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Research Article

Iterative Algorithms for Mixed Equilibrium Problems, System of Quasi-Variational Inclusion, and Fixed Point Problem in Hilbert Spaces

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We introduce a new iterative algorithm for approximating a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Strong convergence of the proposed iterative algorithm is obtained. Our results generalize, extend, and improve the results of Peng and Yao, 2009, Qin et al. 2010 and many authors.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. They use $F(T)$ to denote the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in [0, 1)$, and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [1] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{MEP}(F, \varphi)$. We see that x is a solution of problem (1) which implies that $x \in \operatorname{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi \equiv 0$, then the mixed equilibrium problem (1) becomes the following *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $\mathrm{EP}(F)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (2). Some methods have been proposed to solve the equilibrium problem (see [2–14]).

Let $B : C \to H$ be a mapping. The *variational inequality problem,* denoted by $\mathrm{VI}(C, B)$, is for finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \tag{3}$$

for all $y \in C$. The variational inequality problem has been extensively studied in the literature. See, for example, [15, 16] and the references therein. A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \tag{4}$$

for all $x, y \in C$. B is called *β-inverse-strongly monotone* if there exists a positive real number $\beta > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2. \tag{5}$$

We consider a *system of quasi-variational inclusion* for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (B_1 y^* + M_1 x^*), \\ \theta \in y^* - x^* + \rho_2 (B_2 x^* + M_2 y^*), \end{gathered} \tag{6}$$

where $B_i : H \to H$ and $M_i : H \to 2^H$ are nonlinear mappings for each $i = 1, 2$. The set of solutions of problem (6) is denoted by $\mathrm{SQVI}(B_1, M_1, B_2, M_2)$. As special cases of problem (6), we have the following.

(1) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then problem (6) is reduced to (7) for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (By^* + Mx^*), \\ \theta \in y^* - x^* + \rho_2 (Bx^* + My^*). \end{gathered} \tag{7}$$

(2) Further, if $x^* = y^*$, then problem (7) is reduced to (8) for finding $x^* \in H$ such that

$$\theta \in Bx^* + Mx^*, \tag{8}$$

where θ is the zero vector in H. The set of solutions of problem (8) is denoted by $I(B, M)$. A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$. Let B be a monotone mapping of C into H and let $N_C \overline{y}$ be the *normal cone* to C at $\overline{y} \in C$; that is, $N_C \overline{y} = \{w \in H : \langle u - \overline{y}, w \rangle \leq 0, \forall u \in C\}$, and define

$$M\overline{y} = \begin{cases} B\overline{y} + N_C \overline{y}, & \overline{y} \in C; \\ \emptyset, & \overline{y} \notin C. \end{cases} \tag{9}$$

Then, M is the *maximal monotone* and $\theta \in M\overline{y}$ if and only if $\overline{y} \in \mathrm{VI}(C, B)$; see [17].

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping; then, the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda} x^* = (I + \lambda M)^{-1} x^*, \quad x^* \in H \tag{10}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. The following characterizes the resolvent operator.

(R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$; that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H, \ \forall \lambda > 0. \tag{11}$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; see [18]; that is,

$$\begin{aligned} &\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \\ &\quad \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle, \quad \forall x, y \in H. \end{aligned} \tag{12}$$

(R3) The solution of problem (8) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also [19]; that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{13}$$

(R4) If $0 < \lambda \leq 2\beta$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \to H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Let A be a strongly positive linear bounded operator on H; that is, there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{14}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{15}$$

where A is a strongly positive linear bounded operator and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Plubtieng and Punpaeng [20] proposed the following iterative algorithm:

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) T u_n. \end{gathered} \tag{16}$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f) z, x - z \rangle \geq 0, \quad \forall x \in F(T) \cap \mathrm{EP}(F), \tag{17}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)\cap \mathrm{EP}(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \quad (18)$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [21] introduced an iterative algorithm based on extragradient method which solves the problem for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings, and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. The sequences generated by $v \in C$ are

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \\ \forall y \in C, \\ y_n = P_C(u_n - \gamma_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \lambda_n B y_n), \end{gathered} \quad (19)$$

for all $n \ge 1$, where W_n is W-mapping. They proved the strong convergence theorems under some mild conditions.

In 2010, Qin et al. [22] introduced an iterative method for finding solutions of a generalized equilibrium problem, the set of fixed points of a family of nonexpansive mappings, and the common variational inclusions. The sequences generated by $x_1 \in C$ and $\{x_n\}$ are a sequence generated by

$$\begin{gathered} F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \\ \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \ge 1, \end{gathered} \quad (20)$$

where f is a contraction and A_i is inverse-strongly monotone mappings for $i = 1, 2, 3$ and W_n is called a W-mapping generated by $S_n, S_{n_1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$. They proved the strong convergence theorems under some mild conditions. Liou [23] introduced an algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of variational inclusion in a real Hilbert space. The sequences generated by $x_0 \in C$ are

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) \\ + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \ge 0, \quad \forall y \in C, \quad (21) \\ x_{n+1} = P_C[(1 - \alpha_n A) J_{M,\lambda}(u_n - \lambda B u_n)], \end{gathered}$$

for all $n \ge 1$, where A is a strongly positive bounded linear operator and B, Q are inverse-strongly monotone. They proved the strong convergence theorems under some suitable conditions.

Next, Petrot et al. [24] introduced the new following iterative process for finding the set of solutions of quasi-variational inclusion problem and the set of fixed point of a nonexpansive mapping. The sequence is generated by

$$\begin{gathered} x_0 \in H, \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n, \\ z_n = J_{M,\lambda}(y_n - \lambda A y_n), \\ y_n = J_{M,\rho}(x_n - \rho A x_n), \end{gathered} \quad (22)$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$. They proved that $\{x_n\}$ generated by (22) converges strongly to z_0 which is the unique solution in $F(S) \cap I(A, M)$.

In 2011, Jitpeera and Kumam [25] introduced a shrinking projection method for finding the common element of the common fixed points of nonexpansive semigroups, the set of common fixed point for an infinite family, the set of solutions of a system of mixed equilibrium problems, and the set of solution of the variational inclusion problem. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C, C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$, and

$$\begin{gathered} x_0 = x \in C \quad \text{chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \cdots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \left\{ z \in C_n : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n (1 - \alpha_n) \right. \\ \left. \times \left\| v_n - \frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n ds \right\|^2 \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{gathered} \quad (23)$$

where $K^{F_k}_{r_k} : C \to C$, $k = 1, 2, \ldots, N$. We proved the strong convergence theorem under certain appropriate conditions.

In this paper, motivated by the above results, we introduce a new iterative method for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusions, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Then, we prove strong convergence theorems which are connected with [5, 26–29]. Our results extend and improve the corresponding results of

Jitpeera and Kumam [25], Liou [23], Plubtieng and Punpaeng [20], Petrot et al. [24], Peng and Yao [21], Qin et al. [22], and some authors.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then,

$$\begin{gathered}\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle,\\ \|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\\ \times\|x-y\|^2,\quad\forall x,y\in H,\ \lambda\in[0,1].\end{gathered}\tag{24}$$

For every point $x\in H$, there exists a unique *nearest point* in C, denoted by P_Cx, such that

$$\|x-P_Cx\|\le\|x-y\|,\quad\forall y\in C.\tag{25}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y,P_Cx-P_Cy\rangle\ge\|P_Cx-P_Cy\|^2,\quad\forall x,y\in H.\tag{26}$$

Moreover, P_Cx is characterized by the following properties: $P_Cx\in C$ and

$$\langle x-P_Cx,y-P_Cx\rangle\le0,\tag{27}$$

$$\|x-y\|^2\ge\|x-P_Cx\|^2+\|y-P_Cx\|^2,\quad\forall x\in H,\ y\in C.\tag{28}$$

Let B be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (27) implies the following:

$$u\in\mathrm{VI}(C,B)\Longleftrightarrow u=P_C(u-\lambda Bu),\quad\lambda>0.\tag{29}$$

It is also known that H satisfies the Opial condition [30]; that is, for any sequence $\{x_n\}\subset H$ with $x_n\rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|\tag{30}$$

holds for every $y\in H$ with $x\neq y$.

For the infinite family of nonexpansive mappings of $T_1,T_2,\ldots$, and sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $[0,1)$, see [31]; we define the mapping W_n of C into itself as follows:

$$\begin{aligned}U_{n,0}&=I,\\ U_{n,1}&=\lambda_1T_1U_{n,0}+(1-\lambda_1)U_{n,0},\\ U_{n,2}&=\lambda_2T_2U_{n,1}+(1-\lambda_2)U_{n,1},\\ &\vdots\\ U_{n,N-1}&=\lambda_{N-1}T_{N-1}U_{n,N-2}+(1-\lambda_{N-1})U_{n,N-2},\\ W_n=U_{n,N}&=\lambda_NT_NU_{n,N-1}+(1-\lambda_N)U_{n,N-1}.\end{aligned}\tag{31}$$

Lemma 1 (Shimoji and Takahashi [32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{T}=\{T_i\}_{i=1}^{N}$ be a family of infinitely nonexpanxive mappings with $F(\mathcal{T})=\bigcap_{i=1}^{\infty}F(T_i)\neq\emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0<\lambda_i\le b<1$ for every $i\ge1$. Then*

(1) *W_n is nonexpansive and $F(W_n)=\bigcap_{i=1}^{n}F(T_i)$ for each $n\ge1$;*

(2) *for each $x\in C$ and for each positive integer k, the limit $\lim_{n\to\infty}U_{n,k}x$ exists;*

(3) *the mapping $W:C\to C$ defined by $Wx=\lim_{n\to\infty}W_nx=\lim_{n\to\infty}U_{n,1}x$ is a nonexpansive mapping satisfying $F(W)=F(\mathcal{T})$ and it is called the W-mapping generated by $T_1,T_2,\ldots$, and $\lambda_1,\lambda_2,\ldots$;*

(4) *if K is any bounded subset of C, then $\lim_{n\to\infty}\sup_{x\in K}\|Wx-W_nx\|=0$.*

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction $F:C\times C\to\mathbb{R}$ and a proper extended real-valued function $\varphi:C\to\mathbb{R}\cup\{+\infty\}$ satisfies the following conditions:

(A1) $F(x,x)=0$ for all $x\in C$;

(A2) F is monotone; that is, $F(x,y)+F(y,x)\le0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$, $\lim_{t\to0}F(tz+(1-t)x,y)\le F(x,y)$;

(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous;

(A5) for each $y\in C$, $x\mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x\in H$ and $r>0$, there exist a bounded subset $D_x\subseteq C$ and $y_x\in C$ such that for any $z\in C\setminus D_x$,

$$F(z,y_x)+\varphi(y_x)+\frac{1}{r}\langle y_x-z,z-x\rangle<\varphi(z);\tag{32}$$

(B2) C is a bounded set.

We need the following lemmas for proving our main results.

Lemma 2 (Peng and Yao [21]). *Let C be a nonempty closed convex subset of H. Let $F:C\times C\to\mathbb{R}$ be a bifunction that satisfies (A1)–(A5) and let $\varphi:C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r>0$ and $x\in H$, define a mapping $T_r:H\to C$ as follows:*

$$\begin{aligned}T_r(x)=\Big\{z\in C:F(z,y)+\varphi(y)\\ +\frac{1}{r}\langle y-z,z-x\rangle\ge\varphi(z),\forall y\in C\Big\},\end{aligned}\tag{33}$$

for all $x\in H$. Then, the following hold:

(1) *for each $x\in H,T_r(x)\neq\emptyset$;*

(2) *T_r is single-valued;*

(3) T_r *is firmly nonexpansive; that is, for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(4) $F(T_r) = MEP(F, \varphi)$;

(5) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 3 (Xu [33]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \quad n \geq 0, \tag{34}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (Suzuki [34]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 5 (Marino and Xu [35]). *Assume* A *is a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then,* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 6. *For given* $x^*, y^* \in C \times C$, (x^*, y^*) *is a solution of problem* (6) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right], \quad \forall x \in C, \tag{35}$$

where $y^* = J_{M_2,\mu}(x - \mu E_2 x)$, λ, μ *are positive constants, and* $E_1, E_2 : C \to H$ *are two mappings.*

Proof.

$$\begin{aligned} \theta &\in x^* - y^* + \lambda(E_1 y^* + M_1 x^*), \\ \theta &\in y^* - x^* + \mu(E_2 x^* + M_2 y^*) \end{aligned} \tag{36}$$

$\Leftrightarrow$

$$\begin{aligned} x^* &= J_{M_1,\lambda}(y^* - \lambda E_1 y^*), \\ y^* &= J_{M_2,\mu}(x^* - \mu E_2 x^*) \end{aligned} \tag{37}$$

$\Leftrightarrow$

$$\begin{aligned} G(x^*) = J_{M_1,\lambda}&\left[J_{M_2,\mu}(x^* - \mu E_2 x^*)\right. \\ &\left.-\lambda E_1 J_{M_2,\mu}(x^* - \mu E_2 x^*)\right] = x^*. \end{aligned} \tag{38}$$

This completes the proof. □

Now, we prove the following lemmas which will be applied in the main theorem.

Lemma 7. *Let* $G : C \to C$ *be defined as in Lemma 6. If* $E_1, E_2 : C \to H$ *is* η_1, η_2*-inverse-strongly monotone and* $\lambda \in (0, 2\eta_1)$, *and* $\mu \in (0, 2\eta_2)$, *respectively, then* G *is nonexpansive.*

Proof. For any $x, y \in C$ and $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, we have

$$\begin{aligned}
&\|G(x) - G(y)\|^2 \\
&= \left\| J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right]\right. \\
&\qquad \left. - J_{M_1,\lambda}\left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\right\|^2 \\
&\leq \left\|\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right]\right. \\
&\qquad \left. - \left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\right\|^2 \\
&= \left\|\left[J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right]\right. \\
&\qquad \left. - \lambda\left[E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right]\right\|^2 \\
&= \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad - 2\lambda\left\langle J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y),\right. \\
&\qquad\qquad \left. E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\rangle \\
&\quad + \lambda^2\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad - 2\lambda\eta_1\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad + \lambda^2\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&= \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\left\|E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \left\|J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right\|^2 \\
&\leq \left\|(x - \mu E_2 x) - (y - \mu E_2 y)\right\|^2 \\
&= \left\|(x - y) - \mu(E_2 x - E_2 y)\right\|^2 \\
&= \|x - y\|^2 - 2\mu\langle x - y, E_2 x - E_2 y\rangle + \mu^2\|E_2 x - E_2 y\|^2 \\
&\leq \|x - y\|^2 - 2\eta_2\mu\|E_2 x - E_2 y\|^2 + \mu^2\|E_2 x - E_2 y\|^2 \\
&= \|x - y\|^2 + \mu(\mu - 2\eta_2)\|E_2 x - E_2 y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{39}$$

This shows that G is nonexpansive on C. □

3. Main Results

In this section, we show a strong convergence theorem for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of a infinite family of nonexpansive mappings in a real Hilbert space.

Theorem 8. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \varphi(y) - \varphi(u_n) \\ & \quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \\ & \qquad z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & \qquad y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ & \qquad \forall n \geq 0, \end{aligned} \tag{40}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$,$P_\Theta$ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Let $x^* \in \Theta$; that is $T_r(x^* - rQx^*) = J_{M_1,\lambda}[J_{M_2,\mu}(x^* - \mu B_2 x^*) - \lambda B_1 J_{M_2,\mu}(x^* - \mu B_2 x^*)] = T_i(x^*) = x^*, i \geq 1$. Putting $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$, one can see that $x^* = J_{M_1,\lambda}(y^* - \lambda B_1 y^*)$.

We divide our proofs into the following steps:

(1) sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded;

(2) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(3) $\lim_{n\to\infty} \|Qx_n - Qx^*\| = 0$, $\lim_{n\to\infty} \|E_1 z_n - E_1 x^*\| = 0$ and $\lim_{n\to\infty} \|E_2 u_n - E_2 x^*\| = 0$;

(4) $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;

(5) $\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0$, where $x^* = P_\Theta(\gamma f + I - A)x^*$;

(6) $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Step 1. From conditions (C1) and (C2), we may assume that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. By the same argument as that in [9], we can deduce that $(1 - \beta_n)I - \alpha_n A$ is positive and $\|(1 - \beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \overline{\gamma}$. For all $x, y \in C$ and $r \in (0, 2\delta)$. since Q is a δ-inverse-strongly monotone and B_1, B_2 are η_1, η_2-inverse-strongly monotone, we have

$$\begin{aligned} & \|(I - rQ)x - (I - rQ)y\|^2 \\ & \quad = \|(x - y) - r(Qx - Qy)\|^2 \\ & \quad = \|x - y\|^2 - 2r\langle x - y, Qx - Qy\rangle + r^2\|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2 - 2r\delta\|Qx - Qy\|^2 + r^2\|Qx - Qy\|^2 \\ & \quad = \|x - y\|^2 + r(r - 2\delta)\|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2. \end{aligned} \tag{41}$$

It follows that $\|(I - rQ)x - (I - rQ)y\| \leq \|x - y\|$; hence $I - rQ$ is nonexpansive.

In the same way, we conclude that $\|(I - \lambda E_1)x - (I - \lambda E_1)y\| \leq \|x - y\|$ and $\|(I - \mu E_2)x - (I - \mu E_2)y\| \leq \|x - y\|$; hence $I - \lambda E_1, I - \mu E_2$ are nonexpansive. Let $y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n)$, $n \geq 0$. It follows that

$$\begin{aligned} \|y_n - x^*\| & = \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(y^* - \lambda E_1 y^*)\| \\ & \leq \|(z_n - \lambda E_1 z_n) - (y^* - \lambda E_1 y^*)\| \\ & \leq \|z_n - y^*\|, \\ \|z_n - y^*\| & = \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\| \\ & \leq \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\| \\ & \leq \|u_n - x^*\|. \end{aligned} \tag{42}$$

By Lemma 2, we have $u_n = T_r(x_n - rQx_n)$ for all $n \geq 0, \forall x, y \in C$. Then, for $r \in (0, 2\delta)$, we obtain

$$\begin{aligned} \|u_n - x^*\|^2 & = \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\ & \leq \|(x_n - rQx_n) - (x^* - rQx^*)\|^2 \\ & \leq \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\ & \leq \|x_n - x^*\|^2. \end{aligned} \tag{43}$$

Hence, we have

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{44}$$

From (40) and (44), we deduce that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n(x_n-x^*)\\&\quad+((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*)\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x^*\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\|f(x_n)-f(x^*)\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\alpha\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&=(1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-x^*\|\\&\quad+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\\&\le\max\left\{\|x_n-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\}.\end{aligned}\tag{45}$$

It follows by mathematical induction that

$$\|x_{n+1}-x^*\|\le\max\left\{\|x_0-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\},\quad n\ge 0.\tag{46}$$

Hence, $\{x_n\}$ is bounded and also $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{W_ny_n\}$, $\{AW_ny_n\}$, and $\{fx_n\}$ are all bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Putting $t_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)=(\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n)/(1-\beta_n)$, we get $x_{n+1}=(1-\beta_n)t_n+\beta_nx_n$, $n\ge1$. We note that

$$\begin{aligned}t_{n+1}-t_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)\\&\quad+W_{n+1}y_{n+1}-W_ny_n\\&\quad-\frac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}y_{n+1}+\frac{\alpha_n}{1-\beta_n}AW_ny_n\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-AW_{n+1}y_{n+1})\\&\quad+\frac{\alpha_n}{1-\beta_n}(AW_ny_n-\gamma f(x_n))\\&\quad+W_{n+1}y_{n+1}-W_{n+1}y_n+W_{n+1}y_n-W_ny_n.\end{aligned}\tag{47}$$

It follows that

$$\begin{aligned}&\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\&\qquad+\|W_{n+1}y_n-W_ny_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|y_{n+1}-y_n\|+\|W_{n+1}y_n-W_ny_n\|\\&\qquad-\|x_{n+1}-x_n\|.\end{aligned}\tag{48}$$

By the definition of W_n,

$$\begin{aligned}&\|W_{n+1}y_n-W_ny_n\|\\&\quad=\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n+(1-\lambda_{n+1,N})y_n\\&\qquad-\lambda_{n,N}T_NU_{n,N-1}y_n-(1-\lambda_{n,N})y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n-\lambda_{n,N}T_NU_{n,N-1}y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}(T_NU_{n+1,N-1}y_n-T_NU_{n,N-1}y_n)\|\\&\qquad+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}y_n\|\\&\quad\le2M|\lambda_{n+1,N}-\lambda_{n,N}|\\&\qquad+\lambda_{n+1,N}\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|,\end{aligned}\tag{49}$$

where M is an approximate constant such that $M\ge\max\{\sup_{n\ge1}\{\|y_n\|\},\sup_{n\ge1}\{\|T_mU_{n,m-1}y_n\|\}\mid m=1,2,\ldots,N\}$. Since $0<\lambda_{n_i}\le1$ for all $n\ge1$ and $i=1,2,\ldots,N$, we compute

$$\begin{aligned}&\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|\\&\quad=\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n+(1-\lambda_{n+1,N-1})y_n\\&\qquad-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n-(1-\lambda_{n,N-1})y_n\|\end{aligned}$$

$$\begin{aligned}&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|y_n\|\\&\quad+\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n\|\\&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|y_n\|\\&\quad+\|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}y_n-T_{N-1}U_{n,N-2}y_n)\|\\&\quad+|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|T_{N-1}U_{n,N-2}y_n\|\\&\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\|U_{n+1,N-2}y_n-U_{n,N-2}y_n\|.\end{aligned}\tag{50}$$

It follows that

$$\begin{aligned}&\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|\\&\quad\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}|\\&\qquad+\|U_{n+1,N-3}y_n-U_{n,N-3}y_n\|\\&\quad\leq 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|+\|U_{n+1,1}y_n-U_{n,1}y_n\|\\&\quad=2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\\&\qquad\qquad+\|\lambda_{n+1,1}T_1y_n+(1-\lambda_{n+1,1})y_n\\&\qquad\qquad\quad-\lambda_{n,1}T_1y_n-(1-\lambda_{n,1})y_n\|\\&\quad\leq 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|.\end{aligned}\tag{51}$$

Substituting (51) into (49),

$$\begin{aligned}&\|W_{n+1}y_n-W_ny_n\|\\&\quad\leq 2M|\lambda_{n+1,N}-\lambda_{n,N}|+2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\\&\quad\leq 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.\end{aligned}\tag{52}$$

We note that

$$\begin{aligned}&\|y_{n+1}-y_n\|\\&\quad=\|J_{M_1,\lambda}(z_{n+1}-\lambda E_1z_{n+1})-J_{M_1,\lambda}(z_n-\lambda E_1z_n)\|\\&\quad\leq\|(z_{n+1}-\lambda E_1z_{n+1})-(z_n-\lambda E_1z_n)\|\\&\quad\leq\|z_{n+1}-z_n\|\\&\quad=\|J_{M_2,\mu}(u_{n+1}-\mu E_2u_{n+1})-J_{M_2,\mu}(u_n-\mu E_2u_n)\|\\&\quad\leq\|(u_{n+1}-\mu E_2u_{n+1})-(u_n-\mu E_2u_n)\|\\&\quad\leq\|u_{n+1}-u_n\|\\&\quad=\|T_r(x_{n+1}-rDx_{n+1})-T_r(x_n-rDx_n)\|\\&\quad\leq\|(x_{n+1}-rDx_{n+1})-(x_n-rDx_n)\|\\&\quad\leq\|x_{n+1}-x_n\|.\end{aligned}\tag{53}$$

Applying (52) and (53) in (48), we get

$$\begin{aligned}&\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|\\&\quad\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)+\|x_{n+1}-x_n\|\\&\qquad+2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|-\|x_{n+1}-x_n\|.\end{aligned}\tag{54}$$

By conditions (C1)–(C3), imply that

$$\limsup_{n\to\infty}(\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|)\leq 0.\tag{55}$$

Hence, by Lemma 4, we obtain

$$\lim_{n\to\infty}\|t_n-x_n\|=0.\tag{56}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|t_n-x_n\|=0.\tag{57}$$

We obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{58}$$

Step 3. We can rewrite (40) as $x_{n+1}=\alpha_n(\gamma f(x_n)-AW_ny_n)+\beta_n(x_n-W_ny_n)+W_ny_n$. We observe that

$$\begin{aligned}\|x_n-W_ny_n\|&\leq\|x_n-x_{n+1}\|+\|x_{n+1}-W_ny_n\|\\&\leq\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AW_ny_n\|\\&\quad+\beta_n\|x_n-W_ny_n\|;\end{aligned}\tag{59}$$

it follows that

$$\begin{aligned}&\|x_n-W_ny_n\|\\&\quad\leq\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-AW_ny_n\|.\end{aligned}\tag{60}$$

By conditions (C1), (C2), and (58), imply that

$$\lim_{n\to\infty}\|x_n-W_ny_n\|=0.\tag{61}$$

From (42) and (43), we get

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(x^* - \lambda E_1 x^*)\|^2 \\
&\le \|(z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*)\|^2 \\
&\le \|z_n - x^*\|^2 + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|u_n - x^*\|^2 + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\
&\quad + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2.
\end{aligned} \tag{62}$$

By (40), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - W_n y_n) \\
&\quad + (I - \alpha_n A)(W_n y_n - x^*)\|^2 \\
&\le \|(I - \alpha_n A)(W_n y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle \\
&\le \|(I - \alpha_n A)(y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\| \\
&= (1 - \alpha_n\overline{\gamma})^2\|y_n - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{63}$$

Substituting (62) into (63), imply that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 + r(r - 2\delta)\|Qx_n - Qx^*\|^2 \\
&\quad + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{64}$$

Thus,

$$\begin{aligned}
&r(2\delta - r)\|Qx_n - Qx^*\|^2 + \mu(2\eta_2 - \mu)\|E_2 u_n - E_2 x^*\|^2 \\
&\quad + \lambda(2\eta_1 - \lambda)\|E_1 z_n - E_1 x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\| \\
&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\quad + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{65}$$

By conditions (C1), (C2), (58), and (61), we deduce immediately that

$$\begin{aligned}
\lim_{n\to\infty}\|Qx_n - Qx^*\| &= \lim_{n\to\infty}\|E_1 z_n - E_1 x^*\| \\
&= \lim_{n\to\infty}\|E_2 u_n - E_2 x^*\| = 0.
\end{aligned} \tag{66}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Since T_r is firmly nonexpansive, we have

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&= \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\
&\le \langle (x_n - rQx_n) - (x^* - rQx^*), u_n - x^*\rangle \\
&= \frac{1}{2}\left\{\|(x_n - rQx_n) - (x^* - rQx^*)\|^2 + \|u_n - x^*\|^2\right\} \\
&\quad - \frac{1}{2}\left\{\|(x_n - rQx_n) - (x^* - rQx^*) - (u_n - x^*)\|^2\right\} \\
&= \frac{1}{2}\left\{\|x_n - x^*\|^2 + \|u_n - x^*\|^2 \right. \\
&\qquad \left. - \|(x_n - u_n) - r(Qx_n - Qx^*)\|^2\right\}
\end{aligned}$$

$$= \frac{1}{2}\left\{\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \left(\|x_n - u_n\|^2 + r^2\|Qx_n - Qx^*\|^2 - 2r\langle x_n - u_n, Qx_n - Qx^*\rangle\right)\right\}$$
$$\leq \frac{1}{2}\left\{\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - r^2\|Qx_n - Qx^*\|^2 + 2r\|x_n - u_n\|\|Qx_n - Qx^*\|\right\}, \quad (67)$$

which implies that

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Qx_n - Qx^*\|. \quad (68)$$

Since $J_{M_1,\lambda}$ is 1-inverse-strongly monotone, we have

$$\|y_n - x^*\|^2$$
$$= \left\|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(x^* - \lambda E_1 x^*)\right\|^2$$
$$\leq \langle (z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*), y_n - x^*\rangle$$
$$= \frac{1}{2}\left\{\|(z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*)\|^2 + \|y_n - x^*\|^2\right\}$$
$$- \frac{1}{2}\left\{\|(z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*) - (y_n - x^*)\|^2\right\}$$
$$= \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|(z_n - y_n) - \lambda(E_1 z_n - E_1 x^*)\|^2\right\}$$
$$= \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \left(\|z_n - y_n\|^2 + \lambda^2\|E_1 z_n - E_1 x^*\|^2 - 2\lambda\langle z_n - y_n, E_1 z_n - E_1 x^*\rangle\right)\right\}$$
$$\leq \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|z_n - y_n\|^2 - \lambda^2\|E_1 z_n - E_1 x^*\|^2 + 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\|\right\}, \quad (69)$$

which implies that

$$\|y_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\|. \quad (70)$$

In the same way with (70), we can get

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\mu\|u_n - z_n\|\|E_2 u_n - E_2 x^*\|. \quad (71)$$

Substituting (71) into (70), imply that

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\mu\|u_n - z_n\|\|E_2 u_n - E_2 x^*\| - \|z_n - y_n\|^2 + 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\|. \quad (72)$$

Again, substituting (68) into (72), we get

$$\|y_n - x^*\|^2$$
$$\leq \left\{\|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Qx_n - Qx^*\|\right\}$$
$$- \|u_n - z_n\|^2 + 2\mu\|u_n - z_n\|\|E_2 u_n - E_2 x^*\| - \|z_n - y_n\|^2$$
$$+ 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\|. \quad (73)$$

Substituting (73) into (63), imply that

$$\|x_{n+1} - x^*\|^2$$
$$\leq (1 - \alpha_n\overline{\gamma})^2\left\{\|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Qx_n - Qx^*\| - \|u_n - z_n\|^2 + 2\mu\|u_n - z_n\|\|E_2 u_n - E_2 x^*\| - \|z_n - y_n\|^2 + 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\|\right\}$$
$$+ \beta_n^2\|x_n - W_n y_n\|^2$$
$$+ 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\|$$
$$+ 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|. \quad (74)$$

Then, we derive

$$(1 - \alpha_n\overline{\gamma})^2\left(\|x_n - u_n\|^2 + \|u_n - z_n\|^2 + \|z_n - y_n\|^2\right)$$
$$\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r\|x_n - u_n\|\|Qx_n - Qx^*\|$$
$$+ 2\mu\|u_n - z_n\|\|E_2 u_n - E_2 x^*\|$$
$$+ 2\lambda\|z_n - y_n\|\|E_1 z_n - E_1 x^*\| + \beta_n^2\|x_n - W_n y_n\|^2$$
$$+ 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\|$$
$$+ 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|$$

$$\begin{aligned}&\leq \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right)\|x_{n+1} - x_n\| \\ &\quad + 2r\|x_n - u_n\|\|Qx_n - Qx^*\| \\ &\quad + 2\mu\|u_n - z_n\|\|E_2u_n - E_2x^*\| \\ &\quad + 2\lambda\|z_n - y_n\|\|E_1z_n - E_1x^*\| + \beta_n^2\|x_n - W_ny_n\|^2 \\ &\quad + 2\left(1 - \alpha_n\overline{\gamma}\right)\beta_n\|y_n - x^*\|\|x_n - W_ny_n\| \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|x_{n+1} - x^*\|.\end{aligned} \tag{75}$$

By conditions (C1), (C2), (58), (61), and (66), we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|u_n - z_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{76}$$

Observe that

$$\begin{aligned}\|W_ny_n - y_n\| \leq \|W_ny_n - x_n\| + \|x_n - u_n\| \\ + \|u_n - z_n\| + \|z_n - y_n\|.\end{aligned} \tag{77}$$

By (61) and (76), we have

$$\lim_{n\to\infty}\|W_ny_n - y_n\| = 0. \tag{78}$$

Note that

$$\|Wy_n - y_n\| \leq \|Wy_n - W_ny_n\| + \|W_ny_n - y_n\|. \tag{79}$$

From Lemma 1, we get

$$\lim_{n\to\infty}\|Wy_n - W_ny_n\| = 0. \tag{80}$$

By (78) and (80), we have $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. It follows that $\lim_{n\to\infty}\|Wx_n - x_n\| = 0$.

Step 5. We show that $\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(\gamma f + I - A)z$. It is easy to see that $P_\Theta(\gamma f + (I - A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \overline{\gamma}/\alpha$, we have

$$\begin{aligned}&\|P_\Theta(\gamma f + (I - A))x - P_\Theta(\gamma f + (I - A))y\| \\ &\quad \leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\ &\quad \leq \gamma\|f(x) - f(y)\| + |I - A|\|x - y\| \\ &\quad \leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &\quad = (1 - \overline{\gamma} + \gamma\alpha)\|x - y\|.\end{aligned} \tag{81}$$

Since H is complete, there exists a unique fixed point $z \in H$ such that $z = P_\Theta(\gamma f + I - A)(z)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, such that

$$\lim_{i\to\infty}\left\langle(A - \gamma f)z, z - x_{n_i}\right\rangle = \limsup_{n\to\infty}\left\langle(A - \gamma f)z, z - x_n\right\rangle. \tag{82}$$

Also, since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From $\|Wx_n - x_n\| \to 0$, we obtain $Wx_{n_i} \rightharpoonup w$. Then, by the demiclosed principle of nonexpansive mappings, we obtain $w \in \cap_{i=1}^{\infty}F(T_i)$.

Next, we show that $w \in \mathrm{MEP}(F, \varphi)$. Since $u_n = T_r(x_n - rQx_n)$, we obtain

$$\begin{aligned}&F(u_n, y) + \varphi(y) - \varphi(u_n) \\ &\quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C.\end{aligned} \tag{83}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq F(y, u_n), \quad \forall y \in C, \tag{84}$$

and hence,

$$\begin{aligned}&\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r}\right\rangle \\ &\quad \geq F(y, u_{n_i}), \quad \forall y \in C.\end{aligned} \tag{85}$$

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (85) we have

$$\begin{aligned}\langle y_t - u_{n_i}, Qy_t\rangle &\geq \langle y_t - u_{n_i}, Qy_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r}\right\rangle \\ &\quad + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, Qy_t - Qu_{n_i}\rangle \\ &\quad + \langle y_t - u_{n_i}, Qu_{n_i} - Qx_{n_i}\rangle \\ &\quad - \varphi(y_t) + \varphi(u_{n_i}) \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\right\rangle + F(y_t, u_{n_i}).\end{aligned} \tag{86}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Qu_{n_i} - Qx_{n_i}\| \to 0$. Further, from an inverse-strongly monotonicity of Q, we have $\langle y_t - u_{n_i}, Qy_t - Qu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weakly lower semicontinuity of φ, $\langle u_{n_i} - x_{n_i}\rangle/r \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Qy_t\rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w). \tag{87}$$

From (A1), (A4), and (87), we also have

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t)\\&\le tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y)\\&\quad + (1-t)\varphi(w) - \varphi(y_t)\\&= t(F(y_t, y) + \varphi(y) - \varphi(y_t))\\&\quad + (1-t)(F(y_t, w) + \varphi(w) - \varphi(y_t))\\&\le t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle y_t - w, Qy_t\rangle\\&= t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle y - w, Qy_t\rangle,\end{aligned} \tag{88}$$

and hence,

$$0 \le F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Qy_t\rangle. \tag{89}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Qw\rangle \ge 0. \tag{90}$$

This implies that $w \in \mathrm{MEP}(F, \varphi)$.

Lastly, we show that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Since $\|u_n - z_n\| \to 0$ and $\|z_n - y_n\| \to 0$ as $n \to \infty$, we get

$$\|u_n - y_n\| \le \|u_n - z_n\| + \|z_n - y_n\|, \tag{91}$$

we conclude that $\|u_n - y_n\| \to 0$ as $n \to \infty$. Moreover, by the nonexpansivity of G in Lemma 6, we have

$$\begin{aligned}&\|y_n - G(y_n)\|\\&\quad = \left\|J_{M_1,\lambda}\left[J_{M_2,\mu}(u_n - \mu E_2 u_n) - \lambda E_1 J_{M_2,\mu}(u_n - \mu E_2 u_n)\right]\right.\\&\qquad \left. - G(y_n)\right\|\\&\quad = \|G(u_n) - G(y_n)\|\\&\quad \le \|u_n - y_n\|.\end{aligned} \tag{92}$$

Thus, $\lim_{n\to\infty}\|y_n - G(y_n)\| = 0$. According to Lemma 7, we obtain that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Hence, $w \in \Theta$. Since $z = P_\Theta(I - A + \gamma f)(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle &= \limsup_{i\to\infty}\langle(\gamma f - A)z, x_{n_i} - z\rangle\\&= \langle(\gamma f - A)z, w - z\rangle\\&\le 0.\end{aligned} \tag{93}$$

Step 6. We show that $\{x_n\}$ converges strongly to z; we compute that

$$\begin{aligned}&\|x_{n+1} - z\|^2\\&\quad = \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n - z\|^2\\&\quad = \|\alpha_n(\gamma f(x_n) - Az) + \beta_n(x_n - z)\\&\qquad + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2\\&\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2\\&\qquad + \|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2\\&\qquad + 2\langle\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)\\&\qquad\qquad \times(W_n y_n - z), \alpha_n(\gamma f(x_n) - Az)\rangle\\&\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2\\&\qquad + \{\beta_n\|x_n - z\| + (1-\beta_n-\alpha_n\bar{\gamma})\|y_n - z\|\}^2\\&\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - Az\rangle\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\langle W_n y_n - z, \gamma f(x_n) - Az\rangle\\&\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2\\&\qquad + \{\beta_n\|x_n - z\| + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n - z\|\}^2\\&\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - \gamma f(z)\rangle\\&\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\langle W_n y_n - z, \gamma f(x_n) - \gamma f(z)\rangle\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\&\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1-\alpha_n\bar{\gamma})^2\|x_n - z\|^2\\&\qquad + 2\alpha_n\beta_n\gamma\|x_n - z\|\|f(x_n) - f(z)\|\\&\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\gamma\|W_n y_n - z\|\|f(x_n) - f(z)\|\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\&\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1-\alpha_n\bar{\gamma})^2\|x_n - z\|^2\\&\qquad + 2\alpha_n\beta_n\gamma\alpha\|x_n - z\|^2\\&\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\gamma\alpha\|x_n - z\|^2\\&\qquad + 2\alpha_n(1-\beta_n-\alpha_n\bar{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle\\&\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2\end{aligned}$$

$$
\begin{aligned}
&+\left(1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2+2\alpha_n\gamma\alpha-2\alpha_n^2\overline{\gamma}\gamma\alpha\right)\\
&\times\left\|x_n-z\right\|^2+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle\\
&+2\alpha_n\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\langle W_ny_n-z,\gamma f(z)-Az\right\rangle\\
\leq&\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\left\|x_n-z\right\|^2\\
&+\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle\\
&+2\alpha_n\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\langle W_ny_n-z,\gamma f(z)-Az\right\rangle\\
\leq&\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\left\|x_n-z\right\|^2\\
&+\alpha_n\sigma_n,
\end{aligned}
\tag{94}
$$

where $\sigma_n = \alpha_n\|\gamma f(x_n) - Az\|^2 + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n-\alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle$. It is easy to see that $\limsup_{n\to\infty}\sigma_n \leq 0$. Applying Lemma 3 to (94), we conclude that $x_n \to z$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 8 are satisfied.

Example 9. For instance, let $\alpha_n = 1/2(n+1)$, let $\beta_n = (2n+2)/2(2n)$, let $\lambda_n = n/(n+1)$. Then, we will show that the sequences $\{\alpha_n\}$ satisfy condition (C1). Indeed, we take $\alpha_n = 1/2(n+1)$; then, we have

$$
\begin{gathered}
\sum_{n=1}^{\infty}\alpha_n=\sum_{n=1}^{\infty}\frac{1}{2(n+1)}=\infty,\\
\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\frac{1}{2(n+1)}=0.
\end{gathered}
\tag{95}
$$

We will show that the sequences $\{\beta_n\}$ satisfy condition (C2). Indeed, we set $\beta_n = (2n+2)/2(2n) = (1/2) + (1/2n)$. It is easy to see that $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$.

Next, we will show the condition (C3) is satisfied. We take $\lambda_n = n/(n+1)$; then we compute

$$
\begin{aligned}
\lim_{n\to\infty}\left|\lambda_n-\lambda_{n-1}\right|&=\lim_{n\to\infty}\left|\frac{n}{n+1}-\frac{n-1}{(n-1)+1}\right|\\
&=\lim_{n\to\infty}\left|\frac{n(n)-(n-1)(n+1)}{(n+1)n}\right|\\
&=\lim_{n\to\infty}\left|\frac{n^2-n^2+1}{(n+1)n}\right|\\
&=\lim_{n\to\infty}\left|\frac{1}{n(n+1)}\right|.
\end{aligned}
\tag{96}
$$

Then, we have $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$. The sequence $\{\lambda_n\}$ satisfies condition (C3).

Using Theorem 8, we obtain the following corollaries.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1,2,3,\ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i)\cap SQVI(B_1,M_1,B_2,M_2)\cap MEP(F,\varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ,η_1,η_2-inverse-strongly monotone mapping of C into H. Let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$
\begin{gathered}
F(u_n,y)+\varphi(y)-\varphi(u_n)\\
+\frac{1}{r}\left\langle y-u_n,u_n-(x_n-rQx_n)\right\rangle\geq 0,\quad\forall y\in C,\\
z_n=J_{M_2,\mu}\left(u_n-\mu E_2u_n\right),\\
y_n=J_{M_1,\lambda}\left(z_n-\lambda E_1z_n\right),\\
x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+(1-\beta_n-\alpha_n)W_ny_n,\quad\forall n\geq 0,
\end{gathered}
\tag{97}
$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0,2\eta_1)$, $\mu \in (0,2\eta_2)$, and $r \in (0,2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1,2,\ldots,N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(f + I)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Taking $\gamma \equiv 1$ and $A \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be a nonexpansive mappings for all $i = 1,2,3,\ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i)\cap SQVI(B_1,M_1,B_2,M_2)\cap MEP(F,\varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let E_1, E_2 be η_1,η_2-inverse-strongly monotone mapping of C into H. Let A be strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined* *by* (31). *Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ &\qquad\qquad\qquad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ &\qquad\qquad\qquad \forall n \geq 0, \end{aligned} \tag{98}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, \infty)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $Q \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 12. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) and let* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuos and convex function such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* A *be a strongly positive bounded linear self-adjoint on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$, *let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Assume that either* B_1 *or* B_2 *holds, let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) \\ &\quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) y_n, \quad \forall n \geq 0, \end{aligned} \tag{99}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (7), *which is the unique solution of the variational inequality*

$$\langle(\gamma f - A)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \Theta, \tag{100}$$

Proof. Taking $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 13. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5). Let* $T_i : C \to C$ *be nonexpansive mappings for all* $i = 1, 2, 3, \ldots$, *such that* $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* A *be a strongly positive bounded linear self-adjoint on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$, *let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Assume that either* B_1 *or* B_2 *holds and let* W_n *be the* W*-mapping defined by* (31). *Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ &\qquad\qquad\qquad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ &\qquad\qquad\qquad \forall n \geq 0, \end{aligned} \tag{101}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\varphi \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 14. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* A *be a strongly positive bounded linear self-adjoint on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$, *let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Assume that either* B_1 *or* B_2

holds, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ &\qquad\qquad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) y_n, \\ &\qquad\qquad \forall n \geq 0, \end{aligned} \tag{102}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\varphi \equiv 0$ and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 15. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ &\qquad\qquad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n - \alpha_n) y_n, \\ &\qquad\qquad \forall n \geq 0, \end{aligned} \tag{103}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\gamma \equiv 1$, $A \equiv I$, $\varphi \equiv 0$, and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 16. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E *be* δ, η*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ &\qquad\qquad \forall y \in C, \\ &z_n = J_{M_2,\mu}(u_n - \mu E u_n), \\ &y_n = J_{M_1,\lambda}(z_n - \lambda E z_n), \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n - \alpha_n) y_n, \\ &\qquad\qquad \forall n \geq 0, \end{aligned} \tag{104}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta)$, $\mu \in (0, 2\eta)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E x^*)$ *is solution to the problem* (6).

Proof. Taking $E_1 = E_2 = E$ in Corollary 15, we can conclude the desired conclusion easily. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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