k\right| \leq C_2\left(\tau + h^4\right), \tag{63}$$

$$\left\|R_j^k\right\| \leq C_2\sqrt{(M-1)h}\left(\tau + h^4\right) \leq C_2\sqrt{L}\left(\tau + h^4\right). \tag{64}$$

Using (52) and (64) with Lemma 7, we get

$$\begin{aligned}
\left\|\mathscr{E}^k\right\|_2 &\leq C_1 k\tau\exp(M(k-1)\tau)\left\|R^1\right\|_2 \\
&\leq C_1C_2\sqrt{L}k\tau\exp(MT)\left(\tau + h^4\right) = C\left(\tau + h^4\right).
\end{aligned} \tag{65}$$

Due to $k \leq N$,

$$k\tau \leq T, \tag{66}$$

so,

$$\left\|\mathscr{E}^k\right\|_2 \leq C\left(\tau + h^4\right), \tag{67}$$

where $C = C_1C_2T\sqrt{L}\exp(MT)$. This ends the proof. □

*Remark 9.* From above discussion, we know that difference scheme is an implicit scheme and it is unconditionally stable and convergent. If we take

$$\frac{\partial u(x_j, t_k)}{\partial t} = \frac{u(x_j, t_{k+1}) - u(x_j, t_k)}{\tau} + O(\tau) \tag{68}$$

in (1), then we can obtain an explicit scheme and it is conditionally stable and convergent.

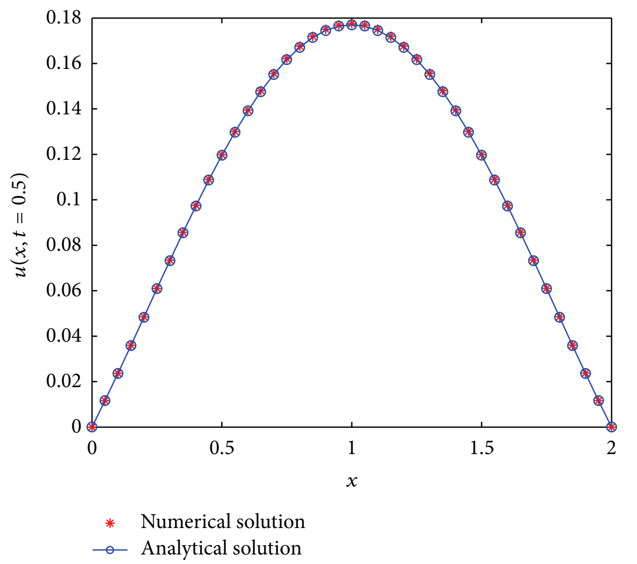


Figure 1: The comparison of the numerical solution with the analytical solution at $t = 0.5$. ($\tau = 1/400, h = 1/20$).

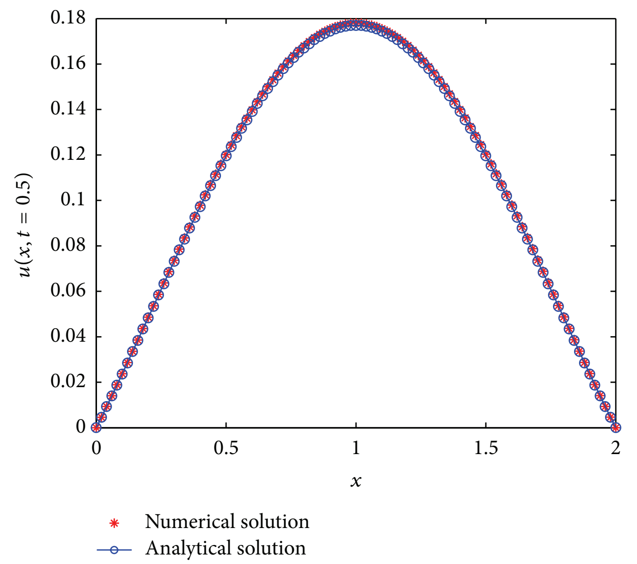


Figure 2: The comparison of the numerical solution with the analytical solution at $t = 0.5$. ($\tau = 1/200, h = 1/50$).

## 5. Numerical Example

In this section, a numerical example is presented to confirm our theoretical results.

*Example 10.* Consider the following equation:

$$\frac{\partial u(x,t)}{\partial t} = \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right] {}_{\mathrm{RL}}D_{0,t}^{1-\alpha} u(x,t) + f(x,t), \quad 0 < x < 2,\ 0 \le t \le 1, \tag{69}$$

with the initial and boundary conditions

$$\begin{gathered} u(x,0) = 0, \quad 0 \le x \le 2, \\ u(0,t) = 0, \quad u(2,t) = 0, \quad 0 < t \le 1, \end{gathered} \tag{70}$$

where $f(x,t) = 2t\sin(x)\sin(2-x) + (2t^{1+\alpha}/\Gamma(2+\alpha))[2\cos 2(x-1) + \sin 2(1-x)]$. The analytical solution of this equation is $u(x,t) = t^2\sin(x)\sin(2-x)$.

The maximum error, temporal, and spatial convergence orders by difference scheme (14) for various $\alpha$ are listed in Table 1. From the obtained results, we can draw the following conclusions: the experimental convergence orders are approximately 1 and 4 in temporal and spatial directions, respectively. Figures 1 and 2 show the comparison of the numerical solution with the analytical solution at $t = 0.5$ and $\alpha = 0.8$ for different temporal and spatial mesh sizes.

By Table 1 and Figures 1 and 2, it can be seen that the numerical solution is in excellent agreement with the analytical solution. These results confirm our theoretical analysis.

## 6. Conclusion

In this paper, a computationally effective numerical method is proposed for simulating the modified time fractional Fokker-Planck equation. It has proven the unconditional stability and solvability of proposed scheme. Also we showed that the method is convergent with order $O(\tau + h^4)$. The numerical results demonstrate the effectiveness of the proposed scheme.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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