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Research Article

Approximation by q-Bernstein Polynomials in the Case $q \to 1+$

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Let $B_{n,q}(f;x)$, $q \in (0,\infty)$ be the q-Bernstein polynomials of a function $f \in C[0,1]$. It has been known that, in general, the sequence $(B_{n,q_n}(f))$ with $q_n \to 1+$ is not an approximating sequence for $f \in C[0,1]$, in contrast to the standard case $q_n \to 1-$. In this paper, we give the sufficient and necessary condition under which the sequence $(B_{n,q_n}(f))$ approximates f for any $f \in C[0,1]$ in the case $q_n > 1$. Based on this condition, we get that if $1 < q_n < 1 + \ln 2/n$ for sufficiently large n, then $(B_{n,q_n}(f))$ approximates f for any $f \in C[0,1]$. On the other hand, if $(B_{n,q_n}(f))$ can approximate f for any $f \in C[0,1]$ in the case $q_n > 1$, then the sequence (q_n) satisfies $\overline{\lim}_{n\to\infty} n(q_n - 1) \le \ln 2$.

1. Introduction

Let $q > 0$. For any nonnegative integer k, the *q-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1}, \quad (k = 1, 2, \ldots), \quad [0]_q := 0, \tag{1}$$

and the *q-factorial* $[k]_q!$ by

$$[k]_q! := [1]_q[2]_q \cdots [k]_q, \quad (k = 1, 2, \ldots), \quad [0]_q! := 1. \tag{2}$$

For integers k, n with $0 \le k \le n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{3}$$

In [1], Phillips proposed the q-Bernstein polynomials: for each positive integer n and $f \in C[0,1]$, the q-Bernstein polynomial of f is

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x), \tag{4}$$

where

$$p_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \tag{5}$$

Note that, for $q = 1$, $B_{n,q}(f;x)$ is the classical Bernstein polynomial $B_n(f;x)$:

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{6}$$

In recent years, the q-Bernstein polynomials have been investigated intensively and a great number of interesting results related to the q-Bernstein polynomials have been obtained. Reviews of the results on q-Bernstein polynomials are given in [2, Chapter 7] and [3, 4].

The q-Bernstein polynomials inherit some of the properties of the classical Bernstein polynomials, for example, the *end-point interpolation* property and the *shape-preserving* properties in the case $0 < q < 1$, representation via divided differences. We can also define the generalized Bézier curve and de Casteljau algorithm, which can be used for evaluating q-Bernstein polynomials iteratively. These properties stipulate the importance of q-Bernstein polynomials for the computer-aided geometric design. Like the classical Bernstein polynomials, the q-Bernstein polynomials reproduce linear functions and are degree reducing on the set of polynomials. Apart from that, the basic q-Bernstein polynomials $p_{nk}(q;x)$ admit a probabilistic interpretation via the stochastic process and the q-binomial distribution in the case $0 < q < 1$; see [5].

On the other hand, when passing from $q = 1$ to $q \neq 1$ convergence properties of the q-Bernstein polynomials dramatically change. More specially, in the case $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0,1]$, and the convergence properties of the q-Bernstein polynomials have been investigated intensively (see, e.g., [6–11]). In the case $q > 1$, $B_{n,q}$ are not positive linear operators on $C[0,1]$, and the lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult. There are many unexpected results concerning convergence of q-Bernstein polynomials in the case $q > 1$ (see [2, 12–17]). For example, the rate of approximation by q-Bernstein polynomials ($q > 1$) in $C[0,1]$ for functions analytic in $\{z : |z| < q+\varepsilon\}$ is q^{-n} versus $1/n$ for the classical Bernstein polynomials, while, for some infinitely differentiable functions on $[0,1]$, their sequences of q-Bernstein polynomials ($q > 1$) may be divergent (see [12]). In [2, 15], strong asymptotic estimates for the norm $\|B_{n,q}\|$ as $n \to \infty$ for fixed $q > 1$ and as $q \to \infty$ are obtained. It was shown in [2] that $\|B_{n,q}\| \to +\infty$ faster than any geometric progression $n \to \infty$ for fixed $q > 1$. This fact provides an explanation for the unpredictable behavior of q-Bernstein polynomials ($q > 1$) with respect to convergence.

This paper is devoted to studying approximation properties of q-Bernstein polynomials for q taking *varying* values that tend to 1. We note that, from the very first papers (see [1]), there was interest in such approximation properties. In the case $0 < q_n < 1$, many interesting results including the convergence, the rate of convergence, Voronvskaya-type theorems, and the direct and converse theorem are obtained (see [1, 6, 8–11]). It was shown in [1, 8] that, in the case $q_n \leq 1$, the condition $q_n \to 1$ is necessary and sufficient for the sequence $(B_{n,q_n}(f))$ to be approximating for any $f \in C[0,1]$.

Naturally, the question arises as to whether the sequence $(B_{n,q_n}(f))$ to be approximating for any $f \in C[0,1]$ as q_n tends to 1 from above. It turns out that, in general, the answer is negative. Indeed, Ostrovska showed in [13] that if $q_n - 1 \downarrow 0$ slower than $(\ln n)/n$, then the sequence $(B_{n,q_n}(f))$ may not be approximating for some $f \in C[0,1]$ (e.g., $f(x) = \sqrt{x}$). However, in [14] Ostrovska showed that if $q_n \to 1^+$ fast enough, the sequence $(B_{n,q_n}(f))$ is approximating for any $f \in C[0,1]$: a sufficient condition is $q_n = 1 + o(n^{-1}3^{-n})$.

In this paper, we continue to study the convergence of the sequence (B_{n,q_n}) as q_n tends to 1 from above. Clearly, the convergence of the sequence (B_{n,q_n}) depends heavily on the operator norms $\|B_{n,q}\|$. We remark that for $\|B_{n,q_n}\| = 1$ for all $0 < q_n < 1$. In contrast to this, $\|B_{n,q_n}\|$ vary with $q_n > 1$. By the delicate analysis of $\|B_{n,q_n}\|$, we obtain the sufficient and necessary condition under which $(B_{n,q_n}(f;\cdot))$ ($q_n > 1$) approximates f for any $f \in C[0,1]$. Based on this condition we get that if $(B_{n,q_n}(f;\cdot))$ can approximate f for any $f \in C[0,1]$, then the sequence (q_n) satisfies $\overline{\lim}_{n\to\infty} n(q_n - 1) \leq \ln 2$. On the other hand, if $1 < q_n \leq 1 + \ln 2/n$ for sufficient large n, then $(B_{n,q_n}(f;\cdot))$ approximates f for any $f \in C[0,1]$.

2. Statement of Results

From here on we assume that $q_n > 1$. The following theorem gives the sufficient and necessary condition for convergence of the sequence $(B_{n,q_n}(f))$ for any $f \in C[0,1]$.

Theorem 1. *Let $q_n > 1$. Then the sequence $(B_{n,q_n}(f))$ converges to f in $C[0,1]$ for any $f \in C[0,1]$ if and only if*

$$\sup_{n\in\mathbb{N}} \sup_{x\in[q_n^{-1},1]} \sum_{k=2}^{n} \left|p_{n\,n-k}(q_n;x)\right| < \infty. \tag{7}$$

Based on Theorem 1, we obtain the following necessary condition for convergence of the sequence $(B_{n,q_n}(f))$. Indeed, we show that if $\overline{\lim}_{n\to\infty} n(q_n - 1) > \ln 2$, then $\sup_{n\in\mathbb{N}} |p_{n\,n-[\ln n]}(q_n;x_0)| = \infty$ with $x_0 = (1+q_n)/2q_n$.

Theorem 2. *Let $q_n > 1$. If the sequence $(B_{n,q_n}(f))$ converges to f in $C[0,1]$, for any $f \in C[0,1]$, then*

$$\overline{\lim_{n\to\infty}}\, n(q_n - 1) \leq \ln 2. \tag{8}$$

Finally, we give the sufficient condition for convergence of the sequence $(B_{n,q_n}(f))$.

Theorem 3. *Let $q_n > 1$. If the sequence (q_n) satisfies $q_n \leq 1 + \ln 2/n$ for sufficiently large n, then, for any $f \in C[0,1]$, $(B_{n,q_n}(f;x))$ converges to $f(x)$ uniformly on $[0,1]$.*

The following corollary follows immediately for Theorem 3.

Corollary 4. *Let $q_n > 1$. If the sequence (q_n) satisfies*

$$\overline{\lim_{n\to\infty}}\, n(q_n - 1) < \ln 2, \tag{9}$$

then, for any $f \in C[0,1]$, $(B_{n,q_n}(f;x))$ converges to $f(x)$ uniformly on $[0,1]$.

Remark 5. Using the same technique as in the proof of Theorem 3, we can prove a slightly stronger conclusion: if

$$1 < q_n \leq 1 + \frac{\ln 2}{n} + \frac{C}{n^2} \tag{10}$$

for some positive constant C and sufficiently large n, then, for any $f \in C[0,1]$, $(B_{n,q_n}(f;x))$ converges to $f(x)$ uniformly on $[0,1]$.

3. Proofs of Theorems 1–3

For $f \in C[0,1]$, we set

$$\begin{gathered} \|f\| := \max_{x\in[0,1]} |f(x)|, \\ \|f\|_s := \|f\|_{C[q_n^{-s-1},q_n^{-s}]} := \max_{x\in[q_n^{-s-1},q_n^{-s}]} |f(x)|. \end{gathered} \tag{11}$$

Let $F_n(x) := \sum_{k=0}^{n} |p_{nk}(q_n;x)|$, $x \in [0,1]$. Clearly,

$$\left\|B_{n,q_n}\right\| = \|F_n\| = \max_{x\in[0,1]} \left(\sum_{k=0}^{n} \left|p_{n\,n-k}(q_n;x)\right| \right). \tag{12}$$

Note that $\sum_{k=0}^{n} p_{nk}(q_n;x)=1$ for $x\in[0,1]$ and $p_{nk}(q_n;x)\ge 0$ for $x\in[0,q_n^{-n+1}]$ and $k=0,1,\dots,n$. This means that

$$F_n(x)=1,\quad x\in\left[0,q_n^{-n+1}\right],\qquad F_n(x)\ge 1,\quad x\in\left[q_n^{-n+1},1\right]. \tag{13}$$

It follows that

$$\left\|B_{n,q_n}\right\|=\|F_n\|=\max_{0\le s\le n-2}\|F_n\|_s. \tag{14}$$

Proof of Theorem 1. From Corollary 7 in [12] we know that, for any polynomial $P(x)$, we have

$$B_{n,q_n}(P;x)\longrightarrow P(x) \tag{15}$$

uniformly in $[0,1]$ as $n\to\infty$. It follows from the well-known Banach-Steinhaus theorem that $(B_{n,q_n}(f))$ $(q_n>1)$ approximates f for any $f\in C[0,1]$ if and only if

$$\sup_{n\in\mathbb{N}}\left\|B_{n,q_n}\right\|=\sup_{n\in\mathbb{N}}\sup_{x\in[0,1]}\left(\sum_{k=0}^{n}\left|p_{n\,n-k}(q_n;x)\right|\right)<+\infty. \tag{16}$$

We set

$$G_{s,n}(x)=\sum_{k=s+2}^{n}\left|p_{n\,n-k}(q_n;x)\right|,\quad s=0,1,\dots,n-2. \tag{17}$$

Since $p_{n\,n-k}(q_n;x)\ge 0$ for $x\in[q_n^{-s-1},q_n^{-s}]$ and $k=0,1,\dots,s+1$, we get, for $x\in[q_n^{-s-1},q_n^{-s}]$,

$$\sum_{k=0}^{s+1}\left|p_{n\,n-k}(q_n;x)\right|=\sum_{k=0}^{s+1}p_{n\,n-k}(q_n;x)=1-\sum_{k=s+2}^{n}p_{n\,n-k}(q_n;x)\le 1+G_{s,n}(x), \tag{18}$$

and, therefore,

$$\|F_n\|_s\le\left\|1+2G_{s,n}\right\|_s=1+2\left\|G_{s,n}\right\|_s,\quad s=0,1,\dots,n-2. \tag{19}$$

Next we will show that

$$\left\|G_{s,n}\right\|_s\le\left\|G_{s-1,n}\right\|_{s-1},\quad s=1,2,\dots,n-2. \tag{20}$$

Note that, for $x\in[q_n^{-s-1},q_n^{-s}]$,

$$G_{s,n}(x)=\sum_{k=s+2}^{n}\left|p_{n\,n-k}(q_n;x)\right|,\qquad G_{s-1,n}(q_nx)=\sum_{k=s+1}^{n}\left|p_{n\,n-k}(q_n;q_nx)\right|. \tag{21}$$

If we show that, for $x\in[q_n^{-s-1},q_n^{-s}]$ and $k=s+1,\dots,n-1$,

$$\left|p_{n\,n-k-1}(q_n;x)\right|\le\left|p_{n\,n-k}(q_n;q_nx)\right|, \tag{22}$$

then

$$G_{s,n}(x)\le G_{s-1,n}(q_nx),\quad x\in\left[q_n^{-s-1},q_n^{-s}\right], \tag{23}$$

and (20) follows. Indeed, for $x\in(q_n^{-s-1},q_n^{-s})$ and $k=s+1,\dots,n-1$,

$$\frac{\left|p_{n\,n-k}(q_n;q_nx)\right|}{\left|p_{n\,n-k-1}(q_n;x)\right|}=\frac{\left[{n\atop k}\right]_{q_n}(q_nx)^{n-k}\prod_{j=0}^{s-1}\left(1-q_n^{j+1}x\right)\prod_{j=s}^{k-1}\left(q_n^{j+1}x+1\right)}{\left[{n\atop k+1}\right]_{q_n}x^{n-k-1}\prod_{j=0}^{s}\left(1-q_n^{j}x\right)\prod_{j=s+1}^{k}\left(q_n^{j}x-1\right)}=\frac{[k+1]_{q_n}q_n^{n-k}x}{[n-k]_{q_n}(1-x)}. \tag{24}$$

Hence, (22) is equivalent to the following inequality:

$$\left(q_n^{k+1}-1\right)q_n^{n-k}x\ge\left(q_n^{n-k}-1\right)(1-x), \tag{25}$$

which is also equivalent to the inequality

$$x\ge\frac{q_n^{n-k}-1}{q_n^{n+1}-1}. \tag{26}$$

For $x\in(q_n^{-s-1},q_n^{-s})$ and $k=s+1,\dots,n-1$, we have

$$x>q_n^{-s-1}\ge\frac{q_n^{-s-1}\left(q_n^{n}-q_n^{s+1}\right)}{q_n^{n+1}-1}=\frac{q_n^{n-s-1}-1}{q_n^{n+1}-1}\ge\frac{q_n^{n-k}-1}{q_n^{n+1}-1}. \tag{27}$$

This proves (26). On the other hand, $p_{n\,n-k-1}(q_n;x)=0=p_{n\,n-k}(q_n;q_nx)$ for $x\in\{q_n^{-s-1},q_n^{-s}\}$, which completes the proof of (20). From (14), (19), and (20), we get

$$\left\|G_{0,n}\right\|_0\le\|F_n\|=\left\|B_{n,q_n}\right\|\le 1+2\left\|G_{0,n}\right\|_0. \tag{28}$$

This implies that (16) is equivalent to

$$\sup_{n\in\mathbb{N}}\left\|G_{0,n}\right\|_0=\sup_{n\in\mathbb{N}}\sup_{x\in[q_n^{-1},1]}\sum_{k=2}^{n}\left|p_{n\,n-k}(q_n;x)\right|<\infty. \tag{29}$$

Theorem 1 is proved. □

Proof of Theorem 2. First we show that

$$q_n-1=O\left(\frac{1}{n}\right). \tag{30}$$

Otherwise, we may assume that

$$\lim_{n\to\infty}n(q_n-1)=+\infty, \tag{31}$$

which implies

$$\lim_{n\to\infty}q_n^{n-1}=\lim_{n\to\infty}\exp\left((n-1)\ln q_n\right)\ge\lim_{n\to\infty}\exp\left((n-1)\min\left\{\frac{(q_n-1)}{2},\ln 2\right\}\right)=+\infty. \tag{32}$$

We have

$$\begin{aligned}\|G_{0,n}\|_0 &\geq \|p_{n\,n-2}(q_n;\cdot)\|_0 \geq \left|p_{n\,n-2}\left(q_n;\frac{q_n+1}{2q_n}\right)\right| \\ &= \frac{(q_n^n-1)(q_n^{n-1}-1)}{(q_n^2-1)(q_n-1)}\left(\frac{1+q_n}{2q_n}\right)^{n-2} \\ &\quad\times\left(1-\frac{1+q_n}{2q_n}\right)\left(q_n\frac{1+q_n}{2q_n}-1\right) \\ &= \frac{\left(1-q_n^{-n+1}\right)(q_n^n-1)}{8}\left(\frac{1+q_n}{2}\right)^{n-3} \\ &\geq \frac{\left(1-q_n^{-n+1}\right)(q_n^n-1)}{8} \longrightarrow +\infty, \quad (\text{as } n\longrightarrow\infty).\end{aligned} \tag{33}$$

This leads to a contradiction by Theorem 1. Hence, (30) holds.

Next, we show Theorem 2. Assume that $\overline{\lim}_{n\to\infty} n(q_n-1) > \ln 2$. Then by (30) we may suppose that, for some A, B, $\ln 2 < A < B < +\infty$,

$$1+\frac{A}{n} \leq q_n \leq 1+\frac{B}{n}. \tag{34}$$

For $0 < a < b$, we set $h(x) = (x^a-1)/(x^b-1)$, $x > 1$. Direct computation gives that

$$h'(x) = \frac{bx^{a-1}\left(x^{b-a}-((b-a)/b)x^b-a/b\right)}{(x^b-1)^2}. \tag{35}$$

Since the function $g(y) = x^y$ is convex on $(-\infty,+\infty)$ for a fixed $x > 0$, we get that

$$x^{b-a} = x^{((b-a)/b)\cdot b+(a/b)\cdot 0} \leq \frac{b-a}{b}x^b+\frac{a}{b}. \tag{36}$$

This means that $h'(x) \leq 0$ and $h(x)$ is nonincreasing on $(1,+\infty)$. Hence, for $x \in (1,\xi_0)$, $\xi_0 > 1$, we have

$$h(\xi_0) \leq h(x) \leq \lim_{x\to 1+} h(x) = \frac{a}{b}. \tag{37}$$

Put $x_0 = (1+q_n)/2q_n \in (q_n^{-1},1)$. Then, for $k_0 = [\ln n]$, we have

$$\begin{aligned}\|G_{0,n}\|_0 &\geq \left\|p_{n\,n-k_0}(q_n;\cdot)\right\|_0 \geq \left|p_{n\,n-k_0}(q_n;x_0)\right| \\ &= \frac{(q_n^n-1)\cdots\left(q_n^{n-k_0+1}-1\right)}{\left(q_n^{k_0}-1\right)\cdots(q_n-1)}x_0^{n-k_0} \\ &\quad\times(1-x_0)\prod_{s=1}^{k_0-1}(q_n^s x_0-1) \\ &\geq \left(q_n^{n-k_0}-1\right)^{k_0}x_0^{n-k_0}(1-x_0) \\ &\quad\times\frac{\left(q_n^{k_0-1}x_0-1\right)\cdots(q_nx_0-1)}{\left(q_n^{k_0}-1\right)\cdots(q_n-1)}.\end{aligned} \tag{38}$$

Using (34), the inequalities

$$\begin{aligned}&\frac{q_n^{s+1}x_0-1}{q_n^s-1} \geq 1, \quad s = 1,\dots,k_0-2, \\ &x_0^{n-k_0}(1-x_0)(q_nx_0-1) \geq q_n^{-n+k_0-1}\frac{(q_n-1)^2}{4} \\ &\qquad\geq \left(1+\frac{B}{n}\right)^{-n}\frac{(q_n-1)^2}{4} \\ &\qquad\geq (q_n-1)^2\frac{\exp(-B)}{4},\end{aligned} \tag{39}$$

and the nonincreasing property of $h(x)$, we continue to obtain that

$$\begin{aligned}&\|G_{0,n}\|_0 \\ &\geq \left(\left(1+\frac{A}{n}\right)^{n-\ln n}-1\right)^{k_0}\frac{\exp(-B)}{4}\frac{(q_n-1)^2}{\left(q_n^{k_0}-1\right)\left(q_n^{k_0-1}-1\right)} \\ &\geq \left(\left(1+\frac{A}{n}\right)^{n-\ln n}-1\right)^{k_0} \\ &\quad\times\frac{\exp(-B)}{4}\frac{(A/n)^2}{\left((1+B/n)^{k_0}-1\right)\left((1+B/n)^{k_0-1}-1\right)}.\end{aligned} \tag{40}$$

We observe that

$$\begin{aligned}&\lim_{n\to\infty}\left(1+\frac{A}{n}\right)^{n-\ln n} \\ &\quad= \exp\left(\lim_{n\to\infty}(n-\ln n)\ln\left(1+\frac{A}{n}\right)\right) \\ &\quad= \exp\left(\lim_{n\to\infty}\frac{A(n-\ln n)}{n}\right) = \exp(A) > 2,\end{aligned} \tag{41}$$

and, for $s = k_0, k_0-1$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{(1+B/n)^s-1}{B\ln n/n} \\ &\quad= \lim_{n\to\infty}\frac{\exp(s\ln(1+B/n))-1}{B\ln n/n} = \lim_{n\to\infty}\frac{s\ln(1+B/n)}{B\ln n/n} = 1.\end{aligned} \tag{42}$$

Thus, for some $a \in (1, e^A-1)$ and sufficiently large n, we have

$$\|G_{0,n}\|_0 \geq \frac{a^{\ln n-1}}{(\ln n)^2}\frac{\exp(-B)A^2}{4B^2} \longrightarrow +\infty. \tag{43}$$

By Theorem 1, we know that there exists a function $f \in C[0,1]$ such that the sequence $(B_{n,q_n}(f))$ does not converge to f in $C[0,1]$. This leads to a contradiction. Hence, $\overline{\lim}_{n\to\infty} n(q_n-1) \leq \ln 2$. Theorem 2 is proved. □

Proof of Theorem 3. From Theorem 1, we know that it is sufficient to show that if $q_n \leq 1 + \ln 2/n$ for sufficiently large n, then

$$\sup_{n\in\mathbb{N}} \|G_{0,n}\|_0 < \infty. \tag{44}$$

For $x \in (q_n^{-1}, 1)$, we set $\alpha = -\log_{q_n} x$. Then $\alpha \in (0,1)$ and $x = q_n^{-\alpha}$. Since, for $k = 2, \ldots, n-1$,

$$\begin{aligned} q_n^{\alpha}\left(q_n^{n-k} - 1\right) &\leq q_n^{n-k+\alpha} - 1 \\ &\leq q_n^n - 1 \leq \left(1 + \frac{\ln 2}{n}\right)^n - 1 \leq 1, \end{aligned} \tag{45}$$

by (37) we get that

$$\begin{aligned} \frac{|p_{n\,n-k-1}(q_n; x)|}{|p_{n\,n-k}(q_n; x)|} &= \frac{\left(q_n^{n-k} - 1\right)\left(q_n^{k-\alpha} - 1\right)}{\left(q_n^{k+1} - 1\right) q_n^{-\alpha}} \\ &\leq \frac{q_n^{k-\alpha} - 1}{q_n^{k+1} - 1} \leq \frac{k-\alpha}{k+1}. \end{aligned} \tag{46}$$

On the other hand, by (37) we have

$$\begin{aligned} |p_{n\,n-2}(q_n; x)| &= \begin{bmatrix} n \\ 2 \end{bmatrix}_{q_n} x^{n-1}\left(\frac{1}{x} - 1\right)(q_n x - 1) \\ &\leq \frac{\left(q_n^n - 1\right)\left(q_n^{n-1} - 1\right)}{\left(q_n^2 - 1\right)(q_n - 1)}\left(q_n^{\alpha} - 1\right)\left(q_n^{1-\alpha} - 1\right) \\ &\leq \frac{\left(q_n^{\alpha} - 1\right)\left(q_n^{1-\alpha} - 1\right)}{2(q_n - 1)^2} \leq \frac{\alpha(1-\alpha)}{2}. \end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$|p_{n\,n-k}(q_n; x)| \leq \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!}. \tag{48}$$

Hence, for $x = q_n^{-\alpha}$, $\alpha \in (0,1)$,

$$G_{0,n}(x) = \sum_{k=2}^{n} |p_{n\,n-k}(q_n; x)| \leq \sum_{k=2}^{\infty} \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!}. \tag{49}$$

Obviously (49) is satisfied for $x \in \{0,1\}$. We note that, for $x \in [0,1]$,

$$(1-x)^{\alpha} = 1 - \alpha x - \sum_{k=2}^{\infty} \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!} x^k. \tag{50}$$

The above formula with $x = 1$ means that

$$\sum_{k=2}^{\infty} \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!} = 1 - \alpha. \tag{51}$$

Thus, by (49),

$$\|G_{0,n}\|_0 \leq \sup_{\alpha\in[0,1]} \sum_{k=2}^{\infty} \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!} = 1. \tag{52}$$

This completes the proof of Theorem 3. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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