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Research Article

Almost Periodic Solution of a Modified Leslie-Gower Predator-Prey Model with Beddington-DeAngelis Functional Response and Feedback Controls

Kerong Zhang,[1] Jianli Li,[1] and Aiwen Yu[2]

[1] *Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China*
[2] *Department of Mathematics, Beiya Middle School, Changsha, Hunan 410008, China*

Correspondence should be addressed to Jianli Li; ljianli@sina.com

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We consider a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis functional response and feedback controls as follows: $\dot{x}(t) = x(t)\left(a_1(t) - b(t)x(t) - c(t)y(t)/\left(\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)\right) - e_1(t)u(t)\right)$, $\dot{u}(t) = -d_1(t)u(t) + p_1(t)x(t-\tau)$, $\dot{y}(t) = y(t)\left(a_2(t) - r(t)y(t)/(x(t) + k(t)) - e_2(t)v(t)\right)$, and $\dot{v}(t) = -d_2(t)v(t) + p_2(t)y(t-\tau)$. Sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained.

1. Introduction

In recent years, the modified predator-prey systems with periodic or almost periodic coefficients have been studied extensively.

Leslie [1] proposed the famous Leslie predator-prey system as follows:

$$\begin{aligned} \dot{x}(t) &= x(a - bx) - p(x)y, \\ \dot{y} &= y\left(e - f\frac{y}{x}\right), \end{aligned} \tag{1}$$

where x and y stand for the population of the prey and the predator at time t, respectively, and $p(x)$ is the so-called predator functional response to the prey. The term y/x is the Leslie-Gower term which measures the loss in the predator population due to rarity of its favorite food.

Global stability of the positive locally asymptotically stable equilibrium in a class of predator-prey systems has been introduced by Hsu and Huang [2], and the system is as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k} - yp(x)\right), \\ \frac{dy}{dx} &= y\left[s\left(1 - \frac{hy}{s}\right)\right], \\ x(0) > 0, &\quad y(0) > 0, \quad r, s, k, h > 0. \end{aligned} \tag{2}$$

When the functional response $p(x)$ equals mx, then (2) turns into a Leslie-Gower system [3].

On the other hand, the periodic solution (almost periodic solution) and some other properties of Leslie-Gower predator-prey models were studied (see [4–9]). In particular, Zhang [10] discussed the almost periodic solution of a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis function response as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r_1(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)}\right), \\ \dot{y}(t) &= y(t)\left(r_2(t) - \frac{d(t)y(t)}{x(t) + k(t)}\right), \end{aligned} \tag{3}$$

where $x(t)$ is the size of prey population and $y(t)$ is the size of predator population.

Stimulated by the above reasons, in this paper, we incorporate the feedback control into model (3) and consider the following model:

$$\begin{aligned}\dot{x}(t) &= x(t)\left(a_1(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)} - e_1(t)u(t)\right),\\ \dot{u}(t) &= -d_1(t)u(t) + p_1(t)x(t-\tau),\\ \dot{y}(t) &= y(t)\left(a_2(t) - \frac{r(t)y(t)}{x(t)+k(t)} - e_2(t)v(t)\right),\\ \dot{v}(t) &= -d_2(t)v(t) + p_2(t)y(t-\tau),\end{aligned} \tag{4}$$

where $\tau > 0$ and all the coefficients $b(t)$, $c(t)$, $r(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $a_i(t)$, $d_i(t)$, $p_i(t)$, and $e_i(t)$ $(i = 1,2)$ are all continuous, almost periodic functions on R.

Associated with (4), we consider a group of initial conditions with the following form (we assume, without loss of generality, that the initial time $t_0 = 0$):

$$\begin{aligned}&x(s) = \phi(s) \ge 0, \quad s \in [-\tau, 0], \ \phi(0) > 0,\\ &y(s) = \varphi(s) \ge 0, \quad s \in [-\tau, 0], \ \varphi(0) > 0,\\ &u(0) > 0, \qquad v(0) > 0.\end{aligned} \tag{5}$$

Let f be a continuous bounded function on R and we set

$$f^l = \inf_{t\in R} f(t), \qquad f^u = \sup_{t\in R} f(t). \tag{6}$$

Throughout this paper, we assume that the coefficients of the almost periodic system (4) satisfy

$$\begin{aligned}&\min_{i=1,2}\left\{b^l, c^l, \alpha^l, \beta^l, \gamma^l, r^l, k^l, a_i^l, d_i^l, p_i^l, e_i^l\right\} > 0,\\ &\max_{i=1,2}\left\{b^u, c^u, \alpha^u, \beta^u, \gamma^u, r^u, k^u, a_i^u, d_i^u, p_i^u, e_i^u\right\} < +\infty.\end{aligned} \tag{7}$$

By constructing a suitable Lyapunov functional, we obtain some sufficient conditions for the existence of a globally attractive positive almost periodic solution of system (4) with initial conditions (5).

2. Permanence

In this section, we give some definitions and results that we will use in the rest of the paper.

Lemma 1 (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \ge (\le)\, x(b - ax)$, when $t \ge 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \ge \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \le \frac{b}{a}\right). \tag{8}$$

Lemma 2 (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \ge (\le)\, b - ax$, when $t \ge 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \ge \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \le \frac{b}{a}\right). \tag{9}$$

Set the following:

$$\begin{aligned}&M_1 = \frac{a_1^u}{b^l}, \qquad L_1 = \frac{p_1^u M_1}{d_1^l},\\ &M_2 = \frac{a_2^u\left(M_1 + k^u\right)}{r^l}, \qquad L_2 = \frac{p_2^u M_2}{d_2^l},\\ &m_1 = \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u}, \qquad l_1 = \frac{p_1^l m_1}{d_1^u},\\ &m_2 = \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right), \qquad l_2 = \frac{p_2^l m_2}{d_2^u}.\end{aligned} \tag{10}$$

Theorem 3. *Suppose that system (4) with initial condition (5) satisfies the following condition:*

$$a_1^l - \frac{c^u}{r^l} - e_1^u L_1 > 0, \qquad a_2^l - e_2^u L_2 > 0. \tag{11}$$

Then system (4) is permanent; that is, any positive solution $(x(t), u(t), y(t), v(t))^T$ of the system (4) satisfies

$$\begin{aligned}&0 < m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1,\\ &0 < l_1 \le \liminf_{t\to+\infty} u(t) \le \limsup_{t\to+\infty} u(t) \le L_1,\\ &0 < m_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le M_2,\\ &0 < l_2 \le \liminf_{t\to+\infty} v(t) \le \limsup_{t\to+\infty} v(t) \le L_2.\end{aligned} \tag{12}$$

Proof. From the first equation of (4), we have the following:

$$\dot{x}(t) \le x(t)\left(a_1^u - b^l x(t)\right). \tag{13}$$

Applying Lemma 1 to (13) leads to

$$\limsup_{t\to+\infty} x(t) \le \frac{a_1^u}{b^l} = M_1. \tag{14}$$

From (14), we know that there exists an enough large $T_1 > 0$ such that

$$x(t) \le M_1, \quad t \ge T_1 > 0, \tag{15}$$

so there exists an enough large $T_2 = T_1 + \tau$ such that

$$x(t-\tau) \le M_1, \quad t \ge T_2 > 0. \tag{16}$$

It follows from (16) and the second equation of system (4) that, for $t \ge T_2$,

$$\dot{u}(t) \le -d^l u(t) + p_1^u M_1. \tag{17}$$

Applying Lemma 2 to (17) leads to

$$\limsup_{t\to+\infty} u(t) \le \frac{p_1^u M_1}{d_1^l} = L_1. \tag{18}$$

By using a similar argument as that in the proof of (14) and (18), we can get the following:

$$\limsup_{t\to+\infty} y(t) \le \frac{a_2^u (M_1 + k^u)}{r^l} = M_2, \qquad \limsup_{t\to+\infty} v(t) \le \frac{p_2^u M_2}{d_2^l} = L_2. \tag{19}$$

From (18) and the first equation of system (4) we know

$$\dot{x}(t) \ge x(t)\left(a_1^l - \frac{c^u}{\gamma^l} - e_1^u L_1 - b^u x(t)\right). \tag{20}$$

Applying Lemma 1 and (11) to the above leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u} = m_1. \tag{21}$$

Therefore, we know that there exists an enough large T_3 such that

$$x(t) \ge m_1, \quad t \ge T_3 > 0. \tag{22}$$

From the second equation of system (4) we have the following:

$$\dot{u}(t) \ge -d_1^u u(t) + p_1^l m_1. \tag{23}$$

Applying Lemma 2 to the above, we obtain the following:

$$\liminf_{t\to+\infty} u(t) \ge \frac{p_1^l m_1}{d_1^u} = l_1. \tag{24}$$

By using a similar method as that in the proof of (21) and (24), it follows that

$$\liminf_{t\to+\infty} y(t) \ge \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right) = m_2, \qquad \liminf_{t\to+\infty} v(t) \ge \frac{p_2^l m_2}{d_2^u} = l_2. \tag{25}$$

This completes the proof. □

We denote by Ω the set of all solutions $z(t) = (x(t), u(t), y(t), v(t))^T$ of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le v(t) \le L_2$ for all $t > 0$.

Theorem 4. *Consider the following:* $\Omega \ne \emptyset$.

Proof. From the properties of almost periodic function there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\begin{aligned}
&a_i(t+t_n) \longrightarrow a_i(t), \qquad d_i(t+t_n) \longrightarrow d_i(t),\\
&e_i(t+t_n) \longrightarrow e_i(t), \qquad p_i(t+t_n) \longrightarrow p_i(t), \quad (i = 1, 2),\\
&b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t),\\
&r(t+t_n) \longrightarrow r(t), \qquad k(t+t_n) \longrightarrow k(t),\\
&\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t),\\
&\gamma(t+t_n) \longrightarrow \gamma(t),
\end{aligned} \tag{26}$$

as $n \to \infty$ uniformly on R. Let $z(t) = (x(t), u(t), y(t), v(t))^T$ be a solution of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le v(t) \le L_2$ for $t > T$. Clearly, the sequence $z(t + t_n)$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore, by the Arzelà-Ascoli theorem, there exists a subsequence $z(t + t_k)$ which converges to a continuous function $z^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))^T$ as $k \to +\infty$ uniformly on each bounded subset of R. Let $T_0 \in R$ be given. We may assume that $t_k + T_0 \ge T$ for all k. For $t \ge 0$, we have the following:

$$\begin{aligned}
x(t+t_k+T_0) &= x(t_k+T_0) + \int_{T_0}^{t+T_0} x(s+t_k)\big(a_1(s+t_k) - b(s+t_k)x(s+t_k)\\
&\quad - (c(s+t_k)y(s+t_k))\\
&\quad \times(\alpha(s+t_k) + \beta(s+t_k)x(s+t_k) + \gamma(s+t_k)y(s+t_k))^{-1}\\
&\quad - e_1(s+t_k)u(s+t_k)\big)\,ds,\\
u(t+t_k+T_0) &= u(t_k+T_0) - \int_{T_0}^{t+T_0} d_1(s+t_k)u(s+t_k) + p_1(s+t_k)x(s+t_k-\tau)\,ds,\\
y(t+t_k+T_0) &= y(t_k+T_0) + \int_{T_0}^{t+T_0} y(s+t_k)\left(a_2(s+t_k) - \frac{r(s+t_k)y(s+t_k)}{x(s+t_k)+k(s+t_k)}\right.\\
&\quad \left. - e_2(s+t_k)v(s+t_k)\right)ds,\\
\dot{v}(t+t_k+T_0) &= v(t_k+T_0) + \int_{T_0}^{t+T_0} -d_2(s+t_k)v(s+t_k) + p_2(s+t_k)y(s+t_k-\tau)\,ds.
\end{aligned} \tag{27}$$

Applying Lebesgue's dominated convergence theorem and letting $k \to +\infty$ in (27), we obtain the following:

$$\begin{aligned}
x^*(t+T_0) &= x^*(T_0) + \int_{T_0}^{t+T_0} x^*(s)\big(a_1(s) - b(s)x^*(s)\\
&\quad - \frac{c(s)y^*(s)}{\alpha(s)+\beta(s)x^*(s)+\gamma(s)y^*(s)}\\
&\quad - e_1(s)u^*(s)\big)\,ds,
\end{aligned}$$

$$u^*(t+T_0) = u^*(T_0) - \int_{T_0}^{t+T_0} d_1(s)\,u^*(s) + p_1(s)\,x^*(s-\tau)\,ds,$$
$$y^*(t+T_0) = y^*(T_0) + \int_{T_0}^{t+T_0} y^*(s)\left(a_2(s) - \frac{r(s)\,y^*(s)}{x^*(s)+k(s)} - e_2(s)\,v^*(s)\right)ds,$$
$$v^*(t+T_0) = v^*(T_0) + \int_{T_0}^{t+T_0} -d_2(s)\,v^*(s) + p_2(s)\,y^*(s-\tau)\,ds. \quad (28)$$

Since $T_0 \in R$ is arbitrarily given, $z^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))^T$ is a solution of system (4) on R. It is clear that $m_1 \le x^*(t) \le M_1, l_1 \le u^*(t) \le L_1, m_2 \le y^*(t) \le M_2, l_2 \le v^*(t) \le L_2$ for $t \in R$. Thus $z^*(t) \in \Omega$. This completes the proof. □

3. Existence of a Unique Almost Periodic Solution

Now let us state several definitions and lemmas which will be useful in the proving of the main result of this section.

Definition 5 (see [12]). A function $f(t,x)$, where f is an m-vector, t is a real scalar, and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in S \subset R^n$, if $f(t,x)$ is continuous in $t \in R$ and $x \in S$ and if, for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a ς such that the inequality

$$|f(t+\varsigma, x) - f(t,x)| < \varepsilon \quad (29)$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in S$. The number ς is called an *ε-translation* number of $f(t,x)$.

Definition 6 (see [12]). A function $f : R \to R$ is said to be asymptotically almost periodic function, if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that $f(t) = q(t) + r(t)$, $t \in R$ and $r(t) \to 0$ as $t \to \infty$.

Lemma 7 (see [13]). *Let f be a nonnegative, integral, and uniformly continuous function defined on $[0,+\infty)$; then $\lim_{t\to+\infty} f(t) = 0$.*

Theorem 8. *Suppose that all conditions of Theorem 3 hold; furthermore assume that*

(H) $\Theta > 0$, where $\Theta = \min\{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$,

$$\Theta_1 = b^l m_1 - p_1^u M_1 - \frac{c^u \beta^u M_1 M_2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} - \frac{r^u M_1 M_2}{(m_1 + k^l)^2} > 0,$$
$$\Theta_2 = \frac{\gamma^l}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2} - \frac{c^u \gamma^u M_2^2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} - p_2^u M_2 > 0, \quad (30)$$
$$\Theta_3 = d_1^l - e_1^u, \qquad \Theta_4 = d_2^l - e_2^u.$$

Then system (4) with initial conditions (5) is globally attractive.

Proof. Let $x(t) = e^{x_1(t)}$, $y(t) = e^{y_1(t)}$, and then system (4) is transformed into

$$\dot{x}_1(t) = a_1(t) - b(t)\,e^{x_1(t)} - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} - e_1(t)\,u(t),$$
$$\dot{u}(t) = -d_1(t)\,u(t) + p_1(t)\,e^{x_1(t-\tau)}, \quad (31)$$
$$\dot{y}_1(t) = a_2(t) - \frac{r(t)\,e^{y_1(t)}}{e^{x_1(t)} + k(t)} - e_2(t)\,v(t),$$
$$\dot{v}(t) = -d_2(t)\,v(t) + p_2(t)\,e^{y_1(t-\tau)}.$$

Suppose that $z_1(t) = (x_1(t), u(t), y_1(t), v(t))^T$ and $z_1^*(t) = (x_1^*(t), u^*(t), y_1^*(t), v^*(t))^T$ are any two positive solutions of (31).

Let $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$, where

$$V_1(t) = |x_1(t) - x_1^*(t)|,$$
$$V_2(t) = |u(t) - u^*(t)| + p_1^u \int_{t-\tau}^{t} \left|e^{x_1(t)} - e^{x_1^*(t)}\right| ds,$$
$$V_3(t) = |y_1(t) - y_1^*(t)|, \quad (32)$$
$$V_4(t) = |v(t) - v^*(t)| + p_2^u \int_{t-\tau}^{t} \left|e^{y_1(t)} - e^{y_1^*(t)}\right| ds.$$

Calculating the right derivative $D^+V_1(t)$ of $V_1(t)$ along the solution of (31), we have the following:

$$D^+V_1(t) = \operatorname{sgn}(x_1(t) - x_1^*(t)) \times \Big[-b(t)\left(e^{x_1(t)} - e^{x_1^*(t)}\right) - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} + \frac{c(t)\,e^{y_1^*(t)}}{\alpha(t) + \beta(t)\,e^{x_1^*(t)} + \gamma(t)\,e^{y_1^*(t)}} - e_1(t)\,(u(t) - u^*(t))\Big]$$

$$
\begin{aligned}
&= \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \Big[-b(t)e^{\xi(t)}\left(x_1(t) - x_1^*(t)\right) \\
&\qquad - \frac{c(t)e^{y_1(t)}}{\alpha(t) + \beta(t)e^{x_1(t)} + \gamma(t)e^{y_1(t)}} \\
&\qquad + \frac{c(t)e^{y_1^*(t)}}{\alpha(t) + \beta(t)e^{x_1(t)} + \gamma(t)e^{y_1(t)}} \\
&\qquad - \frac{c(t)e^{y_1^*(t)}}{\alpha(t) + \beta(t)e^{x_1(t)} + \gamma(t)e^{y_1(t)}} \\
&\qquad + \frac{c(t)e^{y_1^*(t)}}{\alpha(t) + \beta(t)e^{x_1^*(t)} + \gamma(t)e^{y_1^*(t)}} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right)\Big] \\
&\le \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \left[-b^l m_1\left(x_1(t) - x_1^*(t)\right)\right] \\
&\quad - \frac{c(t)}{\alpha(t) + \beta(t)e^{x_1(t)} + \gamma(t)e^{y_1(t)}} \\
&\quad \cdot e^{\eta(t)}\left(y_1(t) - y_1^*(t)\right) \\
&\qquad + \Big(c(t)e^{y_1^*(t)}\Big[\beta(t)\left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad\qquad + \gamma(t)\left(e^{y_1(t)} - e^{y_1^*(t)}\right)\Big]\Big) \\
&\qquad \times \Big(\left(\alpha(t) + \beta(t)e^{x_1^*(t)} + \gamma(t)e^{y_1^*(t)}\right) \\
&\qquad\quad \times \left(\alpha(t) + \beta(t)e^{x_1^*(t)} + \gamma(t)e^{y_1^*(t)}\right)\Big)^{-1} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right) \\
&\le \left(\frac{c^u\beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - b^l m_1\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad + \left(\frac{c^u\gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} + \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2}\right) \\
&\quad \times \left|y_1(t) - y_1^*(t)\right| \\
&\quad + e_1^u\left|u(t) - u^*(t)\right|.
\end{aligned}
\tag{33}
$$

Further, it follows that

$$
\begin{aligned}
D^+V_2(t) &= \operatorname{sgn}\left(u(t) - u^*(t)\right) \\
&\quad \times \Big(-d_1(t)\left(u(t) - u^*(t)\right) \\
&\qquad + p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right) \\
&\qquad + p_1^u\left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad - p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right)\Big) \\
&\le -d_1^l\left|u(t) - u^*(t)\right| + p_1^u M_1\left|x_1(t) - x_1^*(t)\right|, \\
D^+V_3(t) &= \operatorname{sgn}\left(y_1(t) - y_1^*(t)\right) \\
&\quad \times \Big[-\frac{r(t)e^{y_1(t)}}{e^{x_1(t)} + k(t)} + \frac{r(t)e^{y_1^*(t)}}{e^{x_1^*(t)} + k(t)} \\
&\qquad - e_2(t)\left(v(t) - v^*(t)\right)\Big] \\
&\le -\frac{r^l m_2}{M_1 + k^u}\left|y_1(t) - y_1^*(t)\right| \\
&\quad + \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2}\left|x_1(t) - x_1^*(t)\right| \\
&\quad + e_2^u\left|v(t) - v^*(t)\right|, \\
D^+V_4(t) &\le -d_2^l\left|v(t) - v^*(t)\right| + p_2^u M_2\left|y_1(t) - y_1^*(t)\right|.
\end{aligned}
\tag{34}
$$

Therefore, we have the following:

$$
\begin{aligned}
&D^+V(t) \\
&= D^+V_1(T) + D^+V_2(T) + D^+V_3(T) + D^+V_4(T) \\
&\le -\left(b^l m_1 - p_1^u M_1 - \frac{c^u\beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2}\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad - \left(\frac{r^l m_2}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2} - \frac{c^u\gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - p_2^u M_2\right)\left|y_1(t) - y_1^*(t)\right| \\
&\quad - \left(d_1^l - e_1^u\right)\left|u(t) - u^*(t)\right| - \left(d_2^l - e_2^u\right)\left|v(t) - v^*(t)\right| \\
&\le -\Theta\left(\left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| + \left|u(t) - u^*(t)\right| + \left|v(t) - v^*(t)\right|\right).
\end{aligned}
\tag{35}
$$

Integrating the above inequality on internal $[0, t]$, it follows that, for $t \ge 0$,

$$
\begin{aligned}
&V(t) + \Theta\int_0^t \left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| \\
&\qquad + \left|u(t) - u^*(t)\right| + \left|v(t) - v^*(t)\right| ds \\
&\quad \le V(0) < +\infty.
\end{aligned}
\tag{36}
$$

Then, for $t > 0$, we obtain that

$$\int_0^t |x_1(t) - x_1^*(t)| + |y_1(t) - y_1^*(t)| + |u(t) - u^*(t)| + |v(t) - v^*(t)|\, ds \le \frac{V(0)}{\Theta} < +\infty. \quad (37)$$

By Lemma 7, we obtain

$$\lim_{t\to+\infty} |x_1(t) - x_1^*(t)| = 0, \qquad \lim_{t\to+\infty} |y_1(t) - y_1^*(t)| = 0,$$
$$\lim_{t\to+\infty} |u(t) - u^*(t)| = 0, \qquad \lim_{t\to+\infty} |v(t) - v^*(t)| = 0. \quad (38)$$

Then the solution of systems (4) and (5) is globally attractive. □

Theorem 9. *Suppose that all conditions of Theorem 8 hold; then there exists a unique almost periodic solution of systems* (4) *and* (5).

Proof. According to Theorem 4, there exists a bounded positive solution $W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ of (4) and (5). Then there exists a sequence $\{t_k'\}$, $t_k' \to \infty$ as $k \to \infty$, such that $(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T$ is a solution of the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\Big(a_1(t+t_k') - b(t+t_k')x(t) - \frac{c(t+t_k')y(t)}{\alpha(t+t_k') + \beta(t+t_k')x(t) + \gamma(t+t_k')y(t)} - e_1(t+t_k')u(t)\Big),\\
\dot{u}(t) &= -d_1(t+t_k')u(t) + p_1(t+t_k')x(t-\tau),\\
\dot{y}(t) &= y(t)\Big(a_2(t+t_k') - \frac{r(t+t_k')y(t)}{x(t) + k(t+t_k')} - e_2(t+t_k')v(t)\Big),\\
\dot{v}(t) &= -d_2(t+t_k')v(t) + p_2(t+t_k')y(t-\tau).
\end{aligned} \quad (39)$$

According to Theorem 3, we get that not only $\{(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T\}$ but also $\{(\dot{w}_1(t+t_k'), \dot{w}_2(t+t_k'), \dot{w}_3(t+t_k'), \dot{w}_4(t+t_k'))^T\}$ are uniformly bounded and equicontinuous. By Ascoli's theorem there exists a uniformly convergent subsequence $w_i(t+t_k) \subseteq w_i(t+t_k')$ $(i = 1,2,3,4)$ such that, for any $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ with the property that if $m, k \ge K(\varepsilon)$, then

$$|w_i(t+t_m) - w_i(t+t_k)| < \varepsilon, \quad (i = 1,2,3,4). \quad (40)$$

This is to say, $w_i(t+t_k)$ $(i = 1,2,3,4)$ are asymptotically almost periodic functions; hence there exist four almost periodic functions $P_i(t+t_k)$ $(i = 1,2,3,4)$ and four continuous functions $F_i(t+t_k)$ $(i = 1,2,3,4)$ such that

$$w_i(t+t_k) = P_i(t+t_k) + F_i(t+t_k), \quad t \in R,\ i = 1,2,3,4, \quad (41)$$

where

$$\lim_{k\to+\infty} P_i(t+t_k) = P_i(t), \qquad \lim_{k\to+\infty} F_i(t+t_k) = 0, \quad i = 1,2,3,4, \quad (42)$$

$P_i(t)$ $(i = 1,2,3,4)$ are an almost periodic function.

Therefore,

$$\lim_{k\to+\infty} w_i(t+t_k) = P_i(t), \quad (i = 1,2,3,4). \quad (43)$$

On the other hand,

$$\begin{aligned}
\lim_{k\to+\infty} \dot{w}_i(t+t_k) &= \lim_{k\to+\infty}\lim_{h\to0} \frac{w_i(t+t_k+h) - w_i(t+t_k)}{h}\\
&= \lim_{h\to+\infty}\lim_{k\to0} \frac{w_i(t+t_k+h) - w_i(t+t_k)}{h}\\
&= \lim_{h\to0} \frac{P_i(t+h) - P_i(t)}{h}, \quad (i = 1,2,3,4).
\end{aligned} \quad (44)$$

So $\dot{P}_i(t)$ $(i = 1,2,3,4)$ exist. Now we will prove that $(P_1(t), P_2(t), P_3(t), P_4(t))^T$ is an almost periodic solution of system (4).

From properties of almost periodic function, there exits a sequence $\{t_n\}$, $\{t_n\} \to \infty$ as $n \to \infty$, such that

$$\begin{aligned}
&a_i(t+t_n) \longrightarrow a_i(t), \qquad d_i(t+t_n) \longrightarrow d_i(t),\\
&e_i(t+t_n) \longrightarrow e_i(t), \qquad p_i(t+t_n) \longrightarrow p_i(t), \qquad (i = 1,2),\\
&b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t),\\
&r(t+t_n) \longrightarrow r(t), \qquad k(t+t_n) \longrightarrow k(t),\\
&\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t),\\
&\gamma(t+t_n) \longrightarrow \gamma(t),
\end{aligned} \quad (45)$$

as $n \to \infty$ uniformly on R.

It is easy to know that $w_i(t+t_n) \to P_i(t)\,(i=1,2,3,4)$ as $n \to \infty$, and then we have the following:

$$\begin{aligned}
&\dot{P}_1(t) \\
&= \lim_{n\to+\infty} \dot{w}_1(t+t_n) \\
&= \lim_{n\to+\infty} \Big[w_1(t+t_n)\big(a_1(t+t_n) - b(t+t_n)\,w_1(t+t_n) \\
&\qquad - \left(c(t+t_n)\,w_3(t+t_n)\right) \\
&\qquad \times \left(\alpha(t+t_n) + \beta(t+t_n)\,w_1(t+t_n)\right. \\
&\qquad \left. + \gamma(t+t_n)w_3(t+t_n)\right)^{-1} \\
&\qquad - e_1(t+t_n)\,w_2(t+t_n)\big)\Big] \\
&= P_1(t)\Bigg(a_1(t) - b(t)\,P_1(t) \\
&\qquad - \frac{c(t)\,P_3(t)}{\alpha(t)+\beta(t)\,P_1(t)+\gamma(t)\,P_3(t)} - e_1(t)\,P_2(t)\Bigg).
\end{aligned} \tag{46}$$

By using a similar argument as that in the above, we have the following:

$$\begin{aligned}
\dot{P}_2(t) &= -d_1(t)\,P_2(t) + p_1(t)\,P_1(t-\tau), \\
\dot{P}_3(t) &= P_3(t)\left(a_2(t) - \frac{r(t)\,P_3(t)}{P_1(t)+k(t)} - e_2(t)\,P_4(t)\right), \\
\dot{P}_4(t) &= -d_2(t)\,P_4(t) + p_2(t)\,P_3(t-\tau).
\end{aligned} \tag{47}$$

This proves that $P_i(t)\,(i = 1,2,3,4)$ is a nonnegative almost periodic solution of systems (4) and (5); by Theorem 8, it follows that there exists a globally asymptotically stable nonnegative almost periodic solution of system (4). The proof is complete. □

4. An Example

Consider the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left(4 - 2x(t) - \frac{10y(t)}{2+20x(t)+20y(t)} - 2u(t)\right), \\
\dot{u}(t) &= -3u(t) + \frac{1}{5}x(t-\tau), \\
\dot{y}(t) &= y(t)\left(\frac{1}{10} - \frac{20y(t)}{x(t)+23} - \frac{2}{5}v(t)\right), \\
\dot{v}(t) &= -2v(t) + 2y(t-\tau).
\end{aligned} \tag{48}$$

By a simple calculation, we check that all conditions in Theorems 8 and 9 are fulfilled. Therefore, by Theorems 8 and 9, system (48) has a unique globally asymptotically stable nonnegative almost periodic solution (see Figure 1).

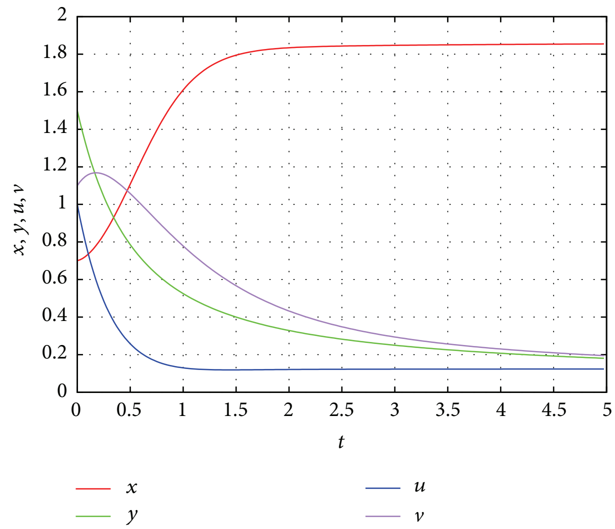

Figure 1: Dynamic behavior of system (48) with the initial $(x(0), y(0), u(0), v(0))^T = (0.7, 1.5, 1.0, 1.1)^T$, for $\tau = 0$, $t \in [0,5]$. From the figure, we could easily see that the solution $(x(t), y(t), u(t), v(t))^T$ is asymptotic to the unique almost periodic solution of the system (48).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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