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*Review Article*

# A Survey of Recent Results for the Generalizations of Ordinary Differential Equations

**Daniel C. Biles,[1] Márcia Federson,[2] and Rodrigo López Pouso[3]**

[1] *Department of Mathematics and Computer Science, Belmont University, 1900 Belmont Boulevard, Nashville, TN 37212, USA*
[2] *Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, 13560-9701900 São Carlos, SP, Brazil*
[3] *Department of Mathematical Analysis, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain*

Correspondence should be addressed to Rodrigo López Pouso; rodrigo.lopez@usc.es

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This is a review paper on recent results for different types of generalized ordinary differential equations. Its scope ranges from discontinuous equations to equations on time scales. We also discuss their relation with inclusion and highlight the use of generalized integration to unify many of them under one single formulation.

## 1. Existence Theory for Differential Equations and Inclusions

There was a series of results which progressively weakened the continuity in the state variable of the classical Carathéodory existence theorem for first-order differential equations; these include [1–7]. Biles and Schechter posed the open problem of proving an existence result for discontinuous systems of differential equations lacking a quasimonotonicity property; see [7, page 3352]. Motivated by that question, Cid and Pouso [8] explored an alternative approach to discontinuous equations which consisted, roughly speaking, of inserting the differential equation into a semicontinuous differential inclusion for which existence results were available, and then positing assumptions on the discontinuities of the former differential equation so that every solution of the inclusion is a solution of the equation. Besides getting an existence result for nonquasimonotone discontinuous systems, the approach in [8] came to unify and extend previous similar results for autonomous equations proven in [9] and for nonautonomous equations proven in [10].

Here and henceforth we work in a real interval $I = [t_0, t_0 + L]$ with $L > 0$.

**Theorem 1** (see [8, Theorem 2.4]). *Assume that $f : I \times \mathbb{R}^m \to \mathbb{R}^m (m \in \mathbb{N})$ and the null set $N \subset I$ satisfy the following conditions.*

(i) *There exists $\psi \in L^1(I)$ such that for all $t \in I \setminus N$ and all $x \in \mathbb{R}^m$ one has $\|f(t,x)\| \le \psi(t)(1 + \|x\|)$, where $\|\cdot\|$ is a norm in $\mathbb{R}^m$.*

(ii) *For all $x \in \mathbb{R}^m$, $f(\cdot, x)$ is measurable.*

(iii) *For all $t \in I \setminus N$, $f(t,\cdot)$ is continuous in $\mathbb{R}^m \setminus K(t)$, where $K(t) = \cup_{n=1}^{\infty} K_n(t)$, and for each $n \in \mathbb{N}$ and $x \in K_n(t)$ one has*

$$\bigcap_{\varepsilon>0} \overline{co} f(t, x + \varepsilon B) \cap DK_n(t,x)(1) \subset \{f(t,x)\}, \tag{1}$$

*where $\overline{co}$ denotes the closed convex hull, $B$ is the unit ball centered at the origin, and $DK_n$ is the contingent derivative of the multivalued map $K_n$ (see [11] for details).*

*Then the set $\mathscr{C}$ of all Carathéodory solutions of the initial value problem*

$$x'(t) = f(t, x(t)), \quad t \in I, \qquad x(t_0) = x_0 \tag{2}$$

*is a nonempty, compact, and connected subset of $\mathscr{C}(I, \mathbb{R}^m)$.*

*Moreover, in the scalar case* ($m = 1$), *one has the following.*

(1) $\mathscr{C}$ *has pointwise maximum,* $x^*$, *and minimum,* $x_*$, *which are the extremal solutions of the initial value problem. Furthermore for each* $t \in I$ *one has*

$$\begin{aligned} x^*(t) &= \max\{v(t) : v \in AC(I), \\ &\qquad v'(s) \le f(s, v(s))\ a.e., v(t_0) \le x_0\}, \\ x_*(t) &= \min\{v(t) : v \in AC(I), \\ &\qquad v'(s) \ge f(s, v(s))\ a.e., v(t_0) \ge x_0\}. \end{aligned} \tag{3}$$

(2) $\mathscr{C}$ *is a funnel; that is, for all* $\bar{t} \in I$ *and* $c \in [x_*(\bar{t}), x^*(\bar{t})]$ *there exists* $x \in \mathscr{C}$ *such that* $x(\bar{t}) = c$.

The simplest case of Theorem 1 occurs in the one-dimensional case, that is, $m = 1$, and when the discontinuity sets $K_n$ are single-valued, that is, $K_n(t) = \{\gamma_n(t)\}$ for, let us say, absolutely continuous functions $\gamma_n$, $n \in \mathbb{N}$. In this situation we have

$$DK_n(t, \gamma_n(t))(1) = \{\gamma_n'(t)\}, \tag{4}$$

and then condition (1) reads simply as follows:

$$\begin{aligned} &\text{either } \gamma_n'(t) \notin \bigcap_{\varepsilon>0} \overline{co} f(t, \gamma_n(t) + \varepsilon B), \\ &\text{or} \quad \gamma_n'(t) = f(t, \gamma_n(t)), \end{aligned} \tag{5}$$

and it is helpful to note that, in the one-dimensional case, we have

$$\begin{aligned} \bigcap_{\varepsilon>0} \overline{co} f(t, x + \varepsilon B) = \Big[ &\min\Big\{f(t, x), \liminf_{y \to x} f(t, y)\Big\}, \\ &\max\Big\{f(t, x), \limsup_{y \to x} f(t, y)\Big\}\Big]. \end{aligned} \tag{6}$$

The first alternative in (5) means that $\gamma_n'(t)$ coincides neither with $f(t, \gamma_n(t))$ nor with any limit value of $f$ when the variables tend to $(t, \gamma_n(t))$. This is a sort of transversality condition between $f(t, x)$ and the discontinuity curve $\gamma_n(t)$, and it is immediately satisfied in case $\gamma_n$ has a sufficiently big (or sufficiently small) slope.

The second alternative is much clearer: it simply means that $\gamma_n$ solves the differential equation at the point $t$. The moral is that we do not have to worry about discontinuities of $f$ when they are located over graphs of solutions of the differential equation (even though these solutions do not satisfy the initial condition or they are not defined on the whole interval $I$).

For simplicity, we have often called *admissible discontinuity curve* any function $\gamma(t)$ satisfying (5), and they have proven to be useful in other contexts; see [12] for singular and discontinuous problems and [13] for a revision of Perron's method using similar curves.

Let us now turn our attention to differential inclusions. The rest of this section is devoted to a somewhat inverse approach to that in the first part: one can get new results for inclusions by means of known results for discontinuous equations.

To start with, we quote [14] where we can find necessary and sufficient conditions for the existence of Carathéodory solutions to

$$x' \in F(x), \qquad x(0) = x_0, \tag{7}$$

where $F$ is an arbitrary multifunction. Biles proves that the necessary and sufficient conditions for (7) to have at least one solution are that $F$ have a selection $f$ such that either $f(x_0) = 0$ or $\int_{x_0}^{\beta} dx/f(x)$ exists (in Lebesgue's sense) for some $\beta \neq x_0$. This uses and generalizes a theorem for differential equations by binding in [15].

We also used known results for equations to study nonautonomous first-order inclusions in [16]. Let us proceed to review the main ideas in that paper.

For a given multifunction $F : I \times \mathbb{R} \to P(\mathbb{R}) \setminus \{\emptyset\}$ we consider the initial value problem

$$\begin{aligned} x'(t) \in F(t, x(t)) \text{ for almost all } (a.a.)\ t \in I, \\ x(t_0) = x_0, \end{aligned} \tag{8}$$

and we look for solutions in the Carathéodory sense, that is, absolutely continuous solutions.

A very usual assumption on the multifunction $F$ is that it assumes compact values for $a.a.$ $t \in I$ and all $x \in \mathbb{R}$, hence the set $F(t, x)$ has minimum and maximum. We simply impose the following condition.

(H1) For $a.a.$ $t \in I$ and all $x \in \mathbb{R}$ the set $F(t, x)$ has a minimum; and now we introduce the following definition.

*Definition 2.* A superfunction (or upper solution) of (8) is any $u \in AC(I)$ such that $u(t_0) \ge x_0$ and for $a.a.$ $t \in I$ one has $u'(t) \ge \min F(t, u(t))$.

We also impose the following.

(H2) There exists $\psi \in L^1(I)$ such that for $a.a.$ $t \in I$ and all $x \in \mathbb{R}$ we have

$$|\min F(t, x)| \le \psi(t), \tag{9}$$

and we restrict, for technical convenience, the set of superfunctions to the following one.

*Definition 3.* The set of admissible superfunctions of (8) is

$$\begin{aligned} U := \{&u \in AC(I) : u \text{ is a superfunction of } (8) \\ &\text{and } |u'| \le \psi + 1\ a.e. \text{ on } I\}. \end{aligned} \tag{10}$$

Notice that $u(t) := x_0 + \int_{t_0}^{t} \psi(r)dr$, $t \in I$, is an admissible superfunction of (8). Thus we can define

$$u_{\inf}(t) := \inf\{u(t) : u \in U\} \quad \forall t \in I. \tag{11}$$

Standard arguments reveal that $u_{\inf}(t_0) = x_0$ and that $u_{\inf} \in AC(I)$.

For simplicity of notation, we also define

$$f_m(t,x) := \min F(t,x) \quad \text{for } a.a.\ t \in I \text{ and all } x \in \mathbb{R}, \tag{12}$$

and we consider the ordinary problem

$$x'(t) = f_m(t, x(t)) \quad \text{for } a.a.\ t \in I, \qquad x(t_0) = x_0. \tag{13}$$

Plainly, solutions of (13) are solutions of (8) by virtue of (H1), but the converse is false in general. Moreover, superfunctions of (8) in the sense of Definition 2 are nothing but the usual superfunctions of (13), and so $u_{\inf}$ is a reasonable candidate for being a solution to (8). Note also that solutions of (8) need not be admissible superfunctions in the sense of Definition 3, so $u_{\inf}$ might not be the least solution of (8).

*Definition 4.* A lower admissible nonquasisemicontinuity curve for (8) (LAD curve, for short) is an absolutely continuous function $\gamma : [a,b] \subset I \to \mathbb{R}$ for which there exist disjoint sets $A, B \subset [a,b]$ such that $A \cup B = [a,b]$ and for $a.a.\ t \in A$ one has

$$\gamma'(t) \in F(t, \gamma(t)), \tag{14}$$

and for $a.a.\ t \in B$ one has

$$\begin{aligned} \gamma'(t) &\geq f_m(t,\gamma(t)) \ \text{ whenever } \gamma'(t) \geq \liminf_{y \to (\gamma(t))^+} f_m(t,y), \\ \gamma'(t) &\leq f_m(t,\gamma(t)) \ \text{ whenever } \gamma'(t) \leq \limsup_{y \to (\gamma(t))^-} f_m(t,y). \end{aligned} \tag{15}$$

*Remark 5.* The sets $A$ or $B$ in Definition 4 might be empty.

A particularly clear case of a LAD curve corresponds to $B = \emptyset$, which means that $A = [a,b]$, so in that case the LAD curve is nothing but a solution of the differential inclusion on $[a,b]$.

In turn, let us point out the following sufficient condition for an absolutely continuous function $\gamma : [a,b] \subset I \to \mathbb{R}$ to be a LAD curve with $B = [a,b]$: there exist $\varepsilon > 0$ and $\rho > 0$ such that for $a.a.\ t \in [a,b]$ we have

$$f_m(t,x) \geq \gamma'(t) + \rho \quad \forall x \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \tag{16}$$

or for $a.a.\ t \in [a,b]$ we have

$$f_m(t,x) \leq \gamma'(t) - \rho \quad \forall x \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon]. \tag{17}$$

Notice that (16) (or (17)) implies that $\gamma$ crosses each solution of $x' = f_m(t,x)$ at most once, so (16) (or (17)) is a transversality condition for $\gamma$ with respect to the differential equation $x' = f_m(t,x)$.

We are now in a position to present the main result in [16].

**Theorem 6** (see [16, Theorem 2.5]). *Assume that conditions (H1) and (H2) hold. Suppose moreover that the following condition is fulfilled.*

(H3) *Either for a.a. $t \in I$ and all $x \in \mathbb{R}$ one has*

$$\limsup_{y \to x^-} f_m(t,y) \leq f_m(t,x) \leq \liminf_{y \to x^+} f_m(t,y) \tag{18}$$

*or there exist countably many LAD curves for* (8), $\gamma_n : [a_n, b_n] \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, *such that for a.a.* $t \in I$ *and all* $x \in \mathbb{R} \setminus \cup_{\{n \mid a_n \leq t \leq b_n\}} \{\gamma_n(t)\}$ *one has* (18).

*Then one has the following results.*

(a) *There exists a null measure set $N \subset I$ such that*

$$\{t \in I : u'_{\inf}(t) \notin F(t, u_{\inf}(t))\} \subset J \cup N, \tag{19}$$

*where* $J = \cup_{n,m \in \mathbb{N}} J_{n,m}$, *and for each* $n, m \in \mathbb{N}$ *the set*

$$\begin{aligned} J_{n,m} := \Big\{ t \in I : {} & u'_{\inf}(t) - \frac{1}{n} \\ & > \sup \Big\{ f_m(t,y) : u_{\inf}(t) - \frac{1}{m} < y < u_{\inf}(t) \Big\} \Big\} \end{aligned} \tag{20}$$

*contains no positive measure subset.*

(b) *The function $u_{\inf}$ is a solution of* (8) *provided that for all $n, m \in \mathbb{N}$ the set $J_{n,m}$ is measurable.*

(c) *If $J_{n,m}$ is measurable for every $n, m \in \mathbb{N}$, then $u_{\inf}$ is the least solution of* (8) *provided that one of the following conditions hold:*

*either for a.a. $t \in I$, all $x \in \mathbb{R}$, and all $y \in F(t,x)$ one has $y \leq \psi(t) + 1$ or the first alternative in (H3) holds, which, furthermore, guarantees that $u_{\inf}$ is the least solution to* (13).

The following result is Lemma 2 in [13], and it is very useful to prove that the $J_{n,m}$'s are measurable in practical situations.

**Lemma 7.** *Let $N \subset I$ be a null measure set and let $g : I \times \mathbb{R} \to \mathbb{R}$ be such that $g(\cdot, q)$ is measurable for each $q \in \mathbb{Q}$.*

*If, moreover, for all $t \in I \setminus N$ and all $x \in \mathbb{R}$ one has*

$$\max\Big\{ \liminf_{y \to x^-} g(t,y), \liminf_{y \to x^+} g(t,y) \Big\} \geq g(t,x), \tag{21}$$

*then the mapping $t \in I \mapsto \sup\{g(t,y) : x_1(t) < y < x_2(t)\}$ is measurable for each pair $x_1, x_2 \in C(I)$ such that $x_1(t) < x_2(t)$ for all $t \in I$.*

Notice that our multifunctions $F$ need not satisfy the usual hypotheses such as monotonicity or upper/lower semicontinuity. Moreover, $F$ need not assume closed or convex values.

An analogous result for the greatest solution to (13) is also given in [16] and existence of solution for a singular version of (8) is considered in [17].

Another example where we used known results for equations to deduce new result for inclusions is [18], which concerns second-order inclusions and relies on the results proven for equations in [19]. In order to present the main result in [18] we need some notations and preliminaries.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, and

$$X = \{u \in \mathcal{C}([0,T]) : u(0) = x_0, u \ \text{ is nondecreasing}\}. \tag{22}$$

For each $u \in X$ we define its "pseudoinverse" $\widehat{u} : \mathbb{R} \to [0, T]$ as

$$\widehat{u}(x) = \begin{cases} 0, & x < x_0, \\ \min u^{-1}(\{x\}), & x_0 \le x \le u(T), \\ T, & u(T) < x. \end{cases} \quad (23)$$

We notice that $\widehat{u}$ is nondecreasing but not necessarily continuous. Moreover, if $u \in X$ is increasing in $I$, then $\widehat{u}(x) = u^{-1}(x)$ for all $x \in [x_0, u(T)]$.

**Theorem 8** (see [18, Theorem 4.1]). *Suppose that for some $R > 0$ the following hypotheses hold.*

(F1) *For each $u \in X$ the multifunction $F_u : \mathbb{R} \to \mathscr{P}(\mathbb{R})\backslash\{\emptyset\}$ defined as $F_u(\cdot) = F(\widehat{u}(\cdot), \cdot)$ has an admissible selection on the right of $x_0$, that is, a selection $f_u : [x_0, x_0+R] \to \mathbb{R}$ such that*

(i) $f_u \in L^1(x_0, x_0 + R)$;

(ii) $x_1^2 + 2\int_{x_0}^{x} f_u(r)dr > 0$ *for a.a.* $x \in [x_0, x_0 + R]$;

(iii) $\max\{1, |f|\}/\sqrt{x_1^2 + 2\int_{x_0}^{\cdot} f_u(r)dr} \in L^1(x_0, x_0 + R)$;

(iv) $\int_{x_0}^{x_0+R}(dx/\sqrt{x_1^2 + 2\int_{x_0}^{x} f_u(r)dr}) \ge T$.

(F2) *There exists $M \in L^1(x_0, x_0 + R)$ such that for all $t \in I$ and all $x \in [x_0, x_0 + R]$ one has*

$$\sup\{y : y \in F(t, x)\} \le M(x). \quad (24)$$

(F3) *For every $u, v \in \widehat{X}$, the relation $u \le v$ on $I$ implies $f_u \le f_v$ on $[x_0, x_0 + R]$.*

*Then the initial value problem*

$$x''(t) \in F(t, x(t)) \quad \text{for a.a. } t \in I := [0, T], \qquad x(0) = x_0, \quad x'(0) = x_1 \ge 0 \quad (25)$$

*has an increasing solution in $W^{2,1}(0, T)$.*

## 2. Dynamic Equations on Time Scales

The study of time scales was formalized in the Ph.D. thesis of Hilger in 1988 [20]. The notions of derivative from differential calculus and the forward jump operator from difference calculus are unified and extended to the delta derivative $f^{\Delta}$ on a time scale $\mathbb{T}$ (an arbitrary set on the real line). These lead to the study of dynamic equations on time scales, unifying differential and difference equations. In addition, these ideas can be applied in situations more general than those for differential and difference equations, such as population problems in which the species alternates between time frames in which they are active and periods of dormancy. The study of time scales yields interesting insight into the special cases. For example, one realizes that the only reason we have the simple derivative from elementary calculus of $t^2$ is $2t$ is because the graininess of real line is zero.

Much of the earlier history of time scales can be found in the books by Bohner and Peterson [21, 22] which are on the bookshelf of every time scales analyst. We refer the reader to these sources for the basic concepts and definitions for time scales. Reference [21] collects much of the information for the linear case. As an example, we will overview the first-order linear case. We define the cylinder transformation $\xi_h$ on $\{z \in \mathbb{C} \mid z \ne -1/h\}$ by $\xi_h(z) = 1/h \log(1 + zh)$ for $h > 0$, where log is the principal logarithm function and $\xi_0(z) = z$. We call a function $p : \mathbb{T} \to \mathbb{R}$ regressive if $1 + \mu(t)p(t)$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu$ is the graininess of the time scale. We can now define the time scales (or generalized) exponential function by $e_p(t, s) = \exp(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau)$, where $s, t \in \mathbb{T}$. The exponential function thus defined enjoys many properties analogous to that of the standard exponential function on the real line. The following can now be proven.

**Theorem 9.** *Suppose $p$ is rd-continuous and regressive, and let $t_0 \in \mathbb{T}$. Then, $e_p(\cdot, t_0)$ is the unique solution to*

$$y^{\Delta} = p(t)\, y, \qquad y(t_0) = 1. \quad (26)$$

Note that this yields the corollaries that $y = e^{\alpha t}$ is the unique solution to $y' = \alpha y$, $y(0) = 1$ on the real line and $y = (1 + \alpha)^t$ is the unique solution to $\Delta y(t) = \alpha y(t)$, $y(0) = 1$ on the integers, where $\Delta y$ represents the forward difference operator from difference calculus.

We note that the nabla derivative on time scales was defined by Atici and Guseinov [23] in 2002, which generalizes the backward difference operator. One might think that the results for nabla derivatives mirror those for the delta case, but this is not true; see, for example, [24]. Recently, work has progressed for dynamic equations with the diamond-alpha derivative initiated in [25] and furthered in [26–28]. In the remainder of this section, without making a claim to being complete, we overview some of the recent work in dynamic equations on time scales to illustrate how many of the ideas from differential and difference equations have been generalized and extended.

Existence of solutions has been proven in a number of cases, such as [29] using fixed point theory, [30] proving a Nagumo-type existence result, and [31] using a fixed point theorem due to Avery and Peterson. (A number of other existence theorems are mentioned in specific contexts below.) As an example, here is the theorem from [30].

**Theorem 10.** *Assume there exist a lower solution $\alpha$ and an upper solution $\beta$ with $\alpha \le \beta$ on $\mathbb{T}$ and*

(a) $f \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$ *satisfies* $f(t, x, y) > 0$ *for all* $t \in \mathbb{T}$, $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$ *and* $y \ne 0$,

(b) *there exists a* $K > 0$ *such that* $f(t, x, y) \le K$ *for all right scattered* $t \in \mathbb{T}$, $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$ *and* $y \in \mathbb{R}$,

(c) $f(t, x, \cdot)$ *is nonincreasing for all right scattered* $t \in \mathbb{T}$ *and* $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$,

(d) $L_1 \in C(\mathbb{R}^4 \times C(\mathbb{T}), \mathbb{R})$ *is nondecreasing in its third variable, nonincreasing in its fourth variable, and nondecreasing in its fifth variable,*

(e) $L_2 \in C(\mathbb{R}^2, \mathbb{R})$ *is nonincreasing in its first variable, and*

(f) $f$ *satisfies a Nagumo condition with respect to the pair* $\alpha$ *and* $\beta$.

*Then, there exists a solution* $y \in [\alpha, \beta]$ *to the problem*

$$\begin{aligned} y^{\Delta\Delta}(t) &= f\left(t, y^{\sigma}(t), y^{\Delta}(t)\right), \quad \textit{for } t \in \mathbb{T}^{\kappa^2}, \\ 0 &= L_1\left(y(a), y^{\Delta}(a), y\left(\sigma^2(b)\right), y^{\Delta}(\sigma(b)), y\right), \\ 0 &= L_2\left(y(a), y\left(\sigma^2(b)\right)\right). \end{aligned} \tag{27}$$

Singular problems have been studied in [32, 33]. Green's functions have been considered in [23, 34]. A Sturm-Liouville eigenvalue problem was studied by [35]. Periodic solutions were investigated in [36]. OscillationS of solutions have been considered in [37–39] using the time scales Taylor formula, [40–45]. Asymptotic behavior of solutions has been studied in [46, 47] using Taylor monomials and in [47]. Laplace transforms on time scales were studied by [48].

Delay equations were studied in [40, 42, 44]. Impulsive problems have been studied in [37, 38, 49–51]. Functional dynamic equations have been studied in [50, 52] using Lyapunov functions [41, 46]. Fractional derivatives have been considered in [53, 54]. Problems in abstract spaces were studied in [55]. Dynamic inclusions have been studied in [24, 37, 50, 56]. Partial differentiation on time scales was introduced in [57] and was continued in [28].

Recently, work has begun on extending stochastic calculus to time scales, for example, [58] for the isolated time scale case, [59, 60] for the delta case, AND [61] for the nabla case.

## 3. Generalized Ordinary Differential Equations

In order to generalize some results on continuous dependence of solutions of ordinary differential equations with respect to the initial data, Jaroslav Kurzweil introduced, in 1957, the notion of generalized ordinary differential equations for functions taking values in Euclidean and Banach spaces. This generalization of the notion of ordinary differential equations uses the concept of the Perron generalized integral, also known as the Kurzweil integral. We refer to these equations as generalized ODEs. See [62–66].

The correspondence between generalized ODEs and classic ODEs is very simple. It is known that the ordinary system

$$\dot{x} = f(x, t), \tag{28}$$

where $\dot{x} = dx/dt$, $\Omega \subset \mathbb{R}^n$ is an open set and $f : \Omega\times\mathbb{R} \to \mathbb{R}^n$, has the integral representation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau), \tau)\, d\tau, \quad t \geq t_0, \tag{29}$$

whenever the integral exists in some sense. It is also known that if the integral in (29) is in the sense of Riemann, Lebesgue (with the equivalent definition given by E. J. McShane), or Henstock-Kurzweil, for instance, then such an integral can be approximated by a Riemann-type sum of the form

$$\sum_{i=1}^{m} f(x(\tau_i), \tau_i)[s_i - s_{i-1}], \tag{30}$$

where $t_0 = s_0 \leq s_1 \leq \cdots \leq s_m = t$ is a fine partition of the interval $[t_0, t]$ and, for each $i = 1, 2, \ldots, m$, $\tau_i$ is sufficiently "*close*" to the interval $[s_{i-1}, s_i]$.

Alternatively, if we define

$$F(x, s) = \int_{s_0}^{s} f(x, \sigma)\, d\sigma, \qquad (x, t) \in \Omega \times \mathbb{R}, \tag{31}$$

then the integral in (29) can be approximated by

$$\begin{aligned} &\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} f(x(\tau_i), \sigma)\, d\sigma \\ &\quad = \sum_{i=1}^{m} [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})]. \end{aligned} \tag{32}$$

In such a case, the right-hand side of (32) approximates the nonabsolute Kurzweil integral which, when considered in (29), gives rise to a "*differential equation*" of type (28), however in a wider sense. Such type of equation is known as generalized ordinary differential equation or *Kurzweil equation*. See [67, 68].

Let $[a, b] \subset \mathbb{R}$ be a compact interval and consider a function $\delta : [a, b] \to \mathbb{R}^+$ (called a gauge on $[a, b]$). A tagged partition of the interval $[a, b]$ with division points $a = s_0 \leq s_1 \leq \cdots \leq s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \ldots, k$, is called $\delta$-fine if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i = 1, \ldots, k. \tag{33}$$

*Definition 11.* Let $X$ be a Banach space. A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is called Kurzweil integrable over $[a, b]$, if there is an element $I \in X$ such that, given $\varepsilon > 0$, there is a gauge $\delta$ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon, \tag{34}$$

for every $\delta$-fine tagged partition of $[a, b]$. In this case, $I$ is called the Kurzweil integral of $U$ over $[a, b]$ and it will be denoted by $\int_a^b DU(\tau, t)$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals, and integrability on subintervals. See, for instance, [68], for these and other interesting properties.

Now, consider a subset $O \subset X$ and a function $G : O \times [a, b] \to X$.

Any function $x : [a,b] \to O$ is called a solution of the generalized ordinary differential equation (we write simply generalized ODE)

$$\frac{dx}{d\tau} = DG(x,t) \tag{35}$$

on the interval $[a,b]$, provided

$$x(d) - x(c) = \int_c^d DG(x(\tau),t), \quad c,d \in [a,b], \tag{36}$$

where the integral is obtained by setting $U(\tau,t) = G(x(\tau),t)$ in the definition of the Kurzweil integral.

As it was done in [69, 70], but using different assumptions, namely, Carathéodory and Lipschitz-type conditions on the indefinite integral, we proved in [71] that retarded functional differential equations (we write RFDEs, for short) can be regarded as abstract generalized ODEs and some applications were investigated.

In [72], together with professor Štefan Schwabik, we proved that RFDEs subject to impulse effects can also be regarded as generalized ODEs taking values in a Banach space.

Recently, in [73], together with Federson et al., we proved that a solution of a measure RFDEs of the form

$$Dy = f(y_t,t)\,Dg, \qquad y_{t_0} = \phi, \tag{37}$$

where $Dy$ and $Dg$ are distributional derivatives in the sense of L. Schwartz with respect to $y$ and $g$, respectively, can be related to a solution of an abstract generalized ODE. More precisely, we considered the integral form of (37) as follows:

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t f(x_s,s)\,dg(s), \quad t \ge t_0, \\ x_{t_0} &= \phi, \end{aligned} \tag{38}$$

where $t_0,\sigma,r$ are given real numbers, with $\sigma,r > 0$, and $y_t(\theta) = y(t+\theta)$, for $\theta \in [-r,0]$. We also considered $O \subset G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ as being an open set and

$$P = \{y_t : y \in O, t \in [t_0,t_0+\sigma]\} \subset G([-r,0],\mathbb{R}^n), \tag{39}$$

where by $G([a,b],X)$ we mean the Banach space of all regulated functions $f : [a,b] \to X$ endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{a \le t \le b} \|f(t)\|, \tag{40}$$

and we assumed that $f : P \times [t_0,t_0+\sigma] \to \mathbb{R}^n$ is a function such that, for each $y \in O$, the mapping $t \mapsto f(y_t,t)$ is Henstock-Kurzweil integrable (or Perron integrable) over $[t_0,t_0+\sigma]$ with respect to a nondecreasing function $g : [t_0,t_0+\sigma] \to \mathbb{R}$. Then, we defined a function $G : O \times [t_0,t_0+\sigma] \to G([t_0,t_0+\sigma],\mathbb{R}^n)$ by

$$G(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(x_s,s)\,dg(s), & t_0 \le \vartheta \le t \le t_0+\sigma, \\ \displaystyle\int_{t_0}^{t} f(x_s,s)\,dg(s), & t \le \vartheta \le t_0+\sigma, \end{cases} \tag{41}$$

and proved the correspondence between a solution of (38) and a solution of the generalized ODE

$$\frac{dx}{d\tau} = DG(x(\tau),t), \tag{42}$$

with initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), & t_0 \le \vartheta \le t_0+\sigma, \end{cases} \tag{43}$$

just by requiring the following conditions.

(A) The integral $\int_{t_0}^{t_0+\sigma} f(y_t,t)dg(t)$ exists, for every $y \in O$.

(B) There exists a function $M : [t_0,t_0+\sigma] \to \mathbb{R}^+$ which is Lebesgue integrable with respect to $g$ such that, for all $y \in O, u_1,u_2 \in [t_0,t_0+\sigma]$, we have

$$\left| \int_{u_1}^{u_2} f(y_s,s)\,dg(s) \right| \le \int_{u_1}^{u_2} M(s)\,dg(s). \tag{44}$$

(C) There exists a function $L : [t_0,t_0+\sigma] \to \mathbb{R}^+$ which is Lebesgue integrable with respect to $g$, such that for all $y,x \in O, u_1,u_2 \in [t_0,t_0+\sigma]$, we have

$$\left| \int_{u_1}^{u_2} [f(x_s,s) - f(y_s,s)]\,dg(s) \right| \le \int_{u_1}^{u_2} L(s)\,\|x_s - y_s\|\,dg(s). \tag{45}$$

Under the above conditions, the paper [73] introduces new concepts of stability for the trivial solutions of (38), with $f(0,t) = 0$, for $t \in [t_0,t_0+\sigma]$, and new results which generalize those from [74–76], for instance.

In [77, 78], together with Federson et al., we proved that measure RFDEs are useful tools in the study of impulsive RFDEs and functional dynamic equations on time scales with or without impulse action. In other words, it was proved that the unique solution of the Cauchy problem for a measure RFDE of type

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t \tilde{f}(x_s,s)\,d\tilde{g}(s), \quad t \in [t_0,t_0+\sigma], \\ x_{t_0} &= \phi \end{aligned} \tag{46}$$

can be regarded, in a one-to-one relation, with the unique solution of the Cauchy problem for the measure RFDE with impulses given by

$$\begin{aligned} x(t) = x(t_0) &+ \int_{t_0}^t f(x_s,s)\,dg(s) \\ &+ \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), \quad t \in [t_0,t_0+\sigma], \\ x_{t_0} = \phi.& \end{aligned} \tag{47}$$

Still in [77, 78], we related the solution of problem (47) to the solution of the following impulsive functional dynamic equation on time scales

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s^*, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}},$$
$$x(t) = \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \tag{48}$$

where $x^*$ is defined as being the extension of $x$ defined by $x^*(t) = x(t^*)$, for $t^* = \inf\{s \in \mathbb{T} : s \geq t\}$.

In order to obtain the correspondences presented in [77, 78], the requirement was mainly that $f$ is Henstock-Kurzweil integrable with respect to a nondecreasing function $g$. Therefore many discontinuities are allowed. Moreover, $f$ does not need to be rd-continuous nor regulated, and yet good results for impulsive functional dynamic equations on time scales can be obtained through these correspondences.

Even more recently, together with Federson et al., we studied, in [79], measure neutral functional differential equations (we write measure NFDEs) of type

$$D[N(y_t, t)] = f(y_t, t)\,Dg,$$
$$y_{t_0} = \phi, \tag{49}$$

where $N$ is a nonautonomous linear operator (i.e., $N(y_t, t) = N(t)y_t$). Besides, we assume that $N$ admits a representation

$$N(t)\varphi = \varphi(0) - \int_{-r}^{0} d_\theta[\mu(t,\theta)]\,\varphi(\theta), \tag{50}$$

where $\mu : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ is a normalized measurable function satisfying

$$\mu(t,\theta) = 0, \quad \theta \geq 0;$$
$$\mu(t,\theta) = \mu(-r), \quad \theta \leq -r, \tag{51}$$

which is continuous to the left on $\theta \in (-r, 0)$ of bounded variation in $\theta \in [-r, 0]$, and the variation of $\mu$ in $[s, 0]$, $\operatorname{var}_{[s,0]}\mu$ tends to zero as $s \to 0$.

In order to obtain a correspondence between solutions of measure NFDEs and solutions of a certain class of generalized ODEs of the form

$$\frac{dx}{d\tau} = DG(x(\tau), t), \tag{52}$$

whose right-hand side is given by

$$G(x,t) = F(x,t) + J(x,t), \quad x \in O,\ t \in [t_0, t_0+\sigma], \tag{53}$$

with $F$ as (47) and $J$ given by

$$J(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{-r}^{0} d_\theta[\mu(\vartheta,\theta)]\,x(\vartheta+\theta) - \int_{-r}^{0} d_\theta[\mu(t_0,\theta)]\,x(t_0+\theta), & t_0 \leq \vartheta \leq t \leq t_0+\sigma, \\ \displaystyle\int_{-r}^{0} d_\theta[\mu(t,\theta)]\,x(t+\theta) - \int_{-r}^{0} d_\theta[\mu(t_0,\theta)]\,x(t_0+\theta), & t \leq \vartheta \leq t_0+\sigma. \end{cases} \tag{54}$$

Besides conditions (A), (B), and (C) presented above, we required that the normalized function $\mu$ satisfies the following:

(D) there exists a Lebesgue integrable function $K$: $[t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left|\int_{-r}^{0} d_\theta\mu(s_2,\theta)\,x(s_2+\theta) - \int_{-r}^{0} d_\theta\mu(s_1,\theta)\,x(s_1+\theta)\right| \leq \int_{s_1}^{s_2} K(s)\int_{-r}^{0} d_\theta\mu(s,\theta)\,|x(s+\theta)|. \tag{55}$$

Thus, in [79], results on the local existence and uniqueness of solutions, as well as continuous dependence of solutions on the initial data, were established.

Clearly there is still much to do to develop the theory of abstract generalized ODEs and to apply the results to other types of differential equations.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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