Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 253102, 12 pages
http://dx.doi.org/10.1155/2014/253102

Research Article

Binary Nonlinearization for AKNS-KN Coupling System

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Received 30 April 2014; Accepted 29 May 2014; Published 15 June 2014

Academic Editor: Yufeng Zhang

The AKNS-KN coupling system is obtained on the base of zero curvature equation by enlarging the spectral equation. Under the Bargmann symmetry constraint, the AKNS-KN coupling system is decomposed into two integrable Hamiltonian systems with the corresponding variables x, t_n and the finite dimensional Hamiltonian systems are Liouville integrable.

1. Introduction

Since the integrable coupling definition is proposed, we have got many integrable coupling systems. Furthermore, the exact solutions, Darboux transformation, and Hamiltonian structure of these coupling systems have been obtained [1–3]. In 2008, the AKNS-KN coupling system is obtained by the loop algebra whose Hamiltonian system is received with the variational identity [4]. In 1994, the binary nonlinearization method was put forward by Li and Ma [5], and then the technique of the binary nonlinearization has been successfully applied to many soliton equations, such as the AKNS hierarchy, the KdV hierarchy, and the super NLS-MKDV hierarchy, but there are few results on binary nonlinearization of the coupling system. In this paper, we design a proper spectrum equation and obtain the AKNS-KN coupling system under the zero curvature equation, but the recursive operator is different from the operator of [4].

This paper is organized as follows. In Section 2, we will consider the AKNS-KN coupling soliton hierarchy. Bargmann symmetry constraint for the AKNS-KN coupling system will be given in Section 3. Section 4 will be devoted to study the AKNS-KN coupling system by employing the binary nonlinearization technique which involves two sets of dependent variables x and t_n. We especially list the special cases, such as AKNS integrable coupling system and KN integrable coupling system.

2. The AKNS-KN Coupling System

We design a spectral problem $\phi_x = U\phi$, where the spectral operator is as follows:

$$U = \begin{pmatrix} \lambda^2 & u_1 + u_3\lambda & 0 & u_5 + u_7\lambda \\ u_2 + u_4\lambda & -\lambda^2 & u_6 + u_8\lambda & 0 \\ 0 & 0 & \lambda^2 & u_1 + u_3\lambda \\ 0 & 0 & u_2 + u_4\lambda & -\lambda^2 \end{pmatrix}, \quad (1)$$

and set

$$V = \begin{pmatrix} A + B\lambda & C + D\lambda & E + F\lambda & G + N\lambda \\ I + M\lambda & -A - B\lambda & K + L\lambda & -E - F\lambda \\ 0 & 0 & A + B\lambda & C + D\lambda \\ 0 & 0 & I + M\lambda & -A - B\lambda \end{pmatrix}, \quad (2)$$

where

$$A = \sum_{i\geq 0} A_i\lambda^{-2i}, \qquad B = \sum_{i\geq 0} B_i\lambda^{-2i},$$

$$C = \sum_{i\geq 0} C_i\lambda^{-2i}, \qquad D = \sum_{i\geq 0} D_i\lambda^{-2i},$$

$$E = \sum_{i\geq 0} E_i\lambda^{-2i}, \qquad F = \sum_{i\geq 0} F_i\lambda^{-2i},$$

$$G=\sum_{i\geq 0}G_i\lambda^{-2i},\qquad N=\sum_{i\geq 0}H_i\lambda^{-2i},$$
$$I=\sum_{i\geq 0}I_i\lambda^{-2i},\qquad M=\sum_{i\geq 0}M_i\lambda^{-2i},$$
$$K=\sum_{i\geq 0}K_i\lambda^{-2i},\qquad L=\sum_{i\geq 0}L_i\lambda^{-2i}. \tag{3}$$

Under the zero curvature equation $V_x=[U\ V]$, we read

$$\begin{aligned}
A_{i,x}&=u_1I_i-u_2C_i+u_3M_{i+1}-u_4D_{i+1},\\
B_{i,x}&=u_1M_i-u_2D_i+u_3I_i-u_4C_i,\\
C_{i,x}&=2C_{i+1}-2u_1A_i-2u_3B_{i+1},\\
D_{i,x}&=2D_{i+1}-2u_1B_i-2u_3A_i,\\
E_{i,x}&=u_1K_i-u_2G_i+u_3L_{i+1}-u_4N_{i+1}\\
&\quad+u_5I_i-u_6C_i+u_7M_{i+1}-u_8D_{i+1},\\
F_{i,x}&=u_1L_i-u_2N_i+u_3K_i-u_4G_i+u_5M_i\\
&\quad-u_6D_i+u_7I_i-u_8C_i,\\
G_{i,x}&=2G_{i+1}-2u_1E_i-2u_3F_{i+1}-2u_5A_i-2u_7B_{i+1},\\
N_{i,x}&=2H_{i+1}-2u_1F_i-2u_3E_i-2u_5B_i-2u_7A_i,\\
I_{i,x}&=-2I_{i+1}+2u_2A_i+2u_4B_{i+1},\\
J_{i,x}&=-2M_{i+1}+2u_2B_i+2u_4A_i,\\
K_{i,x}&=-2K_{i+1}+2u_2E_i+2u_4F_{i+1}+2u_6A_i+2u_8B_{i+1},\\
L_{i,x}&=-2L_{i+1}+2u_2F_i+2u_4E_i+2u_6B_i+2u_8A_i.
\end{aligned}\tag{4}$$

Equation (4) is equivalent to

$$\begin{aligned}
&\left(M_{i+1}+L_{i+1},D_{i+1}+N_{i+1},I_{i+1}+K_{i+1},C_{i+1}\right.\\
&\quad\left.+\,G_{i+1},M_{i+1},D_{i+1},I_{i+1},C_{i+1}\right)\\
&\qquad=\Phi\left(M_i+L_i,D_i+N_i,I_i+K_i,C_i+G_i,J_i,D_i,I_i,C_i\right),\\
&A_i=\partial^{-1}\left(u_1I_i-u_2C_i+u_3M_{i+1}-u_4D_{i+1}\right),\\
&B_i=\partial^{-1}\left(u_1M_i-u_2D_i+u_3I_i-u_4C_i\right),\\
&E_i=\partial^{-1}\left(u_1K_i-u_2G_i+u_3L_{i+1}-u_4N_{i+1}\right.\\
&\qquad\left.+\,u_5I_i-u_6C_i+u_7M_{i+1}-u_8D_{i+1}\right),\\
&F_i=\partial^{-1}\left(u_1L_i-u_2N_i+u_3K_i-u_4G_i+u_5M_i\right.\\
&\qquad\left.-u_6D_i+u_7I_i-u_8C_i\right),
\end{aligned}\tag{5}$$

where

$$\Phi=\begin{pmatrix}
M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8\\
M_9 & M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16}\\
M_{17} & M_{18} & M_{19} & M_{20} & M_{21} & M_{22} & M_{23} & M_{24}\\
M_{25} & M_{26} & M_{27} & M_{28} & M_{29} & M_{30} & M_{31} & M_{32}\\
0 & 0 & 0 & 0 & M_{33} & M_{34} & M_{35} & M_{36}\\
0 & 0 & 0 & 0 & M_{37} & M_{38} & M_{39} & M_{40}\\
0 & 0 & 0 & 0 & M_{41} & M_{42} & M_{43} & M_{44}\\
0 & 0 & 0 & 0 & M_{45} & M_{46} & M_{47} & M_{48}
\end{pmatrix}, \tag{6}$$

$$\begin{aligned}
M_1&=-\frac{\partial^{-1}}{2}+u_2\partial^{-1}u_1+u_4\partial^{-1}u_3\lambda^{-2},\\
M_2&=-u_2\partial^{-1}u_2-u_4\partial^{-1}u_4\lambda^{-2},\\
M_3&=u_2\partial^{-1}u_3+u_4\partial^{-1}u_1,\\
M_4&=-u_2\partial^{-1}u_4-u_4\partial^{-1}u_2,\\
M_5&=u_2\partial^{-1}u_5+u_4\partial^{-1}u_7\lambda^{-2}+u_6\partial^{-1}u_1+u_8\partial^{-1}u_3\lambda^{-2},\\
M_6&=-u_2\partial^{-1}u_6-u_4\partial^{-1}u_8\lambda^{-2}-u_6\partial^{-1}u_2-u_8\frac{\delta}{\delta u}u_4\lambda^{-2},\\
M_7&=u_2\partial^{-1}u_7+u_4\partial^{-1}u_5+u_6\partial^{-1}u_3+u_8\partial^{-1}u_1,\\
M_8&=-u_2\partial^{-1}u_8-u_4\partial^{-1}u_6-u_6\partial^{-1}u_4-u_8\partial^{-1}u_2,\\
M_9&=u_1\partial^{-1}u_1+u_3\partial^{-1}u_3\lambda^{-2},\\
M_{10}&=-u_1\partial^{-1}u_2-u_3\partial^{-1}u_4\lambda^{-2}+\frac{\partial^{-1}}{2},\\
M_{11}&=u_1\partial^{-1}u_3+u_3\partial^{-1}u_1,\\
M_{12}&=-u_1\partial^{-1}u_4-u_3\partial^{-1}u_2,\\
M_{13}&=u_1\partial^{-1}u_5+u_3\partial^{-1}u_7\lambda^{-2}+u_5\partial^{-1}u_1+u_7\partial^{-1}u_3\lambda^{-2},\\
M_{14}&=-u_1\partial^{-1}u_6-u_3\partial^{-1}u_8\lambda^{-2}-u_5\partial^{-1}u_2-u_7\partial^{-1}u_4\lambda^{-2},\\
M_{15}&=u_1\partial^{-1}u_7+u_3\partial^{-1}u_5+u_5\partial^{-1}u_3+u_7\partial^{-1}u_1,\\
M_{16}&=-u_1\partial^{-1}u_8-u_3\partial^{-1}u_6-u_5\partial^{-1}u_3-u_7\partial^{-1}u_2,\\
M_{17}&=u_4\partial^{-1}u_1\lambda^{-2}+u_2\partial^{-1}u_3\lambda^{-2},\\
M_{18}&=-u_4\partial^{-1}u_2\lambda^{-2}-u_2\partial^{-1}u_4\lambda^{-2},\\
M_{19}&=u_4\partial^{-1}u_3\lambda^{-2}+u_2\partial^{-1}u_1-\frac{\partial^{-1}}{2},\\
M_{20}&=-u_4\partial^{-1}u_4\lambda^{-2}-u_2\partial^{-1}u_2,\\
M_{21}&=u_4\frac{\delta}{\delta u}u_5\lambda^{-2}+u_2\frac{\delta}{\delta u}u_7\lambda^{-2}+u_8\partial^{-1}u_1\lambda^{-2}\\
&\quad+u_6\partial^{-1}u_3\lambda^{-2},
\end{aligned}$$

$$\begin{aligned}
M_{22} &= -u_4\partial^{-1}u_6\lambda^{-2} - u_2\partial^{-1}u_8\lambda^{-2} - u_8\partial^{-1}u_2\lambda^{-2} \\
&\quad - u_6\partial^{-1}u_4\lambda^{-2}, \\
M_{23} &= u_4\partial^{-1}u_7\lambda^{-2} + u_2\partial^{-1}u_5 + u_8\partial^{-1}u_3\lambda^{-2} + u_6\partial^{-1}u_1, \\
M_{24} &= -u_4\partial^{-1}u_8\lambda^{-2} - u_2\partial^{-1}u_6 - u_8\partial^{-1}u_4\lambda^{-2} \\
&\quad - u_6\partial^{-1}u_2, \\
M_{25} &= u_3\partial^{-1}u_1\lambda^{-2} + u_1\partial^{-1}u_3\lambda^{-2}, \\
M_{26} &= -u_3\partial^{-1}u_2\lambda^{-2} - u_1\partial^{-1}u_4\lambda^{-2}, \\
M_{27} &= u_3\partial^{-1}u_3\lambda^{-2} + u_1\partial^{-1}u_1, \\
M_{28} &= -u_3\partial^{-1}u_4\lambda^{-2} - u_1\partial^{-1}u_2 + \frac{\partial^{-1}}{2}, \\
M_{29} &= u_3\partial^{-1}u_5\lambda^{-2} + u_1\partial^{-1}u_7\lambda^{-2} \\
&\quad + u_7\partial^{-1}u_1\lambda^{-2} + u_5\partial^{-1}u_3\lambda^{-2}, \\
M_{30} &= -u_3\partial^{-1}u_6\lambda^{-2} - u_1\partial^{-1}u_8\lambda^{-2} \\
&\quad - u_7\partial^{-1}u_2\lambda^{-2} - u_5\partial^{-1}u_4\lambda^{-2}, \\
M_{31} &= u_3\partial^{-1}u_7\lambda^{-2} + u_1\partial^{-1}u_5 \\
&\quad + u_7\partial^{-1}u_3\lambda^{-2} + u_5\partial^{-1}u_1, \\
M_{32} &= -u_3\partial^{-1}u_8\lambda^{-2} - u_1\partial^{-1}u_6 \\
&\quad - u_7\partial^{-1}u_4\lambda^{-2} - u_5\partial^{-1}u_2, \\
M_{33} &= u_2\partial^{-1}u_1 + u_4\partial^{-1}u_3\lambda^{-2} - \frac{\partial^{-1}}{2}, \\
M_{34} &= -u_2\partial^{-1}u_2 - u_4\partial^{-1}u_4\lambda^{-2}, \\
M_{35} &= u_2\partial^{-1}u_3 + u_4\partial^{-1}u_1, \\
M_{36} &= -u_2\partial^{-1}u_4 - u_4\partial^{-1}u_2, \\
M_{37} &= u_1\partial^{-1}u_1 + u_3\partial^{-1}u_3\lambda^{-2}, \\
M_{38} &= -u_1\partial^{-1}u_2 - u_3\partial^{-1}u_4\lambda^{-2} + \frac{\partial^{-1}}{2}, \\
M_{39} &= u_1\partial^{-1}u_3 + u_3\partial^{-1}u_1, \\
M_{40} &= -u_1\partial^{-1}u_4 - u_3\partial^{-1}u_2, \\
M_{41} &= u_4\partial^{-1}u_1\lambda^{-2} + u_2\partial^{-1}u_3\lambda^{-2}, \\
M_{42} &= -u_4\partial^{-1}u_2\lambda^{-2} - u_2\partial^{-1}u_4\lambda^{-2}, \\
M_{43} &= u_4\partial^{-1}u_3\lambda^{-2} + u_2\partial^{-1}u_1 - \frac{\partial^{-1}}{2}, \\
M_{44} &= -u_4\partial^{-1}u_4\lambda^{-2} - u_2\partial^{-1}u_2, \\
M_{45} &= u_3\partial^{-1}u_1\lambda^{-2} + u_1\partial^{-1}u_3\lambda^{-2}, \\
M_{46} &= -u_3\partial^{-1}u_2\lambda^{-2} - u_1\partial^{-1}u_4\lambda^{-2}, \\
M_{47} &= u_3\partial^{-1}u_3\lambda^{-2} + u_1\partial^{-1}u_1, \\
M_{48} &= -u_3\partial^{-1}u_4\lambda^{-2} - u_1\partial^{-1}u_2 + \frac{\partial^{-1}}{2}.
\end{aligned} \tag{7}$$

If we choose the initial conditions as

$$\begin{gathered}
A_0 = 1, \\
B_0 = C_0 = D_0 = E_0 = F_0 = G_0 \\
= N_0 = I_0 = M_0 = K_0 = L_0 = 0,
\end{gathered} \tag{8}$$

then we can get all the other values according to (6). The first few sets are

$$\begin{gathered}
A_1 = 0, \qquad B_1 = 0, \qquad C_1 = u_1, \\
D_1 = u_3, \qquad E_1 = 0, \qquad F_1 = 0, \\
G_1 = u_5, \qquad N_1 = u_7, \qquad I_1 = u_2, \\
M_1 = u_4, \qquad K_1 = u_6, \qquad L_1 = u_8, \\
A_2 = 0, \qquad B_2 = -\left(u_1u_4 + u_2u_3\right), \\
C_2 = u_{1,x} - u_3\left(u_1u_4 + u_2u_3\right), \qquad D_2 = u_{3,x}, \\
E_2 = 0, \qquad F_2 = -\left(u_1u_8 + u_2u_7 + u_3u_6 + u_4u_5\right), \\
G_2 = u_{5,x} - u_3\left(u_1u_8 + u_2u_7 + u_3u_6 + u_4u_5\right) \\
- u_7\left(u_1u_4 + u_2u_3\right), \\
N_2 = u_{7,x}, \qquad M_2 = -u_{4,x}, \\
I_2 = -u_{2,x} - u_4\left(u_1u_4 + u_2u_3\right), \qquad L_2 = -u_{8,x}, \\
K_2 = -u_{6,x} - u_4\left(u_1u_8 + u_2u_7 + u_3u_6 + u_4u_5\right) \\
- u_8\left(u_1u_4 + u_2u_3\right).
\end{gathered} \tag{9}$$

Let us associate (1) with the following problem:

$$\varphi_{t_n} = V^{(n)}\varphi, \tag{10}$$

with

$$V^{(n)} = \sum_{i=0}^{n} \begin{pmatrix} A_i + B_i\lambda & C_i + D_i\lambda & E_i + F_i\lambda & G_i + N_i\lambda \\ I_i + M_i\lambda & -A_i - B_i\lambda & K_i + L_i\lambda & -E_i - F_i\lambda \\ 0 & 0 & A_i + B_i\lambda & C_i + D_i\lambda \\ 0 & 0 & I_i + M_i\lambda & -A_i - B_i\lambda \end{pmatrix} \times \lambda^{2(n-i)}. \tag{11}$$

The compatible condition of the spectral problem (1) and the auxiliary problem (10) is

$$U_{tn} - V_x^{(n)} + \left[U\ V^{(n)}\right] = 0. \tag{12}$$

After a direct calculation, we can get the AKNS-KN coupling system:

$$\begin{aligned}
u_{1,tn} &= \left(2+2u_3\partial^{-1}u_4\right)C_{n+1} - 2u_3\partial^{-1}u_3 I_{n+1} \\
&\quad - \left(2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3\right)M_{n+1} \\
&\quad + \left(2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4\right)D_{n+1}, \\
u_{2,tn} &= \left(-2+2u_4\partial^{-1}u_3\right)I_{n+1} - 2u_4\partial^{-1}u_4 C_{n+1} \\
&\quad + \left(2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3\right)M_{n+1} \\
&\quad - \left(2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4\right)D_{n+1}, \\
u_{3,tn} &= \left(2+2u_3\partial^{-1}u_4\right)D_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3 M_{n+1}, \\
u_{4,tn} &= -2u_4\partial^{-1}u_4 D_{n+1} \\
&\quad + \left(-2+2u_4\partial^{-1}u_3\right)M_{n+1}, \\
u_{5,tn} &= \left(2+2u_3\partial^{-1}u_4\right)G_{n+1} \\
&\quad - \left(2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3\right)L_{n+1} \\
&\quad + \left(2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4\right)H_{n+1} \\
&\quad + \left(2u_3\partial^{-1}u_8 + 2u_7\partial^{-1}u_4\right)C_{n+1} \\
&\quad + \left(2u_3\partial^{-1}u_6 + 2u_1\partial^{-1}u_8 \right. \\
&\qquad \left. + 2u_7\partial^{-1}u_2 + 2u_5\partial^{-1}u_4\right)D_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3 N_{n+1} \\
&\quad - \left(2u_3\partial^{-1}u_5 + 2u_5\partial^{-1}u_3 \right. \\
&\qquad \left. + 2u_1\partial^{-1}u_7 + 2u_7\partial^{-1}u_1\right)M_{n+1} \\
&\quad - \left(2u_3\partial^{-1}u_7 + 2u_7\partial^{-1}u_3\right)I_{n+1}, \\
u_{6,tn} &= \left(-2+2u_4\partial^{-1}u_3\right)K_{n+1} \\
&\quad + \left(2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3\right)L_{n+1} \\
&\quad - \left(2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4\right)N_{n+1} \\
&\quad - \left(2u_4\partial^{-1}u_8 + 2u_8\partial^{-1}u_4\right)C_{n+1} \\
&\quad - \left(2u_4\partial^{-1}u_6 + 2u_2\partial^{-1}u_8 + 2u_8\partial^{-1}u_2 \right. \\
&\qquad \left. + 2u_6\partial^{-1}u_4\right)D_{n+1} \\
&\quad + \left(2u_4\partial^{-1}u_7 + 2u_8\partial^{-1}u_3\right)I_{n+1} - 2u_4\partial^{-1}u_4 G_{n+1} \\
&\quad + \left(2u_4\partial^{-1}u_5 + 2u_6\partial^{-1}u_3 + 2u_8\partial^{-1}u_1 \right. \\
&\qquad \left. + 2u_2\partial^{-1}u_7\right)M_{n+1}, \\
u_{7,tn} &= \left(2+2u_3\partial^{-1}u_4\right)N_{n+1} \\
&\quad + \left(2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8\right)D_{n+1} \\
&\quad - \left(2u_7\partial^{-1}u_3 + 2u_3\partial^{-1}u_7\right)M_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3 L_{n+1}, \\
u_{8,tn} &= \left(-2+2u_4\partial^{-1}u_3\right)L_{n+1} \\
&\quad - \left(2u_4\partial^{-1}u_8 + 2u_8\partial^{-1}u_4\right)D_{n+1} \\
&\quad + \left(2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7\right)M_{n+1} \\
&\quad - 2u_4\partial^{-1}u_4 N_{n+1}.
\end{aligned} \tag{13}$$

Therefore, the AKNS-KN coupling system can be written as

$$\begin{aligned}
u_{t_n} &= J\left(M_{n+1}+L_{n+1}, D_{n+1}+N_{n+1}, I_{n+1}+K_{n+1},\right. \\
&\qquad \left. C_{n+1}+G_{n+1}, M_{n+1}, D_{n+1}, I_{n+1}, C_{n+1}\right) \\
&= J\Phi^n\left(M_1+L_1, D_1+N_1, I_1+K_1,\right. \\
&\qquad \left. C_1+G_1, M_1, D_1, I_1, C_1\right)^T,
\end{aligned} \tag{14}$$

where the operator Φ is determined by (6) and the Hamiltonian operator is as follows:

$$J = \begin{pmatrix} 0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}
Z_1 &= \begin{pmatrix}
-2u_3\partial^{-1}u_1 - 2u_1\partial^{-1}u_3 & 2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4 & -2u_3\partial^{-1}u_3 & 2+2u_3\partial^{-1}u_4 \\
2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3 & -2u_4\partial^{-1}u_2 - 2u_2\partial^{-1}u_4 & -2+2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 \\
-2u_3\partial^{-1}u_3 & 2+2u_3\partial^{-1}u_4 & 0 & 0 \\
-2+2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 & 0 & 0
\end{pmatrix}, \\
Z_2 &= \begin{pmatrix}
Y_1 & Y_2 & Y_3 & Y_4 \\
Y_5 & Y_6 & Y_7 & Y_8 \\
Y_9 & Y_{10} & Y_{11} & Y_{12} \\
Y_{13} & Y_{14} & Y_{15} & Y_{16}
\end{pmatrix},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
Y_1 &= 2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3 - 2u_3\partial^{-1}u_5 \\
&\quad - 2u_5\partial^{-1}u_3 - \left(2u_1\partial^{-1}u_7 - 2u_7\partial^{-1}u_1\right), \\
Y_2 &= -2u_3\partial^{-1}u_2 - 2u_1\partial^{-1}u_4 + 2u_3\partial^{-1}u_6 \\
&\quad + 2u_1\partial^{-1}u_8 + 2u_7\partial^{-1}u_2 + 2u_5\partial^{-1}u_4, \\
Y_3 &= 2u_3\partial^{-1}u_3 - 2u_3\partial^{-1}u_7 - 2u_7\partial^{-1}u_3, \\
Y_4 &= -2 - 2u_3\partial^{-1}u_4 + 2u_3\partial^{-1}u_8 + 2u_7\partial^{-1}u_4, \\
Y_5 &= -2u_4\partial^{-1}u_1 - 2u_2\partial^{-1}u_3 + 2u_4\partial^{-1}u_5 \\
&\quad + 2u_6\partial^{-1}u_3 + \left(2u_8\partial^{-1}u_1 + 2u_2\partial^{-1}u_7\right), \\
Y_6 &= 2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4 - 2u_4\partial^{-1}u_6 \\
&\quad - 2u_2\partial^{-1}u_8 - 2u_8\partial^{-1}u_2 - 2u_6\partial^{-1}u_4, \\
Y_7 &= 2 - 2u_4\partial^{-1}u_3 + 2u_4\partial^{-1}u_7 + 2u_8\partial^{-1}u_3, \\
Y_8 &= 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4, \\
Y_9 &= 2u_3\partial^{-1}u_3 - 2u_7\partial^{-1}u_3 - 2u_3\partial^{-1}u_7, \\
Y_{10} &= -2 - 2u_3\partial^{-1}u_4 + 2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8, \\
Y_{11} &= Y_{12} = Y_{15} = Y_{16} = 0, \\
Y_{13} &= 2 - 2u_4\partial^{-1}u_3 + 2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7, \\
Y_{14} &= 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4.
\end{aligned} \tag{17}$$

To simplify (13), let $u_1 = u_2 = u_5 = u_6 = 0$; then the AKNS integrable coupling system (14) can be written as

$$u_{t_n} = J_1\Phi_1^n(u_4 + u_8, u_3 + u_7, u_4, u_3)^T, \tag{18}$$

where

$$\Phi_1 = \begin{pmatrix}
u_4\partial^{-1}u_3\lambda^{-2} - \dfrac{\partial^{-1}}{2} & -u_4\partial^{-1}u_4\lambda^{-2} & M'_5 & M'_6 \\
u_3\partial^{-1}u_3\lambda^{-2} & -u_3\partial^{-1}u_4\lambda^{-2} + \dfrac{\partial^{-1}}{2} & M'_{13} & M'_{14} \\
0 & 0 & u_4\partial^{-1}u_3\lambda^{-2} - \dfrac{\partial^{-1}}{2} & -u_4\partial^{-1}u_4\lambda^{-2} \\
0 & 0 & u_3\partial^{-1}u_3\lambda^{-2} & -u_3\partial^{-1}u_4\lambda^{-2} + \dfrac{\partial^{-1}}{2}
\end{pmatrix},$$

$$\begin{aligned}
M'_5 &= u_4\partial^{-1}u_7\lambda^{-2} + u_8\partial^{-1}u_3\lambda^{-2}, \\
M'_6 &= -u_4\partial^{-1}u_8\lambda^{-2} - u_8\frac{\delta}{\delta}u_4\lambda^{-2}, \\
M'_{13} &= u_3\partial^{-1}u_7\lambda^{-2} + u_7\partial^{-1}u_3\lambda^{-2}, \\
M'_{14} &= -u_3\partial^{-1}u_8\lambda^{-2} - u_7\partial^{-1}u_4\lambda^{-2},
\end{aligned}$$

$$J_1 = \begin{pmatrix}
0 & 0 & -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 \\
0 & 0 & -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 \\
-2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 & Y_9 & Y_{10} \\
-2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 & Y_{13} & Y_{14}
\end{pmatrix},$$

$$\begin{aligned}
Y_9 &= 2u_3\partial^{-1}u_3 - 2u_7\partial^{-1}u_3 - 2u_3\partial^{-1}u_7, \\
Y_{10} &= -2 - 2u_3\partial^{-1}u_4 + 2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8, \\
Y_{13} &= 2 - 2u_4\partial^{-1}u_3 + 2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7, \\
Y_{14} &= 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4.
\end{aligned} \tag{19}$$

Furthermore, let $u_1 = u_2 = u_5 = u_6 = u_7 = u_8 = 0$, and then $u_{t_n} = J_2\Phi_2^n(u_4, u_3)^T$ is the AKNS system, where

$$\Phi_2 = \begin{pmatrix} u_4\partial^{-1}u_3\lambda^{-2} - \dfrac{\partial^{-1}}{2} & -u_4\partial^{-1}u_4\lambda^{-2} \\ u_3\partial^{-1}u_3\lambda^{-2} & -u_3\partial^{-1}u_4\lambda^{-2} + \dfrac{\partial^{-1}}{2} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 \\ -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 \end{pmatrix}. \quad (20)$$

On the other hand, when $u_3 = u_4 = u_7 = u_8 = 0$, then $u_{t_n} = J_3\Phi_3^n(u_2 + u_6, u_1 + u_5, u_2, u_1)^T$ is the KN integrable coupling system, where

$$\Phi_3 = \begin{pmatrix} u_2\partial^{-1}u_1 - \dfrac{\partial^{-1}}{2} & -u_2\partial^{-1}u_2 & u_2\partial^{-1}u_5 + u_6\partial^{-1}u_1 & -u_2\partial^{-1}u_6 - u_6\partial^{-1}u_2 \\ u_1\partial^{-1}u_1 & -u_1\partial^{-1}u_2 + \dfrac{\partial^{-1}}{2} & u_1\partial^{-1}u_5 + u_5\partial^{-1}u_1 & -u_1\partial^{-1}u_6 - u_5\partial^{-1}u_2 \\ 0 & 0 & u_2\partial^{-1}u_1 - \dfrac{\partial^{-1}}{2} & -u_2\partial^{-1}u_2 \\ 0 & 0 & u_1\partial^{-1}u_1 & -u_1\partial^{-1}u_2 + \dfrac{\partial^{-1}}{2} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ -2 & 0 & 2 & 0 \end{pmatrix}. \quad (21)$$

When $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$, then $u_{t_n} = J_4\Phi_4^n(u_2, u_1)^T$ is the KN system, where

$$\Phi_4 = \begin{pmatrix} u_2\partial^{-1}u_1 - \dfrac{\partial^{-1}}{2} & -u_2\partial^{-1}u_2 \\ u_1\partial^{-1}u_1 & -u_1\partial^{-1}u_2 + \dfrac{\partial^{-1}}{2} \end{pmatrix}, \quad (22)$$

$$J_4 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

3. Bargmann Symmetry Constraint of AKNS-KN Coupling System

In order to get a Bargmann symmetry constraint, we can consider the Lax pairs and the adjoint Lax pairs of the AKNS-KN coupling system. The adjoint Lax pairs of the AKNS-KN coupling system are

$$\psi_x = -U^T\psi = \begin{pmatrix} -\lambda^2 & -(u_2 + u_4\lambda) & 0 & 0 \\ -(u_1 + u_3\lambda) & \lambda^2 & 0 & 0 \\ 0 & -(u_6 + u_8\lambda) & -\lambda^2 & -(u_2 + u_4\lambda) \\ -(u_5 + u_7\lambda) & 0 & -(u_1 + u_3\lambda) & \lambda^2 \end{pmatrix}, \quad (23)$$

$$\psi_{t_n} = \sum_{i=0}^{n} \begin{pmatrix} -A_i - B_i\lambda & -I_i - M_i\lambda & 0 & 0 \\ -C_i - D_i\lambda & A_i + B_i\lambda & 0 & 0 \\ -E_i - F_i\lambda & -K_i - L_i\lambda & -A_i - B_i\lambda & -I_i - M_i\lambda \\ -G_i - N_i\lambda & E_i + F_i\lambda & -C_i - D_i\lambda & A_i + B_i\lambda \end{pmatrix}\lambda^{2(n-i)}, \quad (24)$$

where T means the transpose of matrix and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. It follows from

$$\phi_x = U\phi, \qquad \psi_x = -U^T\psi \tag{25}$$

that

$$\frac{\delta\lambda}{\delta u} = \frac{1}{B}\begin{pmatrix} \langle\psi_1\psi_2\rangle + \langle\psi_3\psi_4\rangle \\ \langle\psi_2\psi_1\rangle + \langle\psi_4\psi_3\rangle \\ (\langle\psi_1\psi_2\rangle + \langle\psi_3\psi_4\rangle)\lambda \\ (\langle\psi_2\psi_1\rangle + \langle\psi_4\psi_3\rangle)\lambda \\ \langle\psi_1\psi_4\rangle \\ \langle\psi_2\psi_3\rangle \\ (\langle\psi_1\psi_4\rangle)\lambda \\ (\langle\psi_2\psi_3\rangle)\lambda \end{pmatrix}, \tag{26}$$

where $B = \int 2(\langle\psi_1\psi_1\rangle - \langle\psi_2\psi_2\rangle + \langle\psi_3\psi_3\rangle - \langle\psi_4\psi_4\rangle)\lambda\, dx$.

According to the zero boundary conditions $\lim_{|x|\to\infty}\phi = \lim_{|x|\to\infty}\psi = 0$, we can get

$$\Phi\frac{\delta\lambda}{\delta u} = \lambda\frac{\delta\lambda}{\delta u}, \tag{27}$$

where Φ and $\delta\lambda/\delta u$ are given by (6) and (26).

Now, let us discuss the spatial systems

$$\begin{aligned}
\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{pmatrix}_x &= \begin{bmatrix} \lambda^2 & u_1 + u_3\lambda & 0 & u_5 + u_7\lambda \\ u_2 + u_4\lambda & -\lambda^2 & u_6 + u_8\lambda & 0 \\ 0 & 0 & \lambda^2 & u_1 + u_3\lambda \\ 0 & 0 & u_2 + u_4\lambda & -\lambda^2 \end{bmatrix}\begin{bmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{bmatrix}, \\
\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{pmatrix}_x &= \begin{bmatrix} -\lambda^2 & -(u_2 + u_4\lambda) & 0 & 0 \\ -(u_1 + u_3\lambda) & \lambda^2 & 0 & 0 \\ 0 & -(u_6 + u_8\lambda) & -\lambda^2 & -(u_2 + u_4\lambda) \\ -(u_5 + u_7\lambda) & 0 & -(u_1 + u_3\lambda) & \lambda^2 \end{bmatrix} \\
&\quad \times \begin{bmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{bmatrix},
\end{aligned} \tag{28}$$

and the temporal systems

$$\begin{aligned}
\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{pmatrix}_{tn} &= \sum_{i=0}^{n}\begin{bmatrix} A_i + B_i\lambda & C_i + D_i\lambda & E_i + F_i\lambda & G_i + N_i\lambda \\ I_i + M_i\lambda & -A_i - B_i\lambda & K_i + L_i\lambda & -E_i - F_i\lambda \\ 0 & 0 & A_i + B_i\lambda & C_i + D_i\lambda \\ 0 & 0 & I_i + M_i\lambda & -A_i - B_i\lambda \end{bmatrix} \\
&\quad \times \lambda^{2(n-i)}\begin{bmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{bmatrix}, \\
\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{pmatrix}_{tn} &= \sum_{i=0}^{n}\begin{bmatrix} -A_i - B_i\lambda & -I_i - M_i\lambda & 0 & 0 \\ -C_i - D_i\lambda & A_i + B_i\lambda & 0 & 0 \\ -E_i - F_i\lambda & -K_i - L_i\lambda & -A_i - B_i\lambda & -I_i - M_i\lambda \\ -G_i - N_i\lambda & E_i + F_i\lambda & -C_i - D_i\lambda & A_i + B_i\lambda \end{bmatrix} \\
&\quad \times \lambda^{2(n-i)}\begin{bmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{bmatrix},
\end{aligned} \tag{29}$$

where $1 \le j \le N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are N distinct parameters. From [6] and [7], the expression of the potential u can be easily calculated:

$$\begin{aligned}
u_1 &= (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle), \\
u_2 &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle), \\
u_3 &= (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)\lambda, \\
u_4 &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)\lambda, \\
u_5 &= (\langle\psi_1\varphi_4\rangle), \\
u_6 &= (\langle\psi_2\varphi_3\rangle), \\
u_7 &= (\langle\psi_1\varphi_4\rangle)\lambda, \\
u_8 &= (\langle\psi_2\varphi_3\rangle)\lambda.
\end{aligned} \tag{30}$$

4. Binary Nonlinearization of AKNS-KN Coupling System

In order to perform binary nonlinearization of AKNS-KN coupling system, let us substitute (30) into the Lax pairs and adjoint Lax pairs (28) and (29); then we can get the following nonlinearized spatial Lax pairs and the adjoint Lax pairs:

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{pmatrix}_x = \begin{bmatrix}\lambda^2 & \tilde{u}_1+\tilde{u}_3\lambda & 0 & \tilde{u}_5+\tilde{u}_7\lambda\\ \tilde{u}_2+\tilde{u}_4\lambda & -\lambda^2 & \tilde{u}_6+\tilde{u}_8\lambda & 0\\ 0 & 0 & \lambda^2 & \tilde{u}_1+\tilde{u}_3\lambda\\ 0 & 0 & \tilde{u}_2+\tilde{u}_4\lambda & -\lambda^2\end{bmatrix}\begin{bmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{bmatrix},$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{pmatrix}_x = \begin{bmatrix}-\lambda^2 & -(\tilde{u}_2+\tilde{u}_4\lambda) & 0 & 0\\ -(\tilde{u}_1+\tilde{u}_3\lambda) & \lambda^2 & 0 & 0\\ 0 & -(\tilde{u}_6+\tilde{u}_8\lambda) & -\lambda^2 & -(\tilde{u}_2+\tilde{u}_4\lambda)\\ -(\tilde{u}_5+\tilde{u}_7\lambda) & 0 & -(\tilde{u}_1+\tilde{u}_3\lambda) & \lambda^2\end{bmatrix} \times \begin{bmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{bmatrix}, \quad (31)$$

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{pmatrix}_{tn} = \sum_{i=0}^{n}\begin{bmatrix}\widetilde{A}_i+\widetilde{B}_i\lambda & \widetilde{C}_i+\widetilde{D}_i\lambda & \widetilde{E}_i+\widetilde{F}_i\lambda & \widetilde{G}_i+\widetilde{N}_i\lambda\\ \widetilde{I}_i+\widetilde{M}_i\lambda & -\widetilde{A}_i-\widetilde{B}_i\lambda & \widetilde{K}_i+\widetilde{L}_i\lambda & -\widetilde{E}_i-\widetilde{F}_i\lambda\\ 0 & 0 & \widetilde{A}_i+\widetilde{B}_i\lambda & \widetilde{C}_i+\widetilde{D}_i\lambda\\ 0 & 0 & \widetilde{I}_i+\widetilde{M}_i\lambda & -\widetilde{A}_i-\widetilde{B}_i\lambda\end{bmatrix} \times \lambda^{2(n-i)}\begin{bmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\\ \varphi_{4j}\end{bmatrix},$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{pmatrix}_{tn} = \sum_{i=0}^{n}\begin{bmatrix}-\widetilde{A}_i-\widetilde{B}_i\lambda & -\widetilde{I}_i-\widetilde{M}_i\lambda & 0 & 0\\ -\widetilde{C}_i-\widetilde{D}_i\lambda & \widetilde{A}_i+\widetilde{B}_i\lambda & 0 & 0\\ -\widetilde{E}_i-\widetilde{F}_i\lambda & -\widetilde{K}_i-\widetilde{L}_i\lambda & -\widetilde{A}_i-\widetilde{B}_i\lambda & -\widetilde{I}_i-\widetilde{M}_i\lambda\\ -\widetilde{G}_i-\widetilde{N}_i\lambda & \widetilde{E}_i+\widetilde{F}_i\lambda & -\widetilde{C}_i-\widetilde{D}_i\lambda & \widetilde{A}_i+\widetilde{B}_i\lambda\end{bmatrix} \times \lambda^{2(n-i)}\begin{bmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \psi_{4j}\end{bmatrix}, \quad (32)$$

where $\widetilde{P}(U)$ means an expression of $P(U)$ under the constraint equation (30). Clearly, (28) can be written:

$$\begin{aligned}
\varphi_{1,x} &= \Lambda^2\varphi_1 + (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)(1+\Lambda^2)\varphi_2 \\
&\quad + \langle\psi_1\varphi_4\rangle(1+\Lambda^2)\varphi_4, \\
\varphi_{2,x} &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)(1+\Lambda^2)\varphi_1 \\
&\quad - \Lambda^2\varphi_2 + \langle\psi_2\varphi_3\rangle(1+\Lambda^2)\varphi_3, \\
\varphi_{3,x} &= \Lambda^2\varphi_3 + (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)(1+\Lambda^2)\varphi_4, \\
\varphi_{4,x} &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)(1+\Lambda^2)\varphi_3 - \Lambda^2\varphi_4, \\
\psi_{1,x} &= -\Lambda^2\psi_1 - \langle\psi_2, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle(1+\Lambda^2), \\
\psi_{2,x} &= \Lambda^2\psi_2 - \langle\psi_1, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle(1+\Lambda^2), \\
\psi_{3,x} &= -\Lambda^2\psi_3 - \langle\psi_4, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle(1+\Lambda^2) \\
&\quad - \langle\psi_2, \langle\psi_2\varphi_3\rangle\rangle(1+\Lambda^2), \\
\psi_{4,x} &= \Lambda^2\psi_4 - \langle\psi_3, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle(1+\Lambda^2) \\
&\quad - \langle\psi_1, \langle\psi_1\varphi_4\rangle\rangle(1+\Lambda^2),
\end{aligned} \quad (33)$$

where $\Lambda = \operatorname{diag}(\lambda_1\lambda_2, \ldots, \lambda_N)$. When $n = 1$, the coupling system equation is exactly system equation with $t_1 = x$. Obviously, system (28) can be written in the following form:

$$\begin{aligned}
\varphi_{1,x} &= \frac{\partial H_1}{\partial\psi_1}, & \varphi_{2,x} &= \frac{\partial H_1}{\partial\psi_2}, \\
\varphi_{3,x} &= \frac{\partial H_1}{\partial\psi_3}, & \varphi_{4,x} &= \frac{\partial H_1}{\partial\psi_4}, \\
\psi_{1,x} &= \frac{\partial H_1}{-\partial,\varphi_1}, & \psi_{2,x} &= \frac{\partial H_1}{-\partial\varphi_2}, \\
\psi_{3,x} &= \frac{\partial H_1}{-\partial\varphi_3}, & \psi_{4,x} &= \frac{\partial H_1}{-\partial\varphi_4},
\end{aligned} \quad (34)$$

where the Hamiltonian form is the following:

$$\begin{aligned}
H_1 &= \frac{(1+\Lambda^2)(((\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle), (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle))}{2} \\
&\quad + \frac{(1+\Lambda^2)(\langle\psi_1\varphi_4\rangle, \langle\psi_1\varphi_4\rangle)}{2} \\
&\quad + \frac{(1+\Lambda^2)(((\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle), (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle))}{2} \\
&\quad + \frac{(1+\Lambda^2)(\langle\psi_2\varphi_3\rangle, \langle\psi_2\varphi_3\rangle)}{2} \\
&\quad + \Lambda^2(\langle\varphi_1\psi_1\rangle - \langle\varphi_2\psi_2\rangle + \langle\varphi_3\psi_3\rangle - \langle\varphi_4\psi_4\rangle).
\end{aligned} \quad (35)$$

When $u_3 = u_4 = u_7 = u_8 = 0$, then (33) can be written as

$$\begin{aligned}
\varphi_{1,x} &= \Lambda^2\varphi_1 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\varphi_2 \\
&\quad + \langle\psi_1\varphi_4\rangle\varphi_4, \\
\varphi_{2,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\varphi_1 - \Lambda^2\varphi_2 \\
&\quad + \langle\psi_2\varphi_3\rangle\varphi_3, \\
\varphi_{3,x} &= \Lambda^2\varphi_3 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\varphi_4, \\
\varphi_{4,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\varphi_3 - \Lambda^2\varphi_4, \\
\psi_{1,x} &= -\Lambda^2\psi_1 - \langle\psi_2, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle, \\
\psi_{2,x} &= \Lambda^2\psi_2 - \langle\psi_1, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle, \\
\psi_{3,x} &= -\Lambda^2\psi_3 - \langle\psi_4, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle \\
&\quad - \langle\psi_2, \langle\psi_2\varphi_3\rangle\rangle, \\
\psi_{4,x} &= \Lambda^2\psi_4 - \langle\psi_3, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle \\
&\quad - \langle\psi_1, \langle\psi_1\varphi_4\rangle\rangle,
\end{aligned} \tag{36}$$

and the Hamiltonian system is given by

$$\begin{aligned}
H_1 &= \frac{\left(\left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right), \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\right)}{2} \\
&\quad + \frac{\left(\langle\psi_1\varphi_4\rangle, \langle\psi_1\varphi_4\rangle\right)}{2} \\
&\quad + \frac{\left(\left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right), \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\right)}{2} \\
&\quad + \frac{\left(\langle\psi_2\varphi_3\rangle, \langle\psi_2\varphi_3\rangle\right)}{2} \\
&\quad + \Lambda^2\left(\langle\varphi_1\psi_1\rangle - \langle\varphi_2\psi_2\rangle + \langle\varphi_3\psi_3\rangle - \langle\varphi_4\psi_4\rangle\right).
\end{aligned} \tag{37}$$

Furthermore, when $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$, then (33) deduces to

$$\begin{aligned}
\varphi_{1,x} &= \Lambda^2\varphi_1 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\varphi_2, \\
\varphi_{2,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\varphi_1 - \Lambda^2\varphi_2, \\
\varphi_{3,x} &= \Lambda^2\varphi_3 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\varphi_4, \\
\varphi_{4,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\varphi_3 - \Lambda^2\varphi_4, \\
\psi_{1,x} &= -\Lambda^2\psi_1 - \langle\psi_2, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle, \\
\psi_{2,x} &= \Lambda^2\psi_2 - \langle\psi_1, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle, \\
\psi_{3,x} &= -\Lambda^2\psi_3 - \langle\psi_4, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle, \\
\psi_{4,x} &= \Lambda^2\psi_4 - \langle\psi_3, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle,
\end{aligned} \tag{38}$$

and the Hamiltonian form is given by

$$\begin{aligned}
H_1 &= \frac{\left(\left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right), \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\right)}{2} \\
&\quad + \frac{\left(\left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right), \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\right)}{2} \\
&\quad + \Lambda^2\left(\langle\varphi_1\psi_1\rangle - \langle\varphi_2\psi_2\rangle + \langle\varphi_3\psi_3\rangle - \langle\varphi_4\psi_4\rangle\right).
\end{aligned} \tag{39}$$

When $u_1 = u_2 = u_5 = u_6 = 0$, (33) is equivalent to the following:

$$\begin{aligned}
\varphi_{1,x} &= \Lambda^2\varphi_1 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\Lambda^2\varphi_2 \\
&\quad + \langle\psi_1\varphi_4\rangle\Lambda^2\varphi_4, \\
\varphi_{2,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\Lambda^2\varphi_1 - \Lambda^2\varphi_2 \\
&\quad + \langle\psi_2\varphi_3\rangle\Lambda^2\varphi_3, \\
\varphi_{3,x} &= \Lambda^2\varphi_3 + \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\Lambda^2\varphi_4, \\
\varphi_{4,x} &= \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\Lambda^2\varphi_3 - \Lambda^2\varphi_4, \\
\psi_{1,x} &= -\Lambda^2\psi_1 - \langle\psi_2, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle\Lambda^2, \\
\psi_{2,x} &= \Lambda^2\psi_2 - \langle\psi_1, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle\Lambda^2, \\
\psi_{3,x} &= -\Lambda^2\psi_3 - \langle\psi_4, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle\Lambda^2 \\
&\quad - \langle\psi_2, \langle\psi_2\varphi_3\rangle\rangle\Lambda^2, \\
\psi_{4,x} &= \Lambda^2\psi_4 - \langle\psi_3, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle\Lambda^2 \\
&\quad - \langle\psi_1, \langle\psi_1\varphi_4\rangle\rangle\Lambda^2,
\end{aligned} \tag{40}$$

and the Hamiltonian system is given by

$$\begin{aligned}
H_1 &= \frac{\Lambda^2\left(\left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right), \left(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\right)\right)}{2} \\
&\quad + \frac{\Lambda^2\left(\langle\psi_1\varphi_4\rangle, \langle\psi_1\varphi_4\rangle\right)}{2} \\
&\quad + \frac{\Lambda^2\left(\left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right), \left(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\right)\right)}{2} \\
&\quad + \frac{\Lambda^2\left(\langle\psi_2\varphi_3\rangle, \langle\psi_2\varphi_3\rangle\right)}{2} \\
&\quad + \Lambda^2\left(\langle\varphi_1\psi_1\rangle - \langle\varphi_2\psi_2\rangle + \langle\varphi_3\psi_3\rangle - \langle\varphi_4\psi_4\rangle\right).
\end{aligned} \tag{41}$$

When $u_1 = u_2 = u_5 = u_6 = u_7 = u_8 = 0$, (33) is equivalent to the following system:

$$\begin{aligned}
\varphi_{1,x} &= \Lambda^2\varphi_1 + (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)\Lambda^2\varphi_2,\\
\varphi_{2,x} &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)\Lambda^2\varphi_1 - \Lambda^2\varphi_2,\\
\varphi_{3,x} &= \Lambda^2\varphi_3 + (\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)\Lambda^2\varphi_4,\\
\varphi_{4,x} &= (\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)\Lambda^2\varphi_3 - \Lambda^2\varphi_4,\\
\psi_{1,x} &= -\Lambda^2\psi_1 - \langle\psi_2, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle\Lambda^2,\\
\psi_{2,x} &= \Lambda^2\psi_2 - \langle\psi_1, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle\Lambda^2,\\
\psi_{3,x} &= -\Lambda^2\psi_3 - \langle\psi_4, \langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle\rangle\Lambda^2,\\
\psi_{4,x} &= \Lambda^2\psi_4 - \langle\psi_3, \langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle\rangle\Lambda^2,
\end{aligned} \tag{42}$$

and the Hamiltonian system is given by

$$\begin{aligned}
H_1 &= \frac{\Lambda^2\left((\langle\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle), (\langle\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle))\right)}{2}\\
&\quad + \frac{\Lambda^2\left((\langle\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle), (\langle\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle))\right)}{2}\\
&\quad + \Lambda^2\left(\langle\varphi_1\psi_1\rangle - \langle\varphi_2\psi_2\rangle + \langle\varphi_3\psi_3\rangle - \langle\varphi_4\psi_4\rangle\right).
\end{aligned} \tag{43}$$

When $n = 2$, the coupling of system (29) is as follows:

$$\begin{aligned}
\varphi_{1,t_2} &= -2\lambda^2(\tilde{u}_1\tilde{u}_2)\varphi_1 + \left[\tilde{u}_{1,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_1\tilde{u}_2)\right]\varphi_2\\
&\quad - 2\lambda^2((\tilde{u}_1\tilde{u}_6) + (u_2u_5))\varphi_3\\
&\quad + \left[\tilde{u}_{5,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_1((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5)))\right]\varphi_4,\\
\varphi_{2,t_2} &= -\left[\tilde{u}_{2,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_2(\tilde{u}_1\tilde{u}_2))\right]\varphi_1 - 2\lambda^2(\tilde{u}_1\tilde{u}_2)\varphi_2\\
&\quad - \left[\tilde{u}_{6,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_2((u_1u_6) + (\tilde{u}_2\tilde{u}_5)))\right]\varphi_3\\
&\quad + 2\lambda^2((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5))\varphi_4,\\
\varphi_{3,t_2} &= -2\lambda^2(\tilde{u}_1\tilde{u}_2)\varphi_3\\
&\quad + \left[\tilde{u}_{1,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_1(\tilde{u}_1\tilde{u}_2))\right]\varphi_4,\\
\varphi_{4,t_2} &= -\left[\tilde{u}_{2,x}\left(1+\lambda^2\right) - 2\tilde{u}_2\lambda^2(\tilde{u}_2, (\tilde{u}_1\tilde{u}_2))\right]\varphi_3\\
&\quad + 2\lambda^2(\tilde{u}_1\tilde{u}_2)\varphi_4,\\
\psi_{1,t_2} &= 2\lambda^2(\tilde{u}_1\tilde{u}_2)\psi_1 + \left[\tilde{u}_{2,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_2(\tilde{u}_1\tilde{u}_2))\right]\psi_2,\\
\psi_{2,t_2} &= -2\lambda^2(\tilde{u}_1\tilde{u}_2)\psi_2 - \left[\tilde{u}_{1,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_1\tilde{u}_2)\right]\psi_1,\\
\psi_{3,t_2} &= 2\lambda^2((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5))\psi_1\\
&\quad + \left[\tilde{u}_{6,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_2((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5)))\right]\psi_2\\
&\quad + 2\lambda^2(\tilde{u}_1\tilde{u}_2)\psi_3\\
&\quad + \left[u_{2,x}\left(1+\lambda^2\right) - 2u_2\lambda^2(\tilde{u}_2(\tilde{u}_1\tilde{u}_2))\right]\psi_4,\\
\psi_{4,t_2} &= -\left[\tilde{u}_{5,x}\left(1+\lambda^2\right) - 2\lambda^2(\tilde{u}_1((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5)))\right]\psi_1\\
&\quad - 2\lambda^2((\tilde{u}_1\tilde{u}_6) + (\tilde{u}_2\tilde{u}_5))\psi_2\\
&\quad - \left[\tilde{u}_{1,x}\left(1+\lambda^2\right) - 2\left(\lambda^2u_1(\tilde{u}_1\tilde{u}_2)\right)\right]\psi_3\\
&\quad - 2\lambda^2(\tilde{u}_1\tilde{u}_2)\psi_4,
\end{aligned} \tag{44}$$

where $\tilde{u}_{1,x}$, $\tilde{u}_{2,x}$, $\tilde{u}_{3,x}$, $\tilde{u}_{4,x}$ are given by

$$\begin{aligned}
\tilde{u}_{1,x} &= -2\Lambda^2(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle) + \left(1+\Lambda^2\right)\\
&\quad \times(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)\\
&\quad \times(\langle\psi_1\varphi_1\rangle - \langle\psi_2\varphi_2\rangle + \langle\psi_3\varphi_3\rangle - \langle\psi_4\varphi_4\rangle)\\
&\quad + \left(1+\Lambda^2\right)\langle\psi_2\varphi_3\rangle(\langle\psi_1\varphi_3\rangle - \langle\psi_2\varphi_4\rangle),\\
\tilde{u}_{2,x} &= 2\Lambda^2(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle) + \left(1+\Lambda^2\right)\\
&\quad \times(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)\\
&\quad \times(\langle\psi_2\varphi_2\rangle - \langle\psi_1\varphi_1\rangle + \langle\psi_3\varphi_3\rangle - \langle\psi_4\varphi_4\rangle)\\
&\quad + \left(1+\Lambda^2\right)\langle\psi_1\varphi_4\rangle(\langle\psi_2\varphi_4\rangle - \langle\psi_1\varphi_3\rangle),\\
\tilde{u}_{3,x} &= \Lambda\tilde{u}_{1,x},\\
\tilde{u}_{4,x} &= \Lambda\tilde{u}_{2,x},\\
\tilde{u}_{5,x} &= -2\Lambda^2\langle\psi_1\varphi_4\rangle + \left(1+\Lambda^2\right)(\langle\psi_2\varphi_1\rangle + \langle\psi_4\varphi_3\rangle)\\
&\quad \times(\langle\psi_1\varphi_3\rangle - \langle\psi_2\varphi_4\rangle),\\
\tilde{u}_{6,x} &= 2\Lambda^2\langle\psi_3\varphi_3\rangle + \left(1+\Lambda^2\right)\\
&\quad \times(\langle\psi_1\varphi_2\rangle + \langle\psi_3\varphi_4\rangle)(\langle\psi_2\varphi_4\rangle - \langle\psi_1\varphi_3\rangle),\\
\tilde{u}_{7,x} &= \Lambda\tilde{u}_{5,x},\\
\tilde{u}_{8,x} &= \Lambda\tilde{u}_{6,x}.
\end{aligned} \tag{45}$$

Then we can get the following Hamiltonian form of (33):

$$\begin{aligned} \varphi_{1,t_2} &= \frac{\partial H_2}{\partial \psi_1}, & \varphi_{2,t_2} &= \frac{\partial H_2}{\partial \psi_2}, \\ \varphi_{3,t_2} &= \frac{\partial H_2}{\partial \psi_3}, & \varphi_{4,t_2} &= \frac{\partial H_2}{\partial \psi_4}, \\ \psi_{1,t_2} &= -\frac{\partial H_2}{\partial \varphi_1}, & \psi_{2,t_2} &= -\frac{\partial H_2}{\partial \varphi_2}, \\ \psi_{3,t_2} &= -\frac{\partial H_2}{\partial \varphi_3}, & \psi_{4,t_2} &= -\frac{\partial H_2}{\partial \varphi_4}, \end{aligned} \tag{46}$$

where the Hamiltonian system is given by

$$\begin{aligned} H_2 = {} & \Lambda^4 \left(\langle \psi_1, \varphi_1\rangle - \langle \psi_2, \varphi_2\rangle + \langle \psi_3, \varphi_3\rangle - \langle \psi_4, \varphi_4\rangle\right) \\ & - 2\Lambda^2 \Big(\left(\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_3\rangle\right)\right. \\ & \quad + 2\left(\Lambda^3 + \Lambda + 1\right) \\ & \quad \times \left(\left(\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\right)\right.\right. \\ & \qquad \times \left(\langle \psi_1, \varphi_1\rangle - \langle \psi_2, \varphi_2\rangle + \langle \psi_3, \varphi_3\rangle\right. \\ & \qquad\quad \left.\left. - \langle \psi_4, \varphi_4\rangle\right)\right) + (1+\Lambda)\left(1+\Lambda^2\right) \\ & \quad \times \left(\left(\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_3\rangle\right)\right.\right. \\ & \qquad + \left(\left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\right)\left(\langle \psi_1, \varphi_4\rangle\right)\right) \\ & \quad \times \left(\left(\left(\langle \psi_1, \varphi_3\rangle + \langle \psi_2, \varphi_4\rangle\right)\right.\right. \\ & \qquad + \left(\langle \psi_2, \varphi_4\rangle - \langle \psi_3, \varphi_1\rangle\right)\right) \\ & \quad + \left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\left(\langle \psi_1, \varphi_4\rangle\right)\right)\Big) \\ & \times \left(\left(\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_1, \varphi_4\rangle\right)\right.\right. \\ & \quad + \left(\left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\left(\langle \psi_2, \varphi_3\rangle\right)\right)\right)\right). \end{aligned} \tag{47}$$

In what follows, we want to prove that (28) is a completed integrable Hamiltonian system in the Liouville sense. In addition, we want prove that (29) is also completed integrable system. From (29) and (5), we can obtain the following form:

$$\begin{aligned} \widetilde{A}_{i+1} &= \Lambda^{2i+1}\left(\langle \psi_1, \varphi_1\rangle - \langle \psi_2, \varphi_2\rangle + \langle \psi_3, \varphi_3\rangle - \langle \psi_4, \varphi_4\rangle\right), \\ \widetilde{B}_{i+1} &= -2\Lambda^{2i+1}\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_1\rangle - \langle \psi_4, \varphi_3\rangle\right), \\ \widetilde{C}_{i+1} &= -2\Lambda^{2i}\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right), \\ \widetilde{D}_{i+1} &= -2\Lambda^{2i+1}\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right), \\ \widetilde{E}_{i+1} &= 0, \\ \widetilde{F}_{i+1} &= -2\Lambda^{2i+1}\left(\left(\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_3\rangle\right)\right.\right. \\ &\qquad \left.\left. + \left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\right)\left(\langle \psi_1, \varphi_4\rangle\right)\right)\right), \\ \widetilde{G}_{i+1} &= -2\Lambda^{2i}\langle \psi_1, \varphi_4\rangle, \\ \widetilde{N}_{i+1} &= -2\Lambda^{2i+1}\left(\langle \psi_1, \varphi_4\rangle\right), \\ \widetilde{I}_{i+1} &= 2\Lambda^{2i}\left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\right), \\ \widetilde{M}_{i+1} &= 2\Lambda^{2i+1}\left(\langle \psi_2, \varphi_1\rangle + \langle \psi_4, \varphi_3\rangle\right), \\ \widetilde{K}_{i+1} &= -2\Lambda^{2i}\langle \psi_2, \varphi_3\rangle, \\ \widetilde{L}_{i+1} &= -2\Lambda^{2i+1}\langle \psi_2, \varphi_3\rangle. \end{aligned} \tag{48}$$

Next, we can check that (31) is a Hamiltonian system. From (48), we know that coadjoint equation $V_x = [U\ V]$ remains true. Furthermore, we know that $\widetilde{V}_x^2 = [\widetilde{U}, \widetilde{V}]$ is also true. Let $F = \operatorname{str}[\widetilde{V}^2]$; it is clear that $F_x = 0$. Let $F = \sum_{n\geq 0} F_n \lambda^{-2n}$; we can get the following formulas of integrable of motion:

$$\begin{aligned} F_n = {} & F_0 + \sum_{i=1}^{n}\Big(\left(\widetilde{A}_i + \widetilde{B}_i\lambda\right)\left(\widetilde{A}_{2n-i} + \widetilde{B}_{2n-i}\lambda\right) \\ & \quad + \left(\widetilde{C}_i + \widetilde{D}_i\lambda\right)\left(\widetilde{I}_{2n-i} + \widetilde{M}_{2n-i}\lambda\right)\Big) \\ = {} & \left(\widetilde{A}_0 + \widetilde{B}_0\lambda\right)^2 \\ & + \sum_{i=1}^{n}\Big(\Lambda^{4n+2}\left(\langle \psi_1, \varphi_1\rangle - \langle \psi_2, \varphi_2\rangle\right. \\ & \qquad \left. + \langle \psi_3, \varphi_3\rangle - \langle \psi_4, \varphi_4\rangle\right) \\ & \qquad -2\Lambda\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_2, \varphi_1\rangle\right)\Big) \\ & \times \left(\langle \psi_1, \varphi_1\rangle - \langle \psi_2, \varphi_2\rangle + \langle \psi_3, \varphi_3\rangle - \langle \psi_4, \varphi_4\rangle\right) \\ & - 2\Lambda\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right)\left(\langle \psi_4, \varphi_3\rangle\right) \\ & - 4\Lambda^{4n}\left(1+\Lambda^2\right)\left(\langle \psi_1, \varphi_2\rangle + \langle \psi_3, \varphi_4\rangle\right) \\ & \times \left(\left(\langle \psi_2, \varphi_1\rangle - \langle \psi_4, \varphi_3\rangle\right)\right). \end{aligned} \tag{49}$$

After a direct calculation, we have

$$\begin{aligned} \varphi_{1,t_n} &= \frac{\partial F_n}{\partial \psi_1}, & \varphi_{2,t_n} &= \frac{\partial F_n}{\partial \psi_2}, \\ \varphi_{3,t_n} &= \frac{\partial F_n}{\partial \psi_3}, & \varphi_{4,t_n} &= \frac{\partial F_n}{\partial \psi_4}, \\ \psi_{1,t_n} &= -\frac{\partial F_n}{\partial \varphi_1}, & \psi_{2,t_n} &= -\frac{\partial F_n}{\partial \varphi_2}, \\ \psi_{3,t_n} &= -\frac{\partial F_n}{\partial \varphi_3}, & \psi_{4,t_n} &= -\frac{\partial F_n}{\partial \varphi_4}, \end{aligned} \tag{50}$$

which means that the AKNS-KN coupling system is a Hamiltonian system. In order to prove that nonlinearized system is completely integrable in the Liouville sense, we choose the following Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{4}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial \phi_{ij}}\frac{\partial G}{\partial \psi_{ij}} - \frac{\partial F}{\partial \psi_{ij}}\frac{\partial G}{\partial \phi_{ij}}\right). \quad (51)$$

At this time, we still have the equality $\widetilde{V}_{t_n} = [\widetilde{U}, \widetilde{V}]$. After a discussion, we know that F is also a generating function of the motion for equation, which makes

$$\{F_{m+1}, F_{n+1}\} = \frac{\partial}{\partial t_n}F_{m+1} = 0. \quad (52)$$

In addition, similar to the method in [6], we know that

$$f_k = \psi_{1k}\phi_{1k} + \psi_{2k}\phi_{2k} + \psi_{3k}\phi_{3k} + \psi_{4k}\phi_{4k} \quad (53)$$

is integrable of motion for (45) and (46). It is easy to see that the 4N functions are involution in pairs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by National Natural Science Foundation of China (no. 11271007), Nature Science Foundation of Shandong Province of China (no. ZR2013AQ017), SDUST Research Fund (no. 2012KYTD105), and Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Science (no. KLOCAW1401).

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