Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 238953, 9 pages
http://dx.doi.org/10.1155/2014/238953

Research Article

Invertibility and Explicit Inverses of Circulant-Type Matrices with k-Fibonacci and k-Lucas Numbers

Zhaolin Jiang, Yanpeng Gong, and Yun Gao

Department of Mathematics, Linyi University, Linyi, Shandong 276005, China

Correspondence should be addressed to Zhaolin Jiang; jzh1208@sina.com

Received 27 March 2014; Accepted 17 April 2014; Published 20 May 2014

Academic Editor: Juntao Sun

Circulant matrices have important applications in solving ordinary differential equations. In this paper, we consider circulant-type matrices with the k-Fibonacci and k-Lucas numbers. We discuss the invertibility of these circulant matrices and present the explicit determinant and inverse matrix by constructing the transformation matrices, which generalizes the results in Shen et al. (2011).

1. Introduction

Circulant matrices play an important role in solving ordinary differential equations. Wilde [1] developed a theory for the solution of ordinary and partial differential equations whose structure involves the algebra of circulants. He showed how the algebra of 2×2 circulants relates to the study of the harmonic oscillator, the Cauchy-Riemann equations, Laplace's equation, the Lorentz transformation, and the wave equation. And he used $n \times n$ circulants to suggest natural generalizations of these equations to higher dimensions. By using the well-known results on circulant matrices for computation of eigenvalues of the perturbation coefficient matrix, in computing the stability criteria, Voorhees and Nip [2] got that the rate constants were all set equal to one. By using a Strang-type block-circulant preconditioner, Zhang et al. [3] speeded up the convergent rate of boundary-value methods. Joy and Tavsanoglu [4] showed that feedback matrices of ring cellular neural networks, which can be described by the ODE, are block circulants. Delgado et al. [5] developed some techniques to obtain global hyperbolicity for a certain class of endomorphisms of $(R^p)^n$ with $p, n \geq 2$; this kind of endomorphisms is obtained from vectorial difference equations where the mapping defining these equations satisfies a circulant matrix condition. In [6], nonsymmetric, large, and sparse linear systems were solved by using the generalized minimal residual (GMRES) method; a circulant-block preconditioner was proposed to speed up the convergence rate of the GMRES method.

Circulant-type matrices include the circulant and left circulant and g-circulant matrices. They have been put on the firm basis with the work of Davis [7], Gray [8], and Jiang and Zhou [9]. In [10], the authors pointed out the processes based on the eigenvalue of the circulant-type matrices with i.i.d. entries. There are discussions about the convergence in probability and in distribution of the spectral norm of circulant-type matrices in [11]. Furthermore, the g-circulant matrices are focused on by many researchers; for details, please refer to [12, 13] and the references therein. Ngondiep et al. showed the singular values of g-circulants in [14].

The k-Fibonacci and k-Lucas number sequences are defined by the following recurrence relations [15, 16], respectively,

$$\begin{aligned} F_{k,n+1} &= kF_{k,n} + F_{k,n-1}, \quad \text{where } F_{k,0} = 0,\ F_{k,1} = 1, \\ L_{k,n+1} &= kL_{k,n} + L_{k,n-1}, \quad \text{where } L_{k,0} = 2,\ L_{k,1} = k. \end{aligned} \tag{1}$$

Let α and β be the roots of the characteristic equation $x^2 - kx - 1 = 0$; then the Binet formulas of the sequences $\{F_{k,n}\}$ and $\{L_{k,n}\}$ have the form

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_{k,n} = \alpha^n + \beta^n, \tag{2}$$

where $\alpha + \beta = k, \alpha\beta = -1$.

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [7, 9]. It is worth pointing out that the computational complexity of these algorithms is very amazing with the order of matrix increasing. However, some authors gave the explicit determinant and inverse of the circulant and skew-circulant involving Fibonacci and Lucas numbers. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses in [17]. Cambini presented an explicit form of the inverse of a particular circulant matrix in [18]. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [19].

The purpose of this paper is to obtain better results for the determinants and inverses of circulant-type matrices by some perfect properties of the k-Fibonacci and k-Lucas numbers.

In this paper, we adopt the following two conventions $0^0 = 1$ and, for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in the case $i > n$.

(1) Circulant matrix: the $n \times n$ circulant matrix (denoted by $\mathrm{Circ}(a_1, a_2, \ldots, a_n)$) with input $\{a_i\}$ is the matrix whose (i, j)th entry is $a_{(j-i+n) \bmod n}$.

(2) Left circulant matrix: this is also a symmetric matrix (denoted by $L\mathrm{Circ}(a_1, a_2, \ldots, a_n)$) where the (i, j)th element of the matrix is $a_{(i+j-2) \bmod n}$.

(3) g-circulant matrix: a g-circulant matrix is an $n \times n$ matrix as in the following form:

$$g - \mathrm{Circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix}_{n\times n}, \tag{3}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

The first row of g-$\mathrm{Circ}(a_1, a_2, \ldots, a_n)$ is $(a_1, a_2, \ldots, a_n)$; its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, $g \bmod n$ positions). Note that $g = 1$ or $g = n + 1$ yields the standard *circulant matrix*. If $g = n - 1$, then we obtain the so-called *left circulant matrix*.

Lemma 1 (see [7, 9]). *Let $A = \mathrm{Circ}(a_1, a_2, \ldots, a_n)$ be a circulant matrix; then we have the following.*

(i) *A is invertible if and only if $f(\omega^k) \neq 0$, $(k = 0, 1, 2, \ldots, n-1)$, where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ and $\omega = \exp(2\pi i/n)$;*

(ii) *If A is invertible, then the inverse A^{-1} of A is a circulant matrix.*

Lemma 2. *Let $\Delta = L\mathrm{Circ}(1, 0, \ldots, 0)$; the matrix Δ is an orthogonal cyclic shift matrix. It holds that*

$$L\mathrm{Circ}(a_1, a_2, \ldots, a_n) = \Delta \mathrm{Circ}(a_1, a_2, \ldots, a_n). \tag{4}$$

Lemma 3 (see [20]). *The $n \times n$ matrix $\mathbb{Q}_g$ is unitary if and only if $(n, g) = 1$, where $\mathbb{Q}_g$ is a g-circulant matrix with the first row $e^* = [1, 0, \ldots, 0]$.*

Lemma 4 (see [20]). *$A_{g,n}$ is a g-circulant matrix with the first row $[a_1, a_2, \ldots, a_n]$ if and only if $A_{g,n} = \mathbb{Q}_g C$, where $C = \mathrm{Circ}(a_1, a_2, \ldots, a_n)$.*

Lemma 5 (see [20]). *The inverse of a nonsingular g-circulant matrix $A_{g,n}$ is an s-circulant B, where s satisfies $gs = kn + 1$, for some integer k.*

2. Circulant Matrix with the k-Fibonacci Numbers

In this section, let $A_{k,n} = \mathrm{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a circulant matrix. Firstly, we give the determinant equation of the matrix $A_{k,n}$. Afterwards, we prove that $A_{k,n}$ is an invertible matrix for $n > 2$, and then we find the inverse of the matrix $A_{k,n}$.

In the following, let $\mu = F_{k,1} - F_{k,n+1}$, $\nu = F_{k,n}/(F_{k,1} - F_{k,n+1})$, $s = L_{k,1} - L_{k,n+1}$, and $t = L_{k,n} - 2$.

Theorem 6. *Let $A_{k,n} = \mathrm{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a circulant matrix; then we have*

$$\det A_{k,n} = \mu^{n-1} + F_{k,n}^{n-2} \sum_{i=1}^{n-1} \frac{F_{k,i}}{\nu^{i-1}}, \tag{5}$$

where $F_{k,n}$ is the nth k-Fibonacci number. Specially, when $k = 1$, this result is the same as Theorem 2.1 in [17].

Proof. Obviously, $\det A_{k,1} = 1$ satisfies formula (5). In the case $n > 1$, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & & \\ -k & & & & & & & 1 \\ -1 & & & & & & 1 & -k \\ 0 & & & 0 & & 1 & -k & -1 \\ \vdots & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & & & 1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ 0 & & 1 & -k & \cdot^{\cdot^{\cdot}} & & 0 & \\ 0 & 1 & -k & -1 & & & & \end{pmatrix}, \tag{6}$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \nu^{n-2} & 0 & \ldots & 0 & 1 \\ 0 & \nu^{n-3} & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \nu & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices; then we have

$$\Gamma A_{k,n} \Pi_1 = \begin{pmatrix} F_{k,1} & f'_{k,n} & F_{k,n-1} & F_{k,n-2} & \cdots & F_{k,2} \\ 0 & f_{k,n} & F_{k,n-2} & F_{k,n-3} & \cdots & F_{k,1} \\ 0 & 0 & \mu & & & \\ 0 & 0 & -F_{k,n} & \mu & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & & 0 & -F_{k,n} & \mu \end{pmatrix}, \tag{7}$$

where

$$f_{k,n} = F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i}\nu^{n-(i+1)},$$
$$f'_{k,n} = \sum_{i=1}^{n-1} F_{k,i+1}\nu^{n-(i+1)}. \tag{8}$$

So we obtain

$$\begin{aligned}\det\Gamma \det A_{k,n} \det\Pi_1 &= F_{k,1}\left[F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i}\nu^{n-(i+1)}\right]\mu^{n-2} \\ &= F_{k,1}\left[F_{k,1} - F_{k,n+1} + \sum_{i=1}^{n-1} F_{k,i}\nu^{n-(i+1)}\right]\mu^{n-2} \\ &= \mu^{n-1} + F_{k,n}^{n-2}\sum_{i=1}^{n-1}\frac{F_{k,i}}{\nu^{i-1}},\end{aligned} \tag{9}$$

while

$$\det\Gamma = \det\Pi_1 = (-1)^{(n-1)(n-2)/2}; \tag{10}$$

hence, we have

$$\det A_{k,n} = \mu^{n-1} + F_{k,n}^{n-2}\sum_{i=1}^{n-1}\frac{F_{k,i}}{\nu^{i-1}}. \tag{11}$$

The proof is completed. □

Theorem 7. *Let $A_{k,n} = \mathrm{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a circulant matrix; if $n > 2$, then $A_{k,n}$ is an invertible matrix. Specially, when $k = 1$, we get Theorem 2.2 in [17].*

Proof. When $n = 3$ in Theorem 6, then we have $\det A_{k,3} = [k^2(k-1) + (-k^2 + k - 1)^2](k^2 + k + 2) \neq 0$; hence, $A_{k,3}$ is invertible. In the case $n > 3$, since $F_{k,n} = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\alpha + \beta = k$, $\alpha \cdot \beta = -1$, we have

$$\begin{aligned} f(\omega^t) &= \sum_{j=1}^{n} F_{k,j}(\omega^t)^{j-1} \\ &= \frac{1}{\alpha-\beta}\sum_{j=1}^{n}(\alpha^j - \beta^j)(\omega^t)^{j-1} \\ &= \frac{1}{\alpha-\beta}\left[\frac{\alpha(1-\alpha^n)}{1-\alpha\omega^t} - \frac{\beta(1-\beta^n)}{1-\beta\omega^t}\right] \\ &\quad \left(\text{because } 1-\alpha\omega^t \neq 0, 1-\beta\omega^t \neq 0\right) \\ &= \frac{1}{\alpha-\beta}\left[\frac{(\alpha-\beta) - (\alpha^{n+1} - \beta^{n+1}) + \alpha\beta(\alpha^n - \beta^n)\omega^t}{1 - (\alpha+\beta)\omega^t + \alpha\beta\omega^{2t}}\right] \\ &= \frac{1 - F_{k,n+1} - F_{k,n}\omega^t}{1 - k\omega^t - \omega^{2t}} \quad (t = 1, 2, \ldots, n-1). \end{aligned} \tag{12}$$

If there exists ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, then we obtain $1 - F_{k,n+1} - F_{k,n}\omega^l = 0$ for $1 - k\omega^l - \omega^{2l} \neq 0$; thus, $\omega^l = (1 - F_{k,n+1})/F_{k,n}$ is a real number. While

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\frac{2l\pi}{n} + i\sin\frac{2l\pi}{n}, \tag{13}$$

hence, $\sin(2l\pi/n) = 0$, so we have $\omega^l = -1$ for $0 < 2l\pi/n < 2\pi$. But $x = -1$ is not the root of the equation $1 - F_{k,n+1} - F_{k,n}x = 0$ $(n > 3)$. Hence, we obtain $f(\omega^t) \neq 0$ for any $\omega^t (t = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} F_{k,j} = -(1/k)(1 - F_{k,n+1} - F_{k,n}) \neq 0$. Hence, by Lemma 1, the proof is completed. □

Lemma 8. *Let the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{i,j} = \begin{cases} \mu, & i = j, \\ -F_{k,n}, & i = j+1, \\ 0, & \textit{otherwise}. \end{cases} \tag{14}$$

Then the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{G}$ is equal to

$$g'_{i,j} = \begin{cases} \dfrac{F_{k,n}^{i-j}}{\mu^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{15}$$

In particular, when $k = 1$, we get Lemma 2.1 in [17].

Proof. Let $c_{i,j} = \sum_{k=1}^{n-2} g_{i,k}g'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$c_{i,i} = g_{i,i}g'_{i,i} = \mu \cdot \frac{1}{\mu} = 1. \tag{16}$$

For $i \geq j+1$, we obtain

$$\begin{aligned} c_{i,j} &= \sum_{k=1}^{n-2} g_{i,k}g'_{k,j} = g_{i,i-1}g'_{i-1,j} + g_{i,i}g'_{i,j} \\ &= -F_{k,n} \cdot \frac{F_{k,n}^{i-j-1}}{\mu^{i-j}} + \mu \cdot \frac{F_{k,n}^{i-j}}{\mu^{i-j+1}} = 0. \end{aligned} \tag{17}$$

Hence, we verify $\mathcal{G}\mathcal{G}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-2}$. □

Theorem 9. *Let $A_{k,n} = \mathrm{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ $(n > 2)$ be a circulant matrix; then we have*

$$\begin{aligned} A_{k,n}^{-1} = \frac{1}{f_{k,n}}\mathrm{Circ}\Bigg(& 1 + \sum_{i=1}^{n-2}\frac{F_{k,n-i}F_{k,n}^{i-1}}{\mu^i}, \\ & -k + \sum_{i=1}^{n-2}\frac{F_{k,n-1-i}F_{k,n}^{i-1}}{\mu^i}, -\frac{1}{\mu}, -\frac{F_{k,n}}{\mu^2}, \\ & -\frac{F_{k,n}^2}{\mu^3}, \ldots, -\frac{F_{k,n}^{n-3}}{\mu^{n-2}}\Bigg), \end{aligned} \tag{18}$$

where $f_{k,n} = F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i}\nu^{n-(i+1)}$. Specially, when $k = 1$, this result is the same as Theorem 2.3 in [17].

Proof. Let

$$\Pi_2 = \begin{pmatrix} 1 & -f'_{k,n} & \rho_{n-2} & \cdots & \rho_1 \\ 0 & 1 & -\frac{F_{k,n-2}}{f_{k,n}} & \cdots & -\frac{F_{k,1}}{f_{k,n}} \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned} f_{k,n} &= F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i}\nu^{n-(i+1)}, \\ f'_{k,n} &= \sum_{i=1}^{n-1} F_{k,i+1}\nu^{n-(i+1)}, \\ \rho_1 &= \frac{f'_{k,n}}{f_{k,n}}F_{k,1} - F_{k,2}, \ldots, \\ \rho_{n-2} &= \frac{f'_{k,n}}{f_{k,n}}F_{k,n-2} - F_{k,n-1}. \end{aligned} \tag{20}$$

Then we have

$$\Gamma A_{k,n}\Pi_1\Pi_2 = \mathscr{D}_1 \oplus \mathscr{G}, \tag{21}$$

where $\mathscr{D}_1 = \mathrm{diag}(F_{k,1}, f_{k,n})$ is a diagonal matrix and $\mathscr{D}_1 \oplus \mathscr{G}$ is the direct sum of $\mathscr{D}_1$ and $\mathscr{G}$. If we denote $\Pi = \Pi_1\Pi_2$, then we obtain

$$A_{k,n}^{-1} = \Pi\left(\mathscr{D}_1^{-1} \oplus \mathscr{G}^{-1}\right)\Gamma. \tag{22}$$

Since the last row elements of matrix Π are $0, 1, -(F_{k,n-2}/f_{k,n})$, $-(F_{k,n-3}/f_{k,n}), \ldots, -(F_{k,2}/f_{k,n})$, $-(F_{k,1}/f_{k,n})$, by Lemma 8, if $A_{k,n}^{-1} = \mathrm{Circ}(x_1, x_2, \ldots, x_n)$, then its last row elements are given by the following equations:

$$\begin{aligned} x_2 &= -\frac{k}{f_{k,n}} + \frac{1}{f_{k,n}}\sum_{i=1}^{n-2}\frac{F_{k,n-1-i}F_{k,n}^{i-1}}{\mu^i}, \\ x_3 &= -\frac{1}{f_{k,n}}\frac{F_{k,1}}{\mu}, \\ x_4 &= -\frac{1}{f_{k,n}}\sum_{i=1}^{2}\frac{F_{k,3-i}F_{k,n}^{i-1}}{\mu^i} + \frac{kF_{k,1}}{f_{k,n}\mu}, \\ x_5 &= -\frac{1}{f_{k,n}}\sum_{i=1}^{3}\frac{F_{k,4-i}F_{k,n}^{i-1}}{\mu^i} \\ &\quad + \frac{k}{f_{k,n}}\sum_{i=1}^{2}\frac{F_{k,3-i}F_{k,n}^{i-1}}{\mu^i} + \frac{F_{k,1}}{f_{k,n}\mu}, \\ &\vdots \\ x_n &= -\frac{1}{f_{k,n}}\sum_{i=1}^{n-2}\frac{F_{k,n-1-i}F_{k,n}^{i-1}}{\mu^i} \\ &\quad + \frac{k}{f_{k,n}}\sum_{i=1}^{n-3}\frac{F_{k,n-2-i}F_{k,n}^{i-1}}{\mu^i} \\ &\quad + \frac{1}{f_{k,n}}\sum_{i=1}^{n-4}\frac{F_{k,n-3-i}F_{k,n}^{i-1}}{\mu^i}, \\ x_1 &= \frac{1}{f_{k,n}} + \frac{k}{f_{k,n}}\sum_{i=1}^{n-2}\frac{F_{k,n-1-i}F_{k,n}^{i-1}}{\mu^i} \\ &\quad + \frac{1}{f_{k,n}}\sum_{i=1}^{n-3}\frac{F_{k,n-2-i}F_{k,n}^{i-1}}{\mu^i}. \end{aligned} \tag{23}$$

Let $C_n^{(j)} = \sum_{i=1}^{j}(F_{k,j+1-i}F_{k,n}^{i-1}/\mu^i)$ $(j = 1, 2, \ldots, n-2)$; then we have

$$\begin{aligned} C_n^{(2)} - kC_n^{(1)} &= \sum_{i=1}^{2}\frac{F_{k,3-i}F_{k,n}^{i-1}}{\mu^i} - \frac{kF_{k,1}}{\mu} = \frac{F_{k,n}}{\mu^2}, \\ kC_n^{(n-2)} &+ C_n^{(n-3)} \\ &= k\sum_{i=1}^{n-2}\frac{F_{k,n-1-i}F_{k,n}^{i-1}}{\mu^i} + \sum_{i=1}^{n-3}\frac{F_{k,n-2-i}F_{k,n}^{i-1}}{\mu^i} \\ &= \frac{kF_{k,1}F_{k,n}^{n-3}}{\mu^{n-2}} + \sum_{i=1}^{n-3}\frac{\left(kF_{k,n-1-i} + F_{k,n-2-i}\right)F_{k,n}^{i-1}}{\mu^i} \\ &= \sum_{i=1}^{n-2}\frac{F_{k,n-i}F_{k,n}^{i-1}}{\mu^i}, \\ C_n^{(j+2)} &- kC_n^{(j+1)} - C_n^{(j)} \\ &= \sum_{i=1}^{j+2}\frac{F_{k,j+3-i}F_{k,n}^{i-1}}{\mu^i} - k\sum_{i=1}^{j+1}\frac{F_{k,j+2-i}F_{k,n}^{i-1}}{\mu^i} \\ &\quad - \sum_{i=1}^{j}\frac{F_{k,j+1-i}F_{k,n}^{i-1}}{\mu^i} \\ &= \frac{F_{k,2}F_{k,n}^{j}}{\mu^{j+1}} + \frac{F_{k,1}F_{k,n}^{j+1}}{\mu^{j+2}} - \frac{kF_{k,1}F_{k,n}^{j}}{\mu^{j+1}} \\ &\quad + \sum_{i=1}^{j}\frac{\left(F_{k,j+3-i} - kF_{k,j+2-i} - F_{k,j+1-i}\right)F_{k,n}^{i-1}}{\mu^i} \\ &= \frac{F_{k,n}^{j+1}}{\mu^{j+2}} \quad (j = 1, 2, \ldots, n-4). \end{aligned} \tag{24}$$

Hence, we obtain

$$\begin{aligned} A_{k,n}^{-1} &= \mathrm{Circ}\left(\frac{1 + kC_n^{(n-2)} + C_n^{(n-3)}}{f_{k,n}}, \frac{C_n^{(n-2)} - k}{f_{k,n}},\right. \\ &\quad -\frac{C_n^{(1)}}{f_{k,n}}, -\frac{C_n^{(2)} - kC_n^{(1)}}{f_{k,n}}, -\frac{C_n^{(3)} - kC_n^{(2)} - C_n^{(1)}}{f_{k,n}}, \ldots, \\ &\quad \left.-\frac{C_n^{(n-2)} - kC_n^{(n-3)} - C_n^{(n-4)}}{f_{k,n}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f_{k,n}} \operatorname{Circ}\left(1 + \sum_{i=1}^{n-2} \frac{F_{k,n-i} F_{k,n}^{i-1}}{\mu^i},\right.\\
&\qquad -k + \sum_{i=1}^{n-2} \frac{F_{k,n-1-i} F_{k,n}^{i-1}}{\mu^i}, -\frac{1}{\mu}, -\frac{F_{k,n}}{\mu^2},\\
&\qquad \left. -\frac{F_{k,n}^2}{\mu^3}, \dots, -\frac{F_{k,n}^{n-3}}{\mu^{n-2}}\right).
\end{aligned} \tag{25}
$$

□

3. Circulant Matrix with the k-Lucas Numbers

In this section, let $B_{k,n} = \operatorname{Circ}(L_{k,1}, L_{k,2}, \dots, L_{k,n})$ be a circulant matrix. Firstly, we give a determinant formula for the matrix $B_{k,n}$. Afterwards, we prove that $B_{k,n}$ is an invertible matrix for any positive integer n, and then we find the inverse of the matrix $B_{k,n}$.

Theorem 10. *Let $B_{k,n} = \operatorname{Circ}(L_{k,1}, L_{k,2}, \dots, L_{k,n})$ be a circulant matrix; then we have*

$$
\begin{aligned}
\det B_{k,n} &= ks^{n-1}\\
&\quad + kt^{n-2} \sum_{i=1}^{n-1} \left[\left(L_{k,i+2} - \frac{k^2+2}{k} L_{k,i+1} \right) \left(\frac{s}{t} \right)^{i-1} \right],
\end{aligned} \tag{26}
$$

where $L_{k,n}$ is the nth k-Lucas number. In particular, when $k = 1$, we get Theorem 3.1 in [17].

Proof. Obviously, $\det B_{k,1} = k$ satisfies formula (26); when $n > 1$, let

$$
\Sigma = \begin{pmatrix}
1 & & & & & & & \\
-\dfrac{k^2+2}{k} & & & & & & & 1 \\
-1 & & & & & & 1 & -k \\
0 & & & 0 & & 1 & -k & -1 \\
\vdots & & & & \iddots & \iddots & \iddots & \\
0 & & & 1 & \iddots & \iddots & & \\
0 & & 1 & -k & \iddots & & 0 & \\
0 & 1 & -k & -1 & & & &
\end{pmatrix}, \tag{27}
$$

$$
\Omega_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left(\frac{t}{s}\right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left(\frac{t}{s}\right)^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{t}{s} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

be two $n \times n$ matrices; then the form of $\Sigma B_{k,n} \Omega_1$ is given as follows:

$$
\begin{pmatrix}
L_{k,1} & l'_{k,n} & L_{k,n-1} & L_{k,n-2} & \dots & L_{k,2} \\
0 & l_{k,n} & \delta_n & \delta_{n-1} & \dots & \delta_3 \\
0 & 0 & s & & & \\
0 & 0 & -t & s & & \\
\vdots & \vdots & & \ddots & \ddots & \\
0 & 0 & \dots & \dots & -t & s
\end{pmatrix}, \tag{28}
$$

where

$$
\begin{aligned}
\delta_i &= L_{k,i} - \frac{k^2+2}{k} L_{k,i-1}, \quad i = 3, \dots, n,\\
l_{k,n} &= L_{k,1} - \frac{k^2+2}{k} L_{k,n} + \sum_{i=1}^{n-2} \delta_{i+2} \left(\frac{t}{s} \right)^{n-(i+1)},\\
l'_{k,n} &= \sum_{i=1}^{n-1} L_{k,i+1} \left(\frac{t}{s} \right)^{n-(i+1)}.
\end{aligned} \tag{29}
$$

Hence, we obtain

$$
\begin{aligned}
&\det \Sigma \det B_{k,n} \det \Omega_1\\
&\quad = L_{k,1} \left[L_{k,1} - \frac{k^2+2}{k} L_{k,n} + \sum_{i=1}^{n-2} \delta_{i+2} \left(\frac{t}{s} \right)^{n-(i+1)} \right] s^{n-2}\\
&\quad = L_{k,1} \left[s + \sum_{i=1}^{n-1} \delta_{i+2} \left(\frac{t}{s} \right)^{n-(i+1)} \right] s^{n-2}\\
&\quad = ks^{n-1} + kt^{n-2} \sum_{i=1}^{n-1} \left(L_{k,i+2} - \frac{k^2+2}{k} L_{k,i+1} \right) \left(\frac{s}{t} \right)^{i-1},
\end{aligned} \tag{30}
$$

while

$$
\det \Sigma = \det \Omega_1 = (-1)^{(n-1)(n-2)/2}; \tag{31}
$$

thus, we have

$$
\begin{aligned}
\det B_{k,n} &= ks^{n-1}\\
&\quad + kt^{n-2} \sum_{i=1}^{n-1} \left[\left(L_{k,i+2} - \frac{k^2+2}{k} L_{k,i+1} \right) \left(\frac{s}{t} \right)^{i-1} \right].
\end{aligned} \tag{32}
$$

□

Theorem 11. *Let $B_{k,n} = \operatorname{Circ}(L_{k,1}, L_{k,2}, \dots, L_{k,n})$ be a circulant matrix; then $B_{k,n}$ is invertible for any positive integer n. Specially, when $k = 1$, we get Theorem 3.2 in [17].*

Proof. Since $L_{k,n} = \alpha^n + \beta^n$, where $\alpha + \beta = k$, $\alpha\beta = -1$, we have

$$
\begin{aligned}
&f(\omega^t)\\
&\quad = \sum_{j=1}^{n} L_{k,j} (\omega^t)^{j-1} = \sum_{j=1}^{n} (\alpha^j + \beta^j)(\omega^t)^{j-1}\\
&\quad = \frac{\alpha(1-\alpha^n)}{1-\alpha\omega^t}\\
&\qquad + \frac{\beta(1-\beta^n)}{1-\beta\omega^t} \quad \left(\text{because } 1-\alpha\omega^t \neq 0, 1-\beta\omega^t \neq 0\right)
\end{aligned}
$$

$$= \frac{(\alpha+\beta)-\left(\alpha^{n+1}+\beta^{n+1}\right)+\alpha\beta\left(\alpha^{n}+\beta^{n}\right)\omega^{t}-2\alpha\beta\omega^{t}}{1-(\alpha+\beta)\omega^{t}+\alpha\beta\omega^{2t}}$$
$$= \frac{k-L_{k,n+1}+\left(2-L_{k,n}\right)\omega^{t}}{1-k\omega^{t}-\omega^{2t}} \quad (t=1,2,\ldots,n-1). \tag{33}$$

If there exists ω^l $(l=1,2,\ldots,n-1)$ such that $f(\omega^l)=0$, then we obtain $k-L_{k,n+1}+(2-L_{k,n})\,\omega^l=0$ for $1-k\omega^l-\omega^{2l}\neq 0$; thus, $\omega^l=(k-L_{k,n+1})/(L_{k,n}-2)$ is a real number. While

$$\omega^{l}=\exp\left(\frac{2l\pi i}{n}\right)=\cos\frac{2l\pi}{n}+i\sin\frac{2l\pi}{n}, \tag{34}$$

hence, $\sin(2l\pi/n)=0$, so we have $\omega^l=-1$ for $0<2l\pi/n<2\pi$. But $x=-1$ is not the root of the equation $k-L_{k,n+1}+(2-L_{k,n})x=0$ for any positive integer n. Hence, we obtain $f(\omega^k)\neq 0$ for any ω^k $(k=1,2,\ldots,n-1)$, while $f(1)=\sum_{j=1}^{n}L_{k,j}=(-1/k)(k+2-L_{k,n+1}-L_{k,n})\neq 0$. Thus, by Lemma 1, the proof is completed. □

Lemma 12. *Let the matrix $\mathscr{H}=[h_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$h_{i,j}=\begin{cases} s, & i=j,\\ t, & i=j+1,\\ 0, & \text{otherwise};\end{cases} \tag{35}$$

then the inverse $\mathscr{H}^{-1}=[h'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathscr{H}$ is equal to

$$h'_{i,j}=\begin{cases} \dfrac{t^{i-j}}{s^{i-j+1}} & i\geq j,\\ 0, & i<j.\end{cases} \tag{36}$$

Specially, when $k=1$, we get Lemma 3.1 in [17].

Proof. Let $r_{i,j}=\sum_{k=1}^{n-2}h_{i,k}h'_{k,j}$. Obviously, $r_{i,j}=0$ for $i<j$. In the case $i=j$, we obtain

$$r_{i,i}=h_{i,i}h'_{i,i}=s\cdot\frac{1}{s}=1. \tag{37}$$

For $i\geq j+1$, we obtain

$$\begin{aligned} r_{i,j}&=\sum_{k=1}^{n-2}h_{i,k}h'_{k,j}=h_{i,i-1}h'_{i-1,j}+h_{i,i}h'_{i,j}\\ &=(-t)\cdot\frac{t^{i-j-1}}{s^{i-j}}+s\cdot\frac{t^{i-j}}{s^{i-j+1}}=0.\end{aligned} \tag{38}$$

Hence, we verify $\mathscr{H}\mathscr{H}^{-1}=I_{n-2}$, where I_{n-2} is an $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathscr{H}^{-1}\mathscr{H}=I_{n-2}$. Thus, the proof is completed. □

Theorem 13. *Let $B_{k,n}=\mathrm{Circ}(L_{k,1},L_{k,2},\ldots,L_{k,n})$ be a circulant matrix; then we have*

$$\begin{aligned} B_{k,n}^{-1} &= \frac{1}{l_{k,n}}\mathrm{Circ}\Bigg(1+\sum_{i=1}^{n-2}\frac{\left(L_{k,n+2-i}-\left(\left(k^2+2\right)/k\right)L_{k,n+1-i}\right)t^{i-1}}{s^{i}},\\ &\quad -\frac{k^2+2}{k}\\ &\quad +\sum_{i=1}^{n-2}\frac{\left(L_{k,n+1-i}-\left(\left(k^2+2\right)/k\right)L_{k,n-i}\right)t^{i-1}}{s^{i}},\\ &\quad \frac{k^2+4}{ks},\frac{\left(k^2+4\right)t}{ks^2},\frac{\left(k^2+4\right)t^2}{ks^3},\ldots,\\ &\quad \frac{\left(k^2+4\right)t^{n-3}}{ks^{n-2}}\Bigg).\end{aligned} \tag{39}$$

In particular, when $k=1$, the result is the same as Theorem 3.3 in [17].

Proof. Let

$$\Omega_2=\begin{pmatrix} 1 & -\dfrac{l'_{k,n}}{k} & \lambda_n & \cdots & \lambda_3\\ 0 & 1 & -\dfrac{\delta_n}{l_{k,n}} & \cdots & -\dfrac{\delta_3}{l_{k,n}}\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}, \tag{40}$$

where

$$\begin{aligned} \lambda_i &= \frac{1}{k}\left[\frac{l'_{k,i}\left(L_{k,i}-\left(\left(k^2+2\right)/k\right)L_{k,i-1}\right)}{l_{k,n}}-L_{k,i-1}\right],\\ &\qquad\qquad i=3,4,\ldots,n,\\ \delta_i &= L_{k,i}-\frac{k^2+2}{k}L_{k,i-1}, \quad i=3,\ldots,n\\ l_{k,n} &= L_{k,1}-\frac{k^2+2}{k}L_{k,n}\\ &\quad +\sum_{i=1}^{n-2}\left(L_{k,i+2}-\frac{k^2+2}{k}L_{k,i+1}\right)\left(\frac{t}{s}\right)^{n-(i+1)},\\ l'_{k,n} &= \sum_{i=1}^{n-1}L_{k,i+1}\left(\frac{t}{s}\right)^{n-(i+1)}.\end{aligned} \tag{41}$$

Then we have

$$\Sigma B_{k,n}\Omega_1\Omega_2=\mathscr{D}_2\oplus\mathscr{H}, \tag{42}$$

where $\mathscr{D}_2 = \text{diag}(L_{k,1}, f_{k,n})$ is a diagonal matrix and $\mathscr{D}_2 \oplus \mathscr{H}$ is the direct sum of $\mathscr{D}_2$ and $\mathscr{H}$. If we denote $\Omega = \Omega_1\Omega_2$, then we obtain

$$B_{k,n}^{-1} = \Omega\left(\mathscr{D}_2^{-1} \oplus \mathscr{H}^{-1}\right)\Sigma. \tag{43}$$

Since the last row elements of the matrix Ω are $0, 1, (((k^2+2)/k)L_{k,n-1} - L_{k,n})/l_{k,n}, (((k^2+2)/k)L_{k,n-2} - L_{k,n-1})/l_{k,n}, \dots, (((k^2+2)/k)L_{k,2} - L_{k,3})/l_{k,n}$, by Lemma 12, if $B_{k,n}^{-1} = \text{Circ}(y_1, y_2, \dots, y_n)$, then its last row elements are given by the following equations:

$$\begin{aligned}
y_2 &= -\frac{k^2+2}{kl_{k,n}} + \frac{1}{l_{k,n}}\sum_{i=1}^{n-2}\frac{\delta_{n+1-i}t^{i-1}}{s^i},\\
y_3 &= -\frac{1}{l_{k,n}}\frac{\delta_3}{s},\\
y_4 &= \frac{k\delta_3}{sl_{k,n}} - \frac{1}{l_{k,n}}\sum_{i=1}^{2}\frac{\delta_{5-i}t^{i-1}}{s^i},\\
y_5 &= \frac{\delta_3}{sl_{k,n}} + \frac{k}{l_{k,n}}\sum_{i=1}^{2}\frac{\delta_{5-i}t^{i-1}}{s^i} - \frac{1}{l_{k,n}}\sum_{i=1}^{3}\frac{\delta_{6-i}t^{i-1}}{s^i},\\
&\vdots\\
y_n &= \frac{1}{l_{k,n}}\sum_{i=1}^{n-4}\frac{\delta_{n-1-i}t^{i-1}}{s^i} + \frac{k}{l_{k,n}}\sum_{i=1}^{n-3}\frac{\delta_{n-i}t^{i-1}}{s^i}\\
&\quad - \frac{1}{l_{k,n}}\sum_{i=1}^{n-2}\frac{\delta_{n+1-i}t^{i-1}}{s^i},\\
y_1 &= \frac{1}{l_{k,n}} + \frac{1}{l_{k,n}}\sum_{i=1}^{n-3}\frac{\delta_{n-i}t^{i-1}}{s^i} + \frac{k}{l_{k,n}}\sum_{i=1}^{n-2}\frac{\delta_{n+1-i}t^{i-1}}{s^i}.
\end{aligned} \tag{44}$$

Let $D_n^{(j)} = \sum_{i=1}^{j}((L_{k,j+3-i} - ((k^2+2)/k)L_{k,j+2-i})t^{i-1}/s^i)$ $(j = 1, 2, \dots, n-2)$; then we have

$$\begin{aligned}
kD_n^{(1)} - D_n^{(2)} &= \frac{k\delta_3}{s} - \sum_{i=1}^{2}\frac{\delta_{5-i}t^{i-1}}{s^i} = \frac{\left(k^2+4\right)t}{ks^2},\\
D_n^{(n-3)} &+ kD_n^{(n-2)}\\
&= \sum_{i=1}^{n-3}\frac{\delta_{n-i}t^{i-1}}{s^i} + \sum_{i=1}^{n-2}\frac{\delta_{n+1-i}t^{i-1}}{s^i}\\
&= \frac{k\delta_3t^{n-3}}{s^{n-2}} + \sum_{i=1}^{n-3}\left[\frac{\delta_{n-i}t^{i-1}}{s^i} + \frac{k\delta_{n+1-i}t^{i-1}}{s^i}\right]\\
&= \frac{\delta_4t^{n-3}}{s^{n-2}} + \sum_{i=1}^{n-3}\frac{\delta_{n+2-i}t^{i-1}}{s^i} = \sum_{i=1}^{n-2}\frac{\delta_{n+2-i}t^{i-1}}{s^i},
\end{aligned}$$

$$\begin{aligned}
D_n^{(j)} &+ kD_n^{(j+1)} - D_n^{(j+2)}\\
&= \sum_{i=1}^{j}\frac{\delta_{j+3-i}t^{i-1}}{s^i} + k\sum_{i=1}^{j+1}\frac{\delta_{j+4-i}t^{i-1}}{s^i} - \sum_{i=1}^{j+2}\frac{\delta_{j+5-i}t^{i-1}}{s^i}\\
&= \frac{k^2+4}{k}\frac{t^{j+1}}{s^{j+2}} \quad (j = 1, 2, \dots, n-4).
\end{aligned} \tag{45}$$

Hence, we obtain

$$\begin{aligned}
&B_{k,n}^{-1}\\
&= \text{Circ}\left(\frac{1 + D_n^{(n-3)} + kD_n^{(n-2)}}{l_{k,n}}, \frac{D_n^{(n-2)} - \left(\left(k^2+2\right)/k\right)}{l_{k,n}},\right.\\
&\qquad -\frac{D_n^{(1)}}{l_{k,n}}, \frac{kD_n^{(1)} - D_n^{(2)}}{l_{k,n}}, \frac{D_n^{(1)} + kD_n^{(2)} - D_n^{(3)}}{l_{k,n}}, \dots,\\
&\qquad \left.\frac{D_n^{(n-4)} + kD_n^{(n-3)} - D_n^{(n-2)}}{l_{k,n}}\right)\\
&= \frac{1}{l_{k,n}}\\
&\quad \times \text{Circ}\left(1 + \sum_{i=1}^{n-2}\frac{\left(L_{k,n+2-i} - \left(\left(k^2+2\right)/k\right)L_{k,n+1-i}\right)t^{i-1}}{s^i},\right.\\
&\qquad -\frac{k^2+2}{k} + \sum_{i=1}^{n-2}\frac{\left(L_{k,n+1-i} - \left(\left(k^2+2\right)/k\right)L_{k,n-i}\right)t^{i-1}}{s^i},\\
&\qquad \left.\frac{k^2+4}{ks}, \frac{\left(k^2+4\right)t}{ks^2}, \frac{\left(k^2+4\right)t^2}{ks^3}, \dots, \frac{\left(k^2+4\right)t^{n-3}}{ks^{n-2}}\right).
\end{aligned} \tag{46}$$

□

4. Left Circulant Matrix with the k-Fibonacci and k-Lucas Numbers

In this section, let $A'_{k,n} = L\text{Circ}(F_{k,1}, F_{k,2}, \dots, F_{k,n})$ and $B'_{k,n} = L\text{Circ}(L_{k,1}, L_{k,2}, \dots, L_{k,n})$ be left circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A'_{k,n}$ and $B'_{k,n}$. Afterwards, we prove that $A'_{k,n}$ is an invertible matrix for $n > 2$ and $B'_{k,n}$ is an invertible matrix for any positive integer n. The inverse of the matrices $A'_{k,n}$ and $B'_{k,n}$ is also presented.

According to Lemma 2, Theorem 6, Theorem 7, and Theorem 9, we can obtain the following theorems.

Theorem 14. *Let $A'_{k,n} = L\text{Circ}(F_{k,1}, F_{k,2}, \dots, F_{k,n})$ be a left circulant matrix; then we have*

$$\det A'_{k,n} = (-1)^{(n-1)(n-2)/2}\left[\mu^{n-1} + F_{k,n}^{n-2}\sum_{i=1}^{n-1}\frac{F_{k,i}}{\nu^{i-1}}\right], \tag{47}$$

where $F_{k,n}$ is the nth k-Fibonacci number.

Theorem 15. *Let $A'_{k,n} = \mathrm{LCirc}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a left circulant matrix; if $n > 2$, then $A'_{k,n}$ is an invertible matrix.*

Theorem 16. *Let $A'_{k,n} = \mathrm{LCirc}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ $(n > 2)$ be a left circulant matrix; then we have*

$$A'^{-1}_{k,n} = \frac{1}{f_{k,n}} \mathrm{LCirc}\left(1 + \sum_{i=1}^{n-2} \frac{F_{k,n-i} F_{k,n}^{i-1}}{\mu^i}, -\frac{F_{k,n}^{n-3}}{\mu^{n-2}}, \ldots, -\frac{F_{k,n}}{\mu^2}, -\frac{1}{\mu}, -k + \sum_{i=1}^{n-2} \frac{F_{k,n-1-i} F_{k,n}^{i-1}}{\mu^i}\right), \tag{48}$$

where

$$f_{k,n} = F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i} \nu^{n-(i+1)}. \tag{49}$$

By Lemma 2, Theorem 10, Theorem 11, and Theorem 13, the following conclusions can be attained.

Theorem 17. *Let $B'_{k,n} = \mathrm{LCirc}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a left circulant matrix; then we have*

$$\det B'_{k,n} = (-1)^{(n-1)(n-2)/2} \times \left[ks^{n-1} + kt^{n-2} \times \sum_{i=1}^{n-1} \left(L_{k,i+2} - \frac{(k^2+2)}{k} L_{k,i+1}\right) \left(\frac{s}{t}\right)^{i-1}\right], \tag{50}$$

where $L_{k,n}$ is the nth k-Lucas number.

Theorem 18. *Let $B'_{k,n} = \mathrm{LCirc}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a left circulant matrix; then $B'_{k,n}$ is invertible for any positive integer n.*

Theorem 19. *Let $B'_{k,n} = \mathrm{LCirc}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a left circulant matrix; then we have*

$$B'^{-1}_{k,n} = \frac{L}{l_{k,n}} \times \mathrm{Circ}\left(1 + \sum_{i=1}^{n-2} \frac{\left(L_{k,n+2-i} - \left(\left(k^2+2\right)/k\right) L_{k,n+1-i}\right) t^{i-1}}{s^i}, \frac{\left(k^2+4\right) t^{n-3}}{ks^{n-2}}, \ldots, \frac{\left(k^2+4\right) t^2}{ks^3}, \frac{\left(k^2+4\right) t}{ks^2}, \frac{k^2+4}{ks}, -\frac{k^2+2}{k} + \sum_{i=1}^{n-2} \frac{\left(L_{k,n+1-i} - \left(\left(k^2+2\right)/k\right) L_{k,n-i}\right) t^{i-1}}{s^i}\right). \tag{51}$$

5. g-Circulant Matrix with the k-Fibonacci and k-Lucas Numbers

In this section, let $A_{g,k,n} = g\text{-Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ and $B_{g,k,n} = g\text{-Circ}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be g-circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A_{g,k,n}$ and $B_{g,k,n}$. Afterwards, we prove that $A_{g,k,n}$ is an invertible matrix for $n > 2$ and $B_{g,k,n}$ is an invertible matrix if $(n, g) = 1$. The inverse of the matrices $A_{g,k,n}$ and $B_{g,k,n}$ is also presented.

From Lemmas 3, 4, and 5 and Theorem 6, Theorem 7, and Theorem 9, we deduce the following results.

Theorem 20. *Let $A_{g,k,n} = g\text{-Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a g-circulant matrix and $(n, g) = 1$; then we have*

$$\det A_{g,k,n} = \det \mathbb{Q}_g \left[\mu^{n-1} + F_{k,n}^{n-2} \sum_{i=1}^{n-1} \frac{F_{k,i}}{\nu^{i-1}}\right], \tag{52}$$

where $F_{k,n}$ is the nth k-Fibonacci number.

Theorem 21. *Let $A_{g,k,n} = g\text{-Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a g-circulant matrix and $(n, g) = 1$; if $n > 2$, then $A_{g,k,n}$ is an invertible matrix.*

Theorem 22. *Let $A_{g,k,n} = g\text{-Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})(n > 2)$ be a g-circulant matrix and $(n, g) = 1$; then we have*

$$A^{-1}_{g,k,n} = \left[\frac{1}{f_{k,n}} \mathrm{Circ}\left(1 + \sum_{i=1}^{n-2} \frac{F_{k,n-i} F_{k,n}^{i-1}}{\mu^i}, -k + \sum_{i=1}^{n-2} \frac{F_{k,n-1-i} F_{k,n}^{i-1}}{\mu^i}, -\frac{1}{\mu}, -\frac{F_{k,n}}{\mu^2}, -\frac{F_{k,n}^2}{\mu^3}, \ldots, -\frac{F_{k,n}^{n-3}}{\mu^{n-2}}\right)\right] \mathbb{Q}_g^T, \tag{53}$$

where $f_{k,n} = F_{k,1} - kF_{k,n} + \sum_{i=1}^{n-2} F_{k,i} \nu^{n-(i+1)}$.

Taking Lemmas 3, 4, and 5 and Theorem 10, Theorem 11 and, Theorem 13 into account, we have the following theorems.

Theorem 23. *Let $B_{g,k,n} = g\text{-Circ}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a g-circulant matrix and $(n, g) = 1$; then we have*

$$\det B_{g,k,n} = \det \mathbb{Q}_g \left[ks^{n-1} + kt^{n-2} \sum_{i=1}^{n-1} \left(\left(L_{k,i+2} - \frac{k^2+2}{k} L_{k,i+1}\right) \times \left(\frac{s}{t}\right)^{i-1}\right)\right], \tag{54}$$

where $L_{k,n}$ is the nth k-Lucas number.

Theorem 24. *Let $B_{g,k,n} = g\text{-Circ}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a g-circulant matrix and $(n, g) = 1$; then $B_{g,k,n}$ is invertible matrix.*

Theorem 25. *Let $B_{g,k,n} = g\text{-Circ}(L_{k,1}, L_{k,2}, \ldots, L_{k,n})$ be a g-circulant matrix and $(n, g) = 1$; then we have*

$$B_{g,k,n}^{-1} = \left[\frac{1}{l_{k,n}} \times \text{Circ}\left(1 + \sum_{i=1}^{n-2} \frac{\left(L_{k,n+2-i} - \left(\left(k^2+2\right)/k\right) L_{k,n+1-i}\right) t^{i-1}}{s^i}, \frac{\left(k^2+4\right) t^{n-3}}{ks^{n-2}}, \ldots, \frac{\left(k^2+4\right) t^2}{ks^3}, \frac{\left(k^2+4\right) t}{ks^2}, \frac{k^2+4}{ks}, -\frac{k^2+2}{k} + \sum_{i=1}^{n-2} \frac{\left(L_{k,n+1-i} - \left(\left(k^2+2\right)/k\right) L_{k,n-i}\right) t^{i-1}}{s^i} \right) \right] \times \mathbb{Q}_g^T. \quad (55)$$

6. Conclusion

Circulant-type matrices have a very nice structure, and the k-Fibonacci and k-Lucas numbers also have amazing properties. The related problem of circulant-type matrices and some famous numbers are studied in this paper. We not only study invertibility of circulant-type matrices with the k-Fibonacci and k-Lucas numbers but also give the explicit determinants and explicit inverses. We would get a lot of good results if we combine famous numbers with circulant-type matrices, and the results would be used in solving ordinary differential equations. We will still focus our attentions on specific matrices and famous numbers in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by the Development Project of Science & Technology of Shandong Province (Grant no. 2012GGX10115) and NSFC (Grant no. 11301251) and the AMEP of Linyi University, China.

References

[1] A. C. Wilde, "Differential equations involving circulant matrices," *The Rocky Mountain Journal of Mathematics*, vol. 13, no. 1, pp. 1–13, 1983.

[2] B. Voorhees and A. Nip, "Ordinary differential equations with star structure," *Journal of Dynamical Systems and Geometric Theories*, vol. 3, no. 2, pp. 121–152, 2005.

[3] C. Zhang, H. Chen, and L. Wang, "Strang-type preconditioners applied to ordinary and neutral differential-algebraic equations," *Numerical Linear Algebra with Applications*, vol. 18, no. 5, pp. 843–855, 2011.

[4] M. P. Joy and V. Tavsanoglu, "Circulant matrices and the stability of a class of CNNs," *International Journal of Circuit Theory and Applications*, vol. 24, no. 1, pp. 7–13, 1996.

[5] J. Delgado, N. Romero, A. Rovella, and F. Vilamajó, "Bounded solutions of quadratic circulant difference equations," *Journal of Difference Equations and Applications*, vol. 11, no. 10, pp. 897–907, 2005.

[6] X.-Q. Jin, V.-K. Sin, and L.-l. Song, "Circulant-block preconditioners for solving ordinary differential equations," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 409–418, 2003.

[7] P. J. Davis, *Circulant Matrices*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 1979.

[8] R. M. Gray, "Toeplitz and circulant matrices: A review," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 3, pp. 155–239, 2006.

[9] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University, Chengdu, China, 1999.

[10] A. Bose, R. S. Hazra, and K. Saha, "Poisson convergence of eigenvalues of circulant type matrices," *Extremes*, vol. 14, no. 4, pp. 365–392, 2011.

[11] A. Bose, R. S. Hazra, and K. Saha, "Spectral norm of circulant-type matrices," *Journal of Theoretical Probability*, vol. 24, no. 2, pp. 479–516, 2011.

[12] C. Erbas and M. M. Tanik, "Generating solutions to the N-queens problem using g-circulants," *Mathematics Magazine*, vol. 68, no. 5, pp. 343–356, 1995.

[13] Y.-K. Wu, R.-Z. Jia, and Q. Li, "g-circulant solutions to the $(0, 1)$ matrix equation $A^m = J_n$," *Linear Algebra and Its Applications*, vol. 345, pp. 195–224, 2002.

[14] E. Ngondiep, S. Serra-Capizzano, and D. Sesana, "Spectral features and asymptotic properties for g-circulants and g-Toeplitz sequences," *SIAM Journal on Matrix Analysis and Applications*, vol. 31, no. 4, pp. 1663–1687, 2009/10.

[15] S. Falcon and A. Plaza, "On k-Fibonacci numbers of arithmetic indexes," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 180–185, 2009.

[16] Y. Yazlik and N. Taskara, "A note on generalized k-Horadam sequence," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 36–41, 2012.

[17] S.-Q. Shen, J.-M. Cen, and Y. Hao, "On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9790–9797, 2011.

[18] A. Cambini, "An explicit form of the inverse of a particular circulant matrix," *Discrete Mathematics*, vol. 48, no. 2-3, pp. 323–325, 1984.

[19] D. Bozkurt and T.-Y. Tam, "Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas Numbers," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 544–551, 2012.

[20] W. T. Stallings and T. L. Boullion, "The pseudoinverse of an r-circulant matrix," *Proceedings of the American Mathematical Society*, vol. 34, pp. 385–388, 1972.