

Research Article

On New Generalized Ostrowski Type Integral Inequalities

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Received 26 February 2014; Revised 30 March 2014; Accepted 6 May 2014; Published 22 May 2014

Academic Editor: Elena Berdysheva

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The Ostrowski inequality expresses bounds on the deviation of a function from its integral mean. The aim of this paper is to establish some new inequalities similar to the Ostrowski's inequality. The current paper obtains bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval in terms of the norms of the second derivative of the function. Some new perturbed results are obtained. Application for cumulative distribution function is also discussed.

1. Introduction

Ostrowski [1] proved his famous inequality in 1938. Milovanović and Pecarić gave the first generalization of Ostrowski's inequality [2]. A number of Ostrowski type inequalities have been derived by Cerone [3], Cheng [4], Dragomir and Barnett [5], Sofo and Dragomir [6], Milovanović [7], and G. V. Milovanović and I. Z. Milovanović [8] with applications in numerical analysis and probability. Several further generalizations are provided by Hussain and Qayyum [9], Qayyum and Hussain [10], and Pecarić [11], respectively. More recent results concerning the generalizations of Ostrowski inequality are given by [4, 5, 12, 13]. Dragomir and Wang [14] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Cheng gave a sharp version of Ostrowski-Grüss inequality in [4]. Cerone [3] and Dragomir and Wang in [14–17] obtained bounds on a particular quadrature rule for differentiable functions which we generalize to two times differentiable functions. This generalization improves bounds on the deviation of a function from a combination of integral means. Some new perturbed results are also discussed. The generalized inequalities obtained in this paper will have applications in approximation theory, probability theory, and numerical

analysis. We will show an application of the inequalities obtained for cumulative distribution function.

In this paper, we will use the usual L_p norms $\|k\|_p$ defined for a function $k \in L_p[a, b]$ as follows:

$$\begin{aligned} \|k\|_\infty &:= \operatorname{ess\,sup}_{t \in [a, b]} |k(t)|, \\ \|k\|_p &:= \left(\int_a^b |k(t)|^p dt \right)^{1/p}, \quad 1 \leq p < \infty. \end{aligned} \quad (1)$$

Let $S(f; a, b)$ be defined by

$$S(f; a, b) := f(x) - M(f; a, b), \quad (2)$$

where

$$M(f; a, b) := \frac{1}{b-a} \int_a^b f(x) dx \quad (3)$$

is the integral mean of f over $[a, b]$. The functional $S(f; a, b)$ represents the deviation of $f(x)$ from its integral mean over $[a, b]$.

Ostrowski [1] proved the following interesting and useful integral inequality.

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) by a constant M . Then,

$$|S(f; a, b)| \leq \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \frac{M}{b-a}, \quad (4)$$

for all $x \in [a, b]$.

In a series of papers, Dragomir and Wang [14–17] proved (4) and some of its variants for $f' \in L_p[a, b]$, when $p \geq 1$, making use of a Peano kernel and Montgomery’s identity [18]. Montgomery’s identity states that for absolutely continuous mappings $f : [a, b] \rightarrow \mathbb{R}$

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b P(x, t) f'(t) dt, \quad (5)$$

where the kernel $P : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$P(x, t) := \begin{cases} t - a, & \text{if } a \leq t \leq x \leq b, \\ t - b, & \text{if } a \leq x < t \leq b. \end{cases} \quad (6)$$

If we assume that $f' \in L_\infty[a, b]$, then M in (4) may be replaced by $\|f'\|_\infty$.

Dragomir and Wang [14–17], utilizing an integration by parts argument, obtained

$$|S(f; a, b)| \leq \begin{cases} \frac{1}{b-a} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty, & f' \in L_\infty[a, b], \\ \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{1/q} \|f'\|_p, & f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{b-a} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1, & f' \in L_1[a, b], \end{cases} \quad (7)$$

where $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and the constants $1/4$, $[1/(q+1)]^{1/q}$, and $1/2$ are sharp.

Cerone [3] proved the following inequality.

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous. Define

$$\tau(x; \alpha, \beta) := f(x) - \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)]. \quad (8)$$

Then,

$$|\tau(x; \alpha, \beta)| \leq \begin{cases} \frac{1}{2(\alpha + \beta)} [\alpha(x-a) + \beta(b-x)] \|f'\|_\infty, & f' \in L_\infty[a, b], \\ \frac{1}{(\alpha + \beta)(q+1)^{1/q}} \times [\alpha^q(x-a) + \beta^q(b-x)]^{1/q} \|f'\|_p, & f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left(1 + \frac{|\alpha - \beta|}{\alpha + \beta} \right) \|f'\|_1, & f' \in L_1[a, b]. \end{cases} \quad (9)$$

In [19], Pachpatte established Čebyšev type inequalities by using Pecarić’s extension of the Montgomery identity [18]. The current paper obtains bounds on the deviation of a function from integral means from the end of the interval which cover the whole interval, in terms of the norms of the second derivative of the function. The paper closely follows ideas from [3], where corresponding results were proved using bounds in terms of the first derivative of the function.

2. Main Results

Denote by $P(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ the kernel given by

$$P(x, t) := \begin{cases} \frac{\alpha}{\alpha + \beta} \frac{1}{x-a} \frac{(t-a)^2}{2}, & a \leq t \leq x, \\ \frac{\beta}{\alpha + \beta} \frac{1}{b-x} \frac{(t-b)^2}{2}, & x < t \leq b, \end{cases} \quad (10)$$

where $\alpha, \beta \in \mathbb{R}$ are nonnegative and not both zero.

Before we state and prove our main result, we will prove the following identity, which will be used to obtain bounds.

Lemma 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be two times differentiable on $[a, b]$. Then,

$$\begin{aligned} & \int_a^b P(x, t) f''(t) dt \\ &= \frac{1}{2(\alpha + \beta)} [\alpha(x-a) - \beta(b-x)] f'(x) - f(x) \\ &+ \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right]. \end{aligned} \quad (11)$$

Proof. From (10), we have

$$\begin{aligned} & \int_a^b P(x, t) f''(t) dt \\ &= \frac{\alpha}{(\alpha + \beta)} \frac{1}{x-a} \int_a^x \frac{(t-a)^2}{2} f''(t) dt \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta}{(\alpha + \beta)} \frac{1}{b - x} \int_x^b \frac{(t - b)^2}{2} f''(t) dt \\
 = & \frac{\alpha}{2(\alpha + \beta)} \frac{1}{x - a} \\
 & \times \left[(x - a)^2 f'(x) - 2 \int_a^x (t - a) f'(t) dt \right] \\
 & + \frac{\beta}{2(\alpha + \beta)} \frac{1}{b - x} \\
 & \times \left[-(x - b)^2 f'(x) - 2 \int_x^b (t - b) f'(t) dt \right].
 \end{aligned} \tag{12}$$

After further simplification, we get the required identity (11). \square

We now give our main result.

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a two times differentiable function. Define

$$\begin{aligned}
 \tau(x; \alpha, \beta) := & \frac{1}{2(\alpha + \beta)} [\alpha(x - a) - \beta(b - x)] f'(x) - f(x) \\
 & + \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)],
 \end{aligned} \tag{13}$$

where $M(f; a, b)$ is the integral mean defined in (3). Then,

$$\begin{aligned}
 & |\tau(x; \alpha, \beta)| \\
 & \leq \left\{ \begin{aligned} & \left[\alpha(x - a)^2 + \beta(b - x)^2 \right] \frac{\|f''\|_\infty}{6(\alpha + \beta)}, \\ & \qquad f'' \in L_\infty[a, b], \\ & \frac{1}{(2q + 1)^{1/q}} \\ & \times \left[\alpha^q(x - a)^{q+1} + \beta^q(b - x)^{q+1} \right]^{1/q} \frac{\|f''\|_p}{2(\alpha + \beta)}, \\ & \qquad f'' \in L_p[a, b], \\ & \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ & (\alpha(x - a) + \beta(b - x) + |\alpha(x - a) - \beta(b - x)|) \\ & \times \frac{\|f''\|_1}{4(\alpha + \beta)}, \quad f'' \in L_1[a, b], \end{aligned} \right.
 \end{aligned} \tag{14}$$

for all $x \in [a, b]$.

Proof. Taking the modulus of (11) and using (13) and (3), we have

$$|\tau(x; \alpha, \beta)| = \left| \int_a^b P(x, t) f''(t) dt \right| \leq \int_a^b |P(x, t)| |f''(t)| dt. \tag{15}$$

Therefore, for $f'' \in L_\infty[a, b]$, we obtain

$$|\tau(x; \alpha, \beta)| \leq \|f''\|_\infty \int_a^b |P(x, t)| dt. \tag{16}$$

Now let us observe that

$$\begin{aligned}
 & \int_a^b |P(x, t)| dt \\
 = & \frac{\alpha}{(\alpha + \beta)} \frac{1}{x - a} \int_a^x \frac{(t - a)^2}{2} dt \\
 & + \frac{\beta}{(\alpha + \beta)} \frac{1}{b - x} \int_x^b \frac{(t - b)^2}{2} dt \\
 = & \frac{\alpha}{(\alpha + \beta)} \frac{1}{x - a} \frac{(x - a)^3}{6} \\
 & + \frac{\beta}{(\alpha + \beta)} \frac{1}{b - x} \frac{(b - x)^3}{6} \\
 = & \frac{\alpha}{(\alpha + \beta)} \frac{(x - a)^2}{6} + \frac{\beta}{(\alpha + \beta)} \frac{(b - x)^2}{6} \\
 = & \frac{1}{6(\alpha + \beta)} [\alpha(x - a)^2 + \beta(b - x)^2].
 \end{aligned} \tag{17}$$

Hence, the first inequality is obtained:

$$|\tau(x; \alpha, \beta)| \leq [\alpha(x - a)^2 + \beta(b - x)^2] \frac{1}{6(\alpha + \beta)} \|f''\|_\infty. \tag{18}$$

Further, using Hölder's integral inequality, from (15), we have for $f'' \in L_p[a, b]$, $p > 1$,

$$|\tau(x; \alpha, \beta)| \leq \|f''\|_p \left(\int_a^b |P(x, t)|^q dt \right)^{1/q}, \tag{19}$$

where $(1/p) + (1/q) = 1$. Now

$$\begin{aligned}
 & (\alpha + \beta) \left(\int_a^b |P(x, t)|^q dt \right)^{1/q} \\
 = & \left[\frac{\alpha^q}{2^q(x - a)^q} \int_a^x (t - a)^{2q} dt \right. \\
 & \left. + \frac{\beta^q}{2^q(b - x)^q} \int_x^b (b - t)^{2q} dt \right]^{1/q} \\
 = & \frac{1}{2(2q + 1)^{1/q}} \\
 & \times [\alpha^q(x - a)^{q+1} + \beta^q(b - x)^{q+1}]^{1/q}.
 \end{aligned} \tag{20}$$

Hence, the second inequality is obtained as below:

$$|\tau(x; \alpha, \beta)| \leq \frac{1}{2(2q+1)^{1/q}(\alpha+\beta)} \times [\alpha^q(x-a)^{q+1} + \beta^q(b-x)^{q+1}]^{1/q} \|f''\|_p. \tag{21}$$

Finally, for $f'' \in L_1[a, b]$, using (10), we have the following from (15). Consider

$$|\tau(x; \alpha, \beta)| \leq \sup_{t \in [a, b]} |P(x, t)| \|f''\|_1, \tag{22}$$

where

$$\begin{aligned} (\alpha + \beta) \sup_{t \in [a, b]} |P(x, t)| &= \max \left\{ \frac{\alpha(x-a)}{2}, \frac{\beta(b-x)}{2} \right\} \\ &= \frac{\alpha(x-a) + \beta(b-x)}{4} \\ &\quad + \left| \frac{\alpha(x-a) - \beta(b-x)}{4} \right|. \end{aligned} \tag{23}$$

This gives us the last inequality as below:

$$\begin{aligned} |\tau(x; \alpha, \beta)| &\leq (\alpha(x-a) + \beta(b-x)) \\ &\quad + |\alpha(x-a) - \beta(b-x)| \\ &\quad \times \frac{\|f''\|_1}{4(\alpha + \beta)}. \end{aligned} \tag{24}$$

This completes the proof of the theorem. \square

Remark 5. We may write

$$\begin{aligned} \alpha M(f; a, x) + \beta M(f; x, b) &= \alpha M(f; a, x) + \frac{\beta}{b-x} \left(\int_a^b f(u) du - \int_a^x f(u) du \right) \\ &= \alpha M(f; a, x) - \frac{\beta}{b-x} \int_a^x f(u) du + \frac{\beta}{b-x} \int_a^b f(u) du \\ &= (\alpha + \beta - \beta\sigma(x)) M(f; a, x) + \beta\sigma(x) M(f; a, b), \end{aligned} \tag{25}$$

where

$$\frac{b-a}{b-x} = \sigma(x). \tag{26}$$

Thus, from (13),

$$\begin{aligned} \tau(x; \alpha, \beta) &= \frac{1}{2(\alpha + \beta)} [\alpha(x-a) - \beta(b-x)] f'(x) - f(x) \\ &\quad + \left[\left(1 - \frac{\beta}{\alpha + \beta} \sigma(x) \right) M(f; a, x) \right. \\ &\quad \left. + \frac{\beta}{\alpha + \beta} \sigma(x) M(f; a, b) \right], \end{aligned} \tag{27}$$

so that, for fixed $[a, b]$, $M(f; a, b)$ is also fixed.

Corollary 6. Let the condition of Theorem 4 hold and $\alpha = \beta$; then,

$$\begin{aligned} &\left| \frac{1}{4} [(x-a) - (b-x)] f'(x) - f(x) \right. \\ &\quad \left. + \frac{1}{2} [M(f; a, x) + M(f; x, b)] \right| \\ &\leq \begin{cases} \left[\frac{(x-a)^2 + (b-x)^2}{12} \right] \frac{\|f''\|_\infty}{12}, \\ \quad f'' \in L_\infty[a, b], \\ \frac{1}{(2q+1)^{1/q}} [(x-a)^{q+1} + (b-x)^{q+1}]^{1/q} \frac{\|f''\|_p}{4}, \\ \quad f'' \in L_p[a, b], p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ \left(b-a + 2 \left| x - \frac{a+b}{2} \right| \right) \frac{\|f''\|_1}{8}, \\ \quad f'' \in L_1[a, b]. \end{cases} \end{aligned} \tag{28}$$

Proof. The proof of the above corollary is a straightforward exercise after putting $\alpha = \beta$ in (14). \square

Corollary 7. Let the conditions of Theorem 4 hold and $x = (a+b)/2$. Then,

$$\begin{aligned} &\left| \frac{b-a}{4(\alpha + \beta)} (\alpha - \beta) f' \left(\frac{a+b}{2} \right) - f \left(\frac{a+b}{2} \right) \right. \\ &\quad \left. + \frac{2}{(b-a)(\alpha + \beta)} \right. \\ &\quad \left. \times \left(\alpha \int_a^{(a+b)/2} f(t) dt + \beta \int_{(a+b)/2}^b f(t) dt \right) \right| \\ &\leq \begin{cases} \frac{(b-a)^2}{24} \|f''\|_\infty, \quad f'' \in L_\infty[a, b], \\ \left(\frac{b-a}{2} \right)^{1+(1/q)} \frac{1}{(2q+1)^{1/q}} \\ \quad \times (\alpha^q + \beta^q)^{1/q} \frac{\|f''\|_p}{2(\alpha + \beta)}, \quad f'' \in L_p[a, b], \\ \quad p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ (b-a) \left(1 + \frac{|\alpha - \beta|}{\alpha + \beta} \right) \frac{\|f''\|_1}{8}, \quad f'' \in L_1[a, b]. \end{cases} \end{aligned} \tag{29}$$

Proof. Placing $x = (a+b)/2$ in (13) and (14) produces the results stated in (29). \square

Corollary 8. *If (28) is evaluated at the midpoint, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \leq \begin{cases} \frac{(b-a)^2}{24} \|f''\|_\infty, & f'' \in L_\infty[a, b], \\ (b-a)^{1+(1/q)} \frac{\|f''\|_p}{8(2q+1)^{1/q}}, & f'' \in L_p[a, b], \\ p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ (b-a) \frac{\|f''\|_1}{8}, & f'' \in L_1[a, b]. \end{cases} \quad (30)$$

The above result can also be obtained by taking $\alpha = \beta$ in (29) or equivalently $\alpha = \beta$ and $x = (a+b)/2$ in (14).

3. Perturbed Results

In 1882, Čebyšev [20] gave the following inequality:

$$|T(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty, \quad (31)$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous functions, which have bounded first derivatives, and

$$\begin{aligned} T(f, g) &:= \frac{1}{b-a} \int_a^b f(x) g(x) dx \\ &\quad - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \\ &= M(fg; a, b) - M(f; a, b) M(g; a, b). \end{aligned} \quad (32)$$

In 1935, Grüss [21] proved the following inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma), \quad (33)$$

provided that f and g are two integrable functions on $[a, b]$ and satisfy the condition

$$\varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma, \quad \text{for all } x \in [a, b]. \quad (34)$$

The constant $1/4$ is the best possible. The perturbed version of the results of Theorem 4 can be obtained by using Grüss type results involving the Čebyšev functional

$$T(f, g) = M(fg; a, b) - M(f; a, b) M(g; a, b), \quad (35)$$

where M is the integral mean defined in (3). Let

$$\kappa := \frac{f'(b) - f'(a)}{b-a}, \quad (36)$$

$$\begin{aligned} N^2(x) &:= \left(\left(\frac{\alpha}{\alpha+\beta} \right)^2 \frac{(x-a)^3}{20} + \left(\frac{\beta}{\alpha+\beta} \right)^2 \frac{(b-x)^3}{20} \right) \frac{1}{b-a} \\ &\quad - \left(\frac{1}{6(b-a)(\alpha+\beta)} [\alpha(x-a)^2 + \beta(b-x)^2] \right)^2. \end{aligned} \quad (37)$$

Theorem 9. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a two times differentiable function, and α, β are nonnegative real numbers. Then,*

$$\begin{aligned} &\left| \tau(x; \alpha, \beta) - \frac{1}{(\alpha+\beta)} [\alpha(x-a)^2 + \beta(b-x)^2] \frac{\kappa}{6} \right| \\ &\leq (b-a) N(x) \left[\frac{1}{b-a} \|f''\|_2^2 - \kappa^2 \right]^{1/2} \\ &\leq \frac{(b-a)(\Gamma-\gamma)}{2(\alpha+\beta)} \max \left\{ \frac{\alpha(x-a)}{2}, \frac{\beta(b-x)}{2} \right\}, \end{aligned} \quad (38)$$

where $\tau(x; \alpha, \beta)$ is as given by (13).

Proof. Associating $f(t)$ with $P(x, t)$ and $g(t)$ with $f''(t)$, we obtain

$$\begin{aligned} &T(P(x, \cdot), f''(\cdot); a, b) \\ &= M(P(x, \cdot) f''(\cdot); a, b) \\ &\quad - M(P(x, \cdot); a, b) M(f''(\cdot); a, b). \end{aligned} \quad (39)$$

Now using identity (11),

$$\begin{aligned} &(b-a) T(P(x, \cdot), f''(\cdot); a, b) \\ &= \tau(x; \alpha, \beta) - (b-a) M(P(x, \cdot); a, b) \kappa, \end{aligned} \quad (40)$$

where κ is the secant slope of f' over $[a, b]$ as given in (36). Now, from (11) and (32),

$$\begin{aligned} &(b-a) M(P(x, \cdot); a, b) \\ &= \int_a^b P(x, t) dt \\ &= \frac{\alpha}{(\alpha+\beta)} \frac{1}{x-a} \int_a^x \frac{(t-a)^2}{2} dt \\ &\quad + \frac{\beta}{(\alpha+\beta)} \frac{1}{b-x} \int_x^b \frac{(t-b)^2}{2} dt \\ &= \frac{1}{6(\alpha+\beta)} [\alpha(x-a)^2 + \beta(b-x)^2]. \end{aligned} \quad (41)$$

Now, combining (41) with (37), the left-hand side of (38) is obtained.

Let $f, g : [a, b] \rightarrow \mathbb{R}$ and $fg : [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$; then [3],

$$\begin{aligned}
 |T(f, g)| &\leq T^{1/2}(f, f)T^{1/2}(g, g) \quad (f, g \in L_2[a, b]) \\
 &\leq \frac{(\Gamma - \gamma)}{2} T^{1/2}(f, f) \quad (\gamma \leq g(x) \leq \Gamma, t \in [a, b]) \\
 &\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) \quad (\varphi \leq f(x) \leq \Phi, t \in [a, b]).
 \end{aligned} \tag{42}$$

Also, note that

$$\begin{aligned}
 0 &\leq T^{1/2}(f''(\cdot), f''(\cdot)) \\
 &= [M(f''(\cdot)^2; a, b) - M^2(f''(\cdot); a, b)]^{1/2} \\
 &= \left[\frac{1}{b-a} \int_a^b |f''(t)|^2 dt - \left(\frac{\int_a^b f''(t) dt}{b-a} \right)^2 \right]^{1/2} \\
 &= \left[\frac{1}{b-a} \|f''\|_2^2 - \kappa^2 \right]^{1/2} \\
 &\leq \frac{(\Gamma - \gamma)}{2},
 \end{aligned} \tag{43}$$

where $\gamma \leq f''(t) \leq \Gamma, t \in [a, b]$. Now, for the bounds on (40), we have to determine $T^{1/2}(P(x, \cdot), P(x, \cdot))$ and ϕ and Φ such that $\varphi \leq P(x, \cdot) \leq \Phi$.

Now from (10), the definition of $P(x, t)$, we have

$$T(P(x, \cdot), P(x, \cdot)) = M(P^2(x, \cdot); a, b) - M^2(P(x, \cdot); a, b). \tag{44}$$

From (41) we obtain

$$\begin{aligned}
 M(P(x, \cdot); a, b) &= \frac{1}{6(b-a)(\alpha + \beta)} [\alpha(x-a)^2 + \beta(b-x)^2], \\
 (b-a)M(P^2(x, \cdot); a, b) &= \left(\frac{\alpha}{\alpha + \beta}\right)^2 \frac{1}{4(x-a)^2} \int_a^x (t-a)^4 dt \\
 &\quad + \left(\frac{\beta}{\alpha + \beta}\right)^2 \frac{1}{4(b-x)^2} \int_x^b (t-b)^4 dt \\
 &= \left(\frac{\alpha}{\alpha + \beta}\right)^2 \frac{(x-a)^3}{20} + \left(\frac{\beta}{\alpha + \beta}\right)^2 \frac{(b-x)^3}{20}.
 \end{aligned} \tag{45}$$

Thus, substituting the above results into (44) gives

$$0 \leq N(x) := T^{1/2}(P(x, \cdot), P(x, \cdot)) \tag{46}$$

which is given explicitly by (37). Combining (40), (43), and (44) gives, from the first inequality in (42), the first inequality

in (38). Now utilizing the inequality in (43) produces the second result in (38). Further, it may be noticed from the definition of $P(x, t)$ in (10) that, for $\alpha, \beta \geq 0$ it gives, for

$$\Phi = \sup_{t \in [a, b]} P(x, t), \quad \varphi = \inf_{t \in [a, b]} P(x, t), \tag{47}$$

the values

$$\Phi = \frac{1}{(\alpha + \beta)} \max \left\{ \frac{\alpha(x-a)}{2}, \frac{\beta(b-x)}{2} \right\}, \quad \varphi = 0. \tag{48}$$

□

4. An Application to the Cumulative Distribution Function

Let $X \in [a, b]$ be a random variable with the cumulative distributive function

$$F(x) = P_r(X \leq x) = \int_a^x f(u) du, \tag{49}$$

where f is the probability density function. In particular,

$$\int_a^b f(u) du = 1. \tag{50}$$

The following theorem holds.

Theorem 10. *Let X and F be as above; then,*

$$\begin{aligned}
 &\left| \frac{1}{2} (x-a)(b-x) [\alpha(x-a) - \beta(b-x)] \right. \\
 &\quad \times f'(x) + [\alpha(b-x) - \beta(x-a)] F(x) \\
 &\quad \left. - (x-a) [(\alpha + \beta)(b-x)f(x) - \beta] \right| \\
 &\leq \begin{cases} \left[\alpha(x-a)^3(b-x) + \beta(b-x)^3(x-a) \right. \\ \quad \times \frac{\|f''\|_\infty}{6}, \quad f'' \in L_\infty[a, b], \\ \frac{1}{(2q+1)^{1/q}} [\alpha^q(b-x)(x-a)^{q+2} \\ \quad \left. + \beta^q(x-a)(b-x)^{q+2}]^{1/q} \frac{\|f''\|_p}{2}, \\ \quad f'' \in L_p[a, b], \\ \quad p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ (b-x)(x-a)(\alpha(x-a) + \beta(b-x)) \\ \quad \left. + |\alpha(x-a) - \beta(b-x)| \frac{\|f''\|_1}{4}, \right. \\ \quad \left. f'' \in L_1[a, b]. \right. \end{cases} \tag{51}
 \end{aligned}$$

Proof. From (13), and by using the definition of probability density function, we have

$$\begin{aligned} \tau(x; \alpha, \beta) &= \frac{1}{2(\alpha + \beta)} [\alpha(x - a) - \beta(b - x)] f'(x) - f(x) \\ &\quad + \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)] \\ &= \frac{1}{2(\alpha + \beta)} [\alpha(x - a) - \beta(b - x)] f'(x) - f(x) \\ &\quad + \frac{1}{\alpha + \beta} \left\{ \left[\frac{\alpha(b - x) - \beta(x - a)}{(x - a)(b - x)} \right] F(x) + \frac{\beta}{(b - x)} \right\} \end{aligned} \tag{52}$$

or

$$\begin{aligned} &(\alpha + \beta)(x - a)(b - x)\tau(x; \alpha, \beta) \\ &= \frac{(x - a)(b - x)}{2} [\alpha(x - a) - \beta(b - x)] f'(x) \\ &\quad - (\alpha + \beta)(x - a)(b - x)f(x) \\ &\quad + \{[\alpha(b - x) - \beta(x - a)]F(x) + \beta(x - a)\}. \end{aligned} \tag{53}$$

Now using (14) and (53), we get our required result (51). \square

Putting $\alpha = \beta = 1/2$ in Theorem 10 gives the following result.

Corollary 11. *Let X be a random variable with the cumulative distributive function $F(x)$ and the probability density function $f(x)$. Then,*

$$\begin{aligned} &\left| \left(x - \frac{a+b}{2} \right) \left[\frac{1}{2}(x - a)(b - x)f'(x) - F(x) \right] \right. \\ &\quad \left. - (x - a) \left[(b - x)f(x) - \frac{1}{2} \right] \right| \\ &\leq \begin{cases} \left[(x - a)^3(b - x) + (b - x)^3(x - a) \right] \frac{\|f''\|_\infty}{12}, & f'' \in L_\infty[a, b], \\ \frac{1}{(2q + 1)^{1/q}} \times [(b - x)(x - a)^{q+2} + (x - a)(b - x)^{q+2}]^{1/q} \\ \quad \times \frac{\|f''\|_p}{4}, & f'' \in L_p[a, b], p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ (b - x)(x - a) \left(b - a + \left| x - \frac{a+b}{2} \right| \right) \frac{\|f''\|_1}{8}, & f'' \in L_1[a, b]. \end{cases} \end{aligned} \tag{54}$$

Remark 12. The above result allows the approximation of $F(x)$ in terms of $f(x)$. The approximation of

$$R(x) = 1 - F(x) \tag{55}$$

could also be obtained by a simple substitution. $R(x)$ is of importance in reliability theory, where $f(x)$ is the probability density function of failure.

Remark 13. We put $\beta = 0$ in (51), assuming that $\alpha \neq 0$, to obtain

$$\begin{aligned} &\left| F(x) + (x - a) \left[\frac{1}{2}(x - a)f'(x) - f(x) \right] \right| \\ &\leq \begin{cases} (x - a)^3 \frac{\|f''\|_\infty}{6}, & f'' \in L_\infty[a, b], \\ \frac{1}{(2q + 1)^{1/q}} (b - x)^{(1/q)-1} (x - a)^{1+(2/q)} \frac{\|f''\|_p}{2}, & f'' \in L_p[a, b], \\ & p > 1, \frac{1}{q} + \frac{1}{p} = 1, \\ (x - a)^2 \frac{\|f''\|_1}{2}, & f'' \in L_1[a, b]. \end{cases} \end{aligned} \tag{56}$$

Further we note that

$$\int_a^b F(u) du = uF(u)|_a^b - \int_a^b xf(x) dx = b - E(X). \tag{57}$$

5. Conclusions

In this paper, we obtained bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval in terms of the L_p norms at the second derivative. The obtained results are applied to approximate the cumulative distribution function. Our results further extend previous inequalities developed in [3, 14–17]. Some perturbed results with the help of Čebyšev and Grüss inequalities are also obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank Professor Elena berdysheva for her useful suggestions which improved the presentation of the paper.

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