

Research Article

Elliptic Travelling Wave Solutions to a Generalized Boussinesq Equation

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Travelling wave solutions for the generalized Boussinesq wave equation are studied by using the Weierstrass elliptic function method. As a result, some previously known solutions are recovered, and at the same time some new ones are also given, as well as integrable ones.

1. Introduction

It is well known that investigating the exact travelling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. In order to obtain the exact solutions, a number of methods have been proposed, such as the homogeneous balance method [1], the hyperbolic function expansion method [2], Jacobi elliptic function method [3], F-expansion method [4], homotopy analysis method [5, 6], the bifurcation theory method of dynamical systems [7, 8], and Weierstrass elliptic function method [9]. Among these methods, Weierstrass elliptic function method is a powerful mathematic tool to solve nonlinear evolution equations. By using this method, many kinds of important nonlinear evolution equations have been solved successfully [10, 11].

To understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [12] introduced and studied a family of fully nonlinear *KdV* equations $K(m, n)$:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3. \quad (1)$$

A class of solitary waves (which they named compactons) with the special property that, after colliding with other compacton solutions, they reemerge with the same coherent shape was presented. From then on, compacton solutions attracted a lot of interest [13–15]. You can refer to [16–18], for

example, for more details about the properties of compacton solutions. Compacton solutions and solitary solutions of other nonlinear evolution equations such as Boussinesq equations and Boussinesq-like $B(m, n)$ equations had been extensively investigated by many authors [19–21]. Recently, Yan [19] introduced a class of fully Boussinesq equations $B(m, n)$

$$u_{tt} = (u^m)_{xx} + (u^n)_{xxxx} = 0, \quad m, n \in \mathbb{R} \quad (2)$$

and presented some of its compacton solutions when $m = n$. Liu et al. [22] utilized the integral approach to investigate its compacton solutions. More recently, Zhu [20, 21] studied Boussinesq-like $B(m, n)$ equations

$$\begin{aligned} u_{tt} + (u^m)_{xx} - (u^n)_{xxxx} &= 0, \quad m, n > 1, \\ u_{tt} - (u^m)_{xx} + (u^n)_{xxxx} &= 0, \quad m, n > 1, \\ u_{tt} + (u^{2n})_{xx} + (u^{2n})_{xxxx} &= 0, \quad n \geq 1, \end{aligned} \quad (3)$$

by using the extended decomposition method.

By making use of the sine-cosine method technique, Wazwaz [23] obtained various forms of travelling wave solutions for improved Boussinesq equations with positive or negative exponents. It was highlighted in [23] that the variants and exponents in the equations directly lead to a qualitative change in the physical structures of the obtained solutions. Lai and Wu [24, 25] proposed an approach for

constructing asymptotic solutions in a Sobolev space for generalized Boussinesq equations. Using the tanh methods, Wazwaz [26] obtained the compacton solutions, solitons, solitary pattern solutions, and periodic solutions for the Boussinesq wave equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} + a_1 u_{xxxx} \tag{4}$$

and its generalized form

$$u_{tt} - u_{xx} + \frac{3}{2}(u^m)_{xx} + a_1 u_{xxxx}, \tag{5}$$

where a_1 and $m \neq \pm 1$ are constants.

In this paper, we will consider the travelling wave solutions of the following generalized Boussinesq equations (simply called GB(m, n) equation):

$$u_{tt} - u_{xx} + a(u^m)_{xx} + b(u^n)_{xxxx}. \tag{6}$$

The objective of this paper is to investigate the travelling wave solutions of (6) systematically, by applying the Weierstrass elliptic function method. It will be shown that some previously known solutions are recovered, and at the same time, some new ones are also given.

The rest of this paper is organized as follows. In Section 2, we first outline the Weierstrass elliptic function method which will be used in the next section. In Section 3, we give some particular travelling wave solutions of (6). Finally, some conclusions are given in Section 4.

2. Weierstrass Elliptic Functions

Let us consider the following nonlinear differential equation:

$$\left(\frac{d\phi(z)}{dz}\right)^2 = a_0\phi^4 + 4a_1\phi^3 + 6a_2\phi^2 + 4a_3\phi + a_4 \equiv f(\phi). \tag{7}$$

As is well known [27, 28], the solutions ϕ of (7) can be expressed in terms of elliptic functions \wp . It reads as

$$\phi(z) = \phi_0 + \frac{f'(\phi_0)}{4[\wp(z, g_2, g_3) - (1/24)f''(\phi_0)]}, \tag{8}$$

where the primes denote differentiation with respect to ϕ and ϕ_0 is a simple root of $f(\phi)$. The invariants g_2, g_3 of elliptic functions $\wp(t, g_2, g_3)$ are related to the coefficients of $f(\phi)$ by [29]

$$\begin{aligned} g_2 &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ g_3 &= a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \end{aligned} \tag{9}$$

When g_2 and g_3 are real and the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{10}$$

is positive, negative, or zero, we have different behavior of $\wp(t)$. The conditions [9]

$$\Delta \neq 0 \quad \text{or} \quad \Delta = 0, \quad g_2 > 0, \quad g_3 > 0 \tag{11}$$

lead to periodic solutions, whereas the conditions

$$\Delta = 0, \quad g_2 \geq 0, \quad g_3 \leq 0 \tag{12}$$

lead to solitary wave solutions.

If $\Delta = 0$, then $\wp(t, g_2, g_3)$ degenerates into trigonometric or hyperbolic functions [30]. Thus, periodic solutions according to (8) are determined by

$$\begin{aligned} \phi(z) &= \phi_0 \\ &+ f'(\phi_0) \times \left(4 \left[-\frac{e_1}{2} - \frac{f''(\phi_0)}{24} \right. \right. \\ &\quad \left. \left. + \frac{3}{2}e_1 \operatorname{csc}^2 \left(\sqrt{\frac{3}{2}e_1 t} \right) \right] \right)^{-1}, \end{aligned} \tag{13}$$

$\Delta = 0, \quad g_3 > 0,$

and solitary wave solutions by

$$\begin{aligned} \phi(z) &= \phi_0 \\ &+ f'(\phi_0) \times \left(4 \left[e_1 - \frac{f''(\phi_0)}{24} \right. \right. \\ &\quad \left. \left. + 3e_1 \operatorname{csch}^2(\sqrt{3e_1 t}) \right] \right)^{-1}, \end{aligned} \tag{14}$$

$\Delta = 0, \quad g_3 < 0,$

where $e_1 = \sqrt[3]{|g_3|}$ in (13) and $e_1 = (1/2)\sqrt[3]{|g_3|}$ in (14).

3. Travelling Wave Solutions of the Generalized Boussinesq Equations

In this section, we consider the travelling wave solutions of (6). Assume that (6) has an exact solution in the form of a travelling wave

$$u(x, t) = u(\xi), \quad \xi = \mu(x - ct - x_0), \tag{15}$$

where $\mu \neq 0, c \neq 0$, and x_0 are arbitrary constants. Substituting (15) into (6), we get

$$(c^2 - 1)u'' + a(u^m)'' + b\mu^2(u^n)'''''. \tag{16}$$

Integrating (16) twice and letting the constants of integration be zero give rise to

$$(c^2 - 1)u + a(u^m) + b\mu^2(u^n)'' = 0. \tag{17}$$

Making transformation, $u^n = \varphi$, and (17) becomes

$$(c^2 - 1)\varphi^{1/n} + a(\varphi^{m/n}) + b\mu^2\varphi'' = 0. \tag{18}$$

Multiply (18) by the integrating factor φ' on both sides, and integrating it with respect to ξ , we have

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2}{b\mu^2} \left[C_1 - \frac{n(c^2 - 1)}{n + 1} \varphi^{1+(1/n)} - \frac{an}{m + n} \varphi^{1+(m/n)} \right], \tag{19}$$

where C_1 is an integration constant.

Making the transformation $\varphi = \phi^p$, $p \neq 0$ we get

$$\begin{aligned} \left(\frac{d\phi}{d\xi}\right)^2 &= \frac{2}{b\mu^2 p^2} \\ &\times \left[C_1 \phi^{2-2p} - \frac{n(c^2 - 1)}{n + 1} \phi^{2+((1/n)-1)p} \right. \\ &\quad \left. - \frac{an}{m + n} \phi^{2+((m/n)-1)p} \right]. \end{aligned} \tag{20}$$

In order to guarantee the integrability of (20), the powers of ϕ have to be integer numbers between 0 and 4, and therefore we have the following parameter conditions:

- (1) if $C_1 = 0, n = 1$, then $p \in \{-2/m, -1/(m - 1), 2/(m - 1), 1/(m - 1)\}$;
- (2) if $C_1 = 0, n \neq 1, n = m + 1$, then $p \in \{m/(m - 1), -m/(m - 1), 2m/(m - 1), -2m/(m - 1)\}$;
- (3) if $C_1 \neq 0, n = 1$, then we have $m = 5$ and $p = \pm 1/2$;
- (4) if $n = p = 1$, then we have $m = 2$ or $m = 3$.

Next, using the basic results on Weierstrass elliptic functions shown in Section 2, we will analyze the solutions of (20) in the above cases.

3.1. Case 1. (i) If $C_1 = 0, n = 1$, and $p = 1/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2(m - 1)^2}{b\mu^2} \left[\frac{(1 - c^2)}{2} \phi^2 - \frac{a}{m + 1} \phi^3 \right] = f(\phi). \tag{21}$$

It is easy to see that $f(\phi)$ has two roots: $\phi_0 = 0$ and $\phi_0 = ((1 - c^2)(m + 1))/2a$. From (8), we can obtain the following solution for (21):

$$\phi = \phi_0 + \frac{6(m - 1)((1 - c^2)\phi_0 - (3a/(m + 1))\phi_0^2)}{12b\mu^2 \wp(\xi) - (m - 1)(1 - c^2 - (6a/(m + 1)))}, \tag{22}$$

where the invariants are

$$g_2 = \frac{(m - 1)^4(1 - c^2)^2}{12(b\mu^2)^2}, \quad g_3 = -\frac{(m - 1)^6(1 - c^2)^3}{216(b\mu^2)^3}. \tag{23}$$

Then the discriminant $\Delta = 0$. The root $\phi_0 = 0$ gives the trivial solution $\phi = 0$, and the nonzero solution of (21) can be found

by taking $\phi_0 = ((1 - c^2)(m + 1))/2a$. Hence, from (11), we have the periodic wave solution to (21)

$$\phi = \frac{(1 - c^2)(m + 1)}{2a} \sec^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{(b\mu^2)}} \xi \right) \tag{24}$$

for $(1 - c^2)/b < 0$. From (12), we have the solitary wave solution

$$\phi = \frac{(1 - c^2)(m + 1)}{2a} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{(b\mu^2)}} \xi \right) \tag{25}$$

for $(1 - c^2)/b > 0$.

Therefore, when $(1 - c^2)/b < 0$, GB($m, 1$) (6) has the following periodic wave solution:

$$\begin{aligned} u(x, t) &= \left(\frac{(1 - c^2)(m + 1)}{2a} \right. \\ &\quad \left. \times \sec^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{(b\mu^2)}} \xi \right) \right)^{1/(m-1)}. \end{aligned} \tag{26}$$

When $(1 - c^2)/b < 0$, GB($m, 1$) (6) has the following solitary wave solution:

$$\begin{aligned} u(x, t) &= \left(\frac{(1 - c^2)(m + 1)}{2a} \right. \\ &\quad \left. \times \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{(b\mu^2)}} \xi \right) \right)^{1/(m-1)}. \end{aligned} \tag{27}$$

Remark 1. When $m > 1$, the solutions (26)-(27) that we obtained are completely the same as (22)-(24) that were obtained by Lai [31].

(ii) If $C_1 = 0, n = 1$, and $p = -1/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2(m - 1)^2}{b\mu^2} \left[\frac{(1 - c^2)}{2} \phi^2 - \frac{a}{m + 1} \phi \right] = f(\phi). \tag{28}$$

It is easy to see that $f(\phi)$ has two roots: $\phi_0 = 0$ and $\phi_0 = 2a/((1 - c^2)(m + 1))$. From (8), we can obtain that the expression for the solutions of (28) is

$$\begin{aligned} \phi = & \left(12b\mu^2\wp_0(\xi) + 5(m - 1)^2(1 - c^2)\phi_0 \right. \\ & \left. - \frac{6a(m - 1)^2}{(m + 1)} \right) \\ & \times \left(12b\mu^2\wp(\xi) - (m - 1)^2(1 - c^2) \right)^{-1}, \end{aligned} \tag{29}$$

where the invariants are given by (23). Taking the root $\phi_0 = 0$ into (29), we get

$$\phi = \frac{-6a(m - 1)^2}{(m + 1) \left[12b\mu^2\wp(\xi) - (m - 1)^2(1 - c^2) \right]}. \tag{30}$$

Since the discriminant $\Delta = 0$, thus from (11), we have the periodic wave solution to (28)

$$\phi = \frac{2a}{(1 - c^2)(m + 1)} \sin^2 \left(\frac{1}{2} \sqrt{-\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right), \tag{31}$$

for $(1 - c^2)/b < 0$. From (12), we have the solitary wave solution

$$\phi = -\frac{2a}{(1 - c^2)(m + 1)} \sinh^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right) \tag{32}$$

for $(1 - c^2)/b > 0$.

Therefore, when $(1 - c^2)/b < 0$, the GB($m, 1$) (6) has the following periodic wave solution:

$$\begin{aligned} u(x, t) &= \left(\frac{(1 - c^2)(m + 1)}{2a} \right. \\ & \left. \times \operatorname{csc}^2 \left(\frac{1}{2} \sqrt{-\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right) \right)^{1/(m-1)}. \end{aligned} \tag{33}$$

When $(1 - c^2)/b > 0$, the GB($m, 1$) (6) has the following solitary wave solution:

$$\begin{aligned} u(x, t) &= \left(-\frac{(1 - c^2)(m + 1)}{2a} \right. \\ & \left. \times \operatorname{csch}^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right) \right)^{1/(m-1)}. \end{aligned} \tag{34}$$

When substituting the second root $\phi_0 = 2a/((1 - c^2)(m + 1))$ into (29), we have

$$\begin{aligned} \phi = & \left(24ab\mu^2\wp(\xi) + 5(m + 1)(m - 1)^2(1 - c^2)^2 \right. \\ & \left. - 6a(m - 1)^2(1 - c^2) \right) \\ & \times \left((m + 1)(1 - c^2) \left[12b\mu^2\wp(\xi) - (m - 1)^2(1 - c^2) \right] \right)^{-1}. \end{aligned} \tag{35}$$

So from (11), we have the periodic wave solution to (28):

$$\phi = \frac{2a}{(1 - c^2)(m + 1)} \cos^2 \left(\frac{1}{2} \sqrt{-\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right), \tag{36}$$

for $(1 - c^2)/b < 0$. From (12), we have the solitary wave solution to (28):

$$\phi = \frac{2a}{(1 - c^2)(m + 1)} \cosh^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right). \tag{37}$$

Therefore, when $(1 - c^2)/b < 0$, GB($m, 1$) (6) has the following periodic wave solution:

$$\begin{aligned} u(x, t) &= \left(\frac{2a}{(1 - c^2)(m + 1)} \right. \\ & \left. \times \cos^2 \left(\frac{1}{2} \sqrt{-\frac{(m - 1)^2(1 - c^2)}{b\mu^2}} \xi \right) \right)^{1/(1-m)}. \end{aligned} \tag{38}$$

When $(1 - c^2)/b > 0$, the GB($m, 1$) (6) has the following solitary wave solution:

$$u(x, t) = \left(\frac{2a}{(1 - c^2)(m + 1)} \times \cosh^2 \left(\frac{1}{2} \sqrt{\frac{(m - 1)^2 (1 - c^2)}{b\mu^2}} \xi \right) \right)^{1/(1-m)} \tag{39}$$

Remark 2. When $m > 1$, the solutions (33)-(34) that we obtained are completely the same as (23)-(25) that were obtained by Lai [31]; when $m < 1$ and $m \neq -1$, the solutions (38)-(39) that we obtained are similar as the solutions (26)-(28) that were obtained by Lai [31].

(iii) If $C_1 = 0, n = 1$, and $p = 2/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi} \right)^2 = \frac{(m - 1)^2}{2b\mu^2} \left[\frac{(1 - c^2)}{2} \phi^2 - \frac{a}{m + 1} \phi^4 \right] = f(\phi). \tag{40}$$

According to (8), the solutions of (40) read

$$\phi = \phi_0 + \left(3(m - 1)^2 \left[(1 - c^2) \phi_0 - \frac{4a}{m + 1} \phi_0^3 \right] \times \left(24b\mu^2 \wp(\xi) - (m - 1)^2 \times \left(\frac{(1 - c^2)}{2} - \frac{6a}{m + 1} \phi_0^2 \right) \right)^{-1} \right), \tag{41}$$

where the invariants are

$$g_2 = \frac{(m - 1)^4 (1 - c^2)^2}{192(b\mu^2)^2}, \quad g_3 = -\frac{(m - 1)^6 (1 - c^2)^3}{(24b\mu^2)^3}. \tag{42}$$

Taking the root $\phi_0 = 0$ into (41), we get the trivial solution $\phi = 0$. However, when taking the root $\phi = \pm \sqrt{(1 - c^2)(m + 1)/2a}$ into (41), we can obtain the solutions to (40)

$$\phi = \pm \sqrt{\frac{(1 - c^2)(m + 1)}{2a} \frac{48b\mu^2 \wp(\xi) - (m - 1)^2 (1 - c^2)}{48b\mu^2 \wp(\xi) + 5(m - 1)^2 (1 - c^2)}}. \tag{43}$$

Since the discriminant $\Delta = 0$, it is easy to see from (11), (12) that the above solutions (43) will generate the same periodic and solitary wave solutions to (6) as (26), (27).

(iv) If $C_1 = 0, n = 1$, and $p = -2/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi} \right)^2 = \frac{(m - 1)^2}{2b\mu^2} \left[\frac{(1 - c^2)}{2} \phi^2 - \frac{a}{m + 1} \right] = f(\phi). \tag{44}$$

Using the same arguments as the above, we can deduce that it would give exactly the same solutions of (6) as Case 1 (ii).

3.2. Case 2. (i) If $C_1 = 0, n \neq 1, n = m$, and $p = -m/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi} \right)^2 = \frac{2(m - 1)^2}{b\mu^2 m^2} \left[\frac{m(1 - c^2)}{m + 1} \phi^3 - \frac{a}{2} \phi^2 \right] = f(\phi). \tag{45}$$

Using similar arguments to Case 1 (i), we can deduce that when $a/b > 0$, the GB(m, m) (6) has the following periodic wave solution:

$$u(x, t) = \left(\frac{a(m + 1)}{2m(1 - c^2)} \sec^2 \left(\frac{1}{2} \left| \frac{m - 1}{m} \right| \sqrt{\frac{a}{b\mu^2}} \xi \right) \right)^{-1/(m-1)}. \tag{46}$$

When $a/b < 0$, the GB(m, m) (6) has the following solitary wave solution:

$$u(x, y, t) = \left[\frac{a(m + 1)}{2m(1 - c^2)} \operatorname{sech}^2 \left(\frac{1}{2} \left| \frac{m - 1}{m} \right| \sqrt{\frac{a}{b\mu^2}} \xi \right) \right]^{-1/(m-1)}. \tag{47}$$

Remark 3. When $m < 1$, the solutions (46)-(47) that we obtained are completely the same as (62)-(64) that were obtained by Lai [31].

(ii) If $C_1 = 0, n \neq 1, n = m$, and $p = m/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi} \right)^2 = \frac{2(m - 1)^2}{b\mu^2 m^2} \left[\frac{m(1 - c^2)}{m + 1} \phi - \frac{a}{2} \phi^2 \right] = f(\phi). \tag{48}$$

Using similar arguments to Case 1 (ii), we can deduce that when $a/b > 0$, the GB(m, m) (6) has the following periodic wave solution:

$$u(x, t) = \left[\frac{2m(1 - c^2)}{a(m + 1)} \sin^2 \left(\frac{1}{2} \left| \frac{m - 1}{m} \right| \sqrt{\frac{a}{b\mu^2}} \xi \right) \right]^{1/(m-1)}. \tag{49}$$

When $a/b < 0$, the GB(m, m) (6) has the following solitary wave solution:

$$u(x, t) = \left[\frac{2m(1 - c^2)}{a(m + 1)} \sinh^2 \left(\frac{1}{2} \left| \frac{m - 1}{m} \right| \sqrt{\frac{a}{b\mu^2}} \xi \right) \right]^{1/(m-1)}. \tag{50}$$

Remark 4. When $m > 1$, solutions (49)-(50) that we obtained are completely the same as (59)-(61) that were obtained by Lai [31].

(iii) If $C_1 = 0, n \neq 1, n = m$, and $p = -2m/(m - 1)$, (20) becomes

$$\left(\frac{d\phi}{d\xi} \right)^2 = \frac{(m - 1)^2}{2b\mu^2 m^2} \left[\frac{m(1 - c^2)}{m + 1} \phi^4 - \frac{a}{2} \phi^2 \right] = f(\phi). \tag{51}$$

Using similar arguments to Case 2 (i), we can deduce that it would give exactly the same solutions of the GB(m, m) (6) as those given by (45) and (46).

(iv) If $C_1 = 0, n \neq 1, n = m$, and $p = 2m/(m - 1)$,

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{(m-1)^2}{2b\mu^2 m^2} \left[\frac{m(1-c^2)}{m+1} - \frac{a}{2}\phi^2 \right] = f(\phi). \quad (52)$$

Using similar arguments to Case 2 (ii), we can deduce that it would give exactly the same solutions of the GB(m, m) (6) as those given by (48) and (49).

3.3. Case 3. (i) If $C_1 \neq 0, n = 1, m = 5$, and $p = 1/2$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{8}{b\mu^2} \left[C_1\phi + \frac{(1-c^2)}{2}\phi^2 - \frac{a}{6}\phi^4 \right] = f(\phi). \quad (53)$$

According to (8), the solutions of (52) read

$$\phi = \frac{3b\mu^2\phi_0\wp(\xi) + 5(1-c^2)\phi_0 - 2a\phi_0^3 + 6C_1}{3b\mu^2\wp(\xi) - (1-c^2) + 2a\phi_0^2}, \quad (54)$$

where the invariants are given by

$$g_2 = \frac{4(1-c^2)^2}{3(b\mu^2)^2}, \quad g_3 = -\frac{8(1-c^2)^3}{27(b\mu^2)^3} + \frac{16aC_1^2}{3(b\mu^2)^3}. \quad (55)$$

So we can obtain the general expressions for the solutions to the GB(5, 1) (6):

$$u(x, t) = \left[\frac{3b\mu^2\phi_0\wp(\xi) + 5(1-c^2)\phi_0 - 2a\phi_0^3 + 6C_1}{3b\mu^2\wp(\xi) - (1-c^2) + 2a\phi_0^2} \right]^{1/2}. \quad (56)$$

For example, substituting the simplest root $\phi = 0$ of $f(\phi)$ into (56), we get

$$u(x, t) = \left[\frac{6C_1}{3b\mu^2\wp(\xi) - (1-c^2)} \right]^{1/2}. \quad (57)$$

(ii) If $C_1 \neq 0, n = 1, m = 5$, and $p = -1/2$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{8}{b\mu^2} \left[C_1\phi^3 + \frac{(1-c^2)}{2}\phi^2 - \frac{a}{6} \right] = f(\phi). \quad (58)$$

Using similar arguments to Case 3 (i), we can get the following general expression for the solutions to GB(5, 1) (6):

$$u(x, t) = \left[\frac{3b\mu^2\phi_0\wp(\xi) + 3(1-c^2)\phi_0 + 12C_1\phi_0^2}{3b\mu^2\wp(\xi) - (1-c^2) - 6C_1} \right]^{-1/2}, \quad (59)$$

where the invariants are given by (55).

3.4. Case 4. (i) If $n = 1, m = 2$, and $p = 1$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2}{b\mu^2} \left[C_1 + \frac{(1-c^2)}{2}\phi^2 - \frac{a}{3}\phi^3 \right] = f(\phi). \quad (60)$$

Following the same procedure as mentioned above, we can get the general solutions to GB(2, 1) (6):

$$u(x, t) = \left[\frac{12b\mu^2\phi_0\wp(\xi) + 5(1-c^2)\phi_0 - 4a\phi_0^2}{12b\mu^2\wp(\xi) - (1-c^2) + 2a\phi_0} \right], \quad (61)$$

where ϕ_0 is the real root of $f(\phi) = 0$ and the invariants are given by

$$g_2 = 3\frac{4(1-c^2)^2}{6(b\mu^2)^2}, \quad g_3 = -\frac{(1-c^2)^3}{216(b\mu^2)^3} - \frac{a^2C_1}{108(b\mu^2)^3}. \quad (62)$$

(ii) If $n = 1, m = 3$, and $p = 1$, (20) becomes

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2}{b\mu^2} \left[C_1 + \frac{(1-c^2)}{2}\phi^2 - \frac{a}{4}\phi^4 \right] = f(\phi). \quad (63)$$

Likewise, we can get the general solutions to GB(3, 1) (6):

$$u(x, t) = \left[\frac{12b\mu^2\phi_0\wp(\xi) + 5(1-c^2)\phi_0 - 3a\phi_0^3}{12b\mu^2\wp(\xi) - (1-c^2) + 3a\phi_0^2} \right], \quad (64)$$

where ϕ_0 is the real root of $f(\phi) = 0$.

4. Conclusion

From the above discussion, we find the traveling wave solutions of the generalized Boussinesq equation GB(m, n), which are expressed by the hyperbolic functions and trigonometric functions, out without the aid of mathematical software. The results show that the Weierstrass function method is a powerful mathematical tool to search for exact solutions to nonlinear differential equations, especially solitary ones. It may be advantageous that this quite general method can lead to free parameters as shown in the solution. We believe that this approach can also be used to solve other nonlinear equations.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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