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Research Article

Some Generalizations and Modifications of Iterative Methods for Solving Large Sparse Symmetric Indefinite Linear Systems

Yu-Chien Li,[1] Jen-Yuan Chen,[1] and David R. Kincaid[2]

[1] *Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan*
[2] *Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX 78712, USA*

Correspondence should be addressed to Jen-Yuan Chen; jchen@nknu.edu.tw

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We discuss a variety of iterative methods that are based on the Arnoldi process for solving large sparse symmetric indefinite linear systems. We describe the SYMMLQ and SYMMQR methods, as well as generalizations and modifications of them. Then, we cover the Lanczos/MSYMMLQ and Lanczos/MSYMMQR methods, which arise from a double linear system. We present pseudocodes for these algorithms.

The authors dedicate this paper to the memory of Professor David M. Young, Jr., for his pioneering research, inspirational teaching, and exceptional life

1. Introduction

Frequently, when computing numerical solutions of partial differential equations, one needs to solve systems of very large sparse linear algebraic equations of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{b}$ is an $n \times 1$ vector, and one seeks a numerical solution vector $\mathbf{x}$ or a good approximation of it. Particularly for large linear systems arising from partial differential equations in three dimensions, well-known direct methods, such as Gaussian elimination, may become prohibitively expensive in terms of both computer storage and computer time. On the other hand, a variety of iterative methods may avoid these difficulties.

For linear systems involving symmetric positive definite (SPD) matrices, the conjugate gradient (CG) method (and variations of it) may work well. On the other hand, when solving linear systems, where the coefficient matrix $\mathbf{A}$ is symmetric indefinite, the choice of a suitable iterative method is *not* at all clear. On the other hand, the SYMMLQ and MINRES methods have been shown to be useful in certain situations (see Paige and Saunders [1]). For nonsymmetric systems, Saad and Schultz [2] generalized the MINRES method to obtain the GMRES method.

In Section 2, we review the Arnoldi process. In Sections 3 and 4, we describe the SYMMLQ and SYMMQR methods. Then we can generalize them, in Section 5, and we outline the modified SYMMLQ method, in Section 6. Next, in Section 7, we discuss applying the MSYMMLQ and MSYMMQR methods applied to a double linear system. Finally, we present pseudocodes in Sections 8–11.

2. Arnoldi Process

We begin with a review of the Arnoldi process.

Theorem 1. *Suppose that $\mathbf{A}$ is an $n \times n$ symmetric matrix. One can generate orthonormal vectors $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}$ using this short-term recurrence*

$$\widetilde{\mathbf{w}}^{(j+1)} \equiv \mathbf{A}\mathbf{w}^{(j)} - \alpha_j \mathbf{w}^{(j)} - \beta_j \mathbf{w}^{(j-1)} \quad (0 \le j \le n-2)$$

$$\mathbf{w}^{(j+1)} = \left(\frac{1}{\sigma_{j+1}}\right)\widetilde{\mathbf{w}}^{(j+1)}, \quad \textit{where } \sigma_{j+1} = \sqrt{\left\langle \widetilde{\mathbf{w}}^{(j+1)}, \widetilde{\mathbf{w}}^{(j+1)} \right\rangle}, \tag{2}$$

where

$$\alpha_j = \left\langle \mathbf{A}\mathbf{w}^{(j)}, \mathbf{w}^{(j)} \right\rangle,$$
$$\beta_j = \left\langle \mathbf{A}\mathbf{w}^{(j)}, \mathbf{w}^{(j-1)} \right\rangle. \tag{3}$$

Here, one assumes that $w^{(-1)} = 0$ *and* $\sigma_j \neq 0$*, for all* j*. Then the following properties hold, for* $(0 \leq i,\ j \leq n-1)$*:*

$$\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle = \delta_{ij}, \qquad \beta_j = \sigma_j. \tag{4}$$

Proof. If we let $\widetilde{\mathbf{w}}^{(0)} \equiv \mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{u}^{(0)}$, then the subspace

$$\text{Span}\left\{\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}\right\} \tag{5}$$

is equivalent to the Krylov subspace

$$\mathscr{K}_n\left(\mathbf{r}^{(0)}, \mathbf{A}\right) = \text{Span}\left\{\mathbf{r}^{(0)}, \mathbf{A}\mathbf{r}^{(0)}, \dots, \mathbf{A}^{n-2}\mathbf{r}^{(0)}, \mathbf{A}^{n-1}\mathbf{r}^{(0)}\right\}. \tag{6}$$

We obtain

$$\begin{aligned}\beta_j &= \left\langle \mathbf{A}\mathbf{w}^{(j)}, \mathbf{w}^{(j-1)} \right\rangle \quad (\text{by } (3)) \\ &= \left\langle \mathbf{w}^{(j)}, \mathbf{A}\mathbf{w}^{(j-1)} \right\rangle \quad \left(\mathbf{A}^T = \mathbf{A}\right) \\ &= \left\langle \mathbf{w}^{(j)}, \left(\sigma_j \mathbf{w}^{(j)} + \alpha_{j-1}\mathbf{w}^{(j-1)} + \beta_{j-1}\mathbf{w}^{(j-2)}\right) \right\rangle \quad (\text{by } (2)) \\ &= \sigma_j\end{aligned} \tag{7}$$

since $\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \rangle = \delta_{ij}$. □

From Theorem 1, in matrix form, it follows that

$$\begin{aligned}\mathbf{A}\mathbf{W}_{n-1} &= \mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n \mathbf{w}^{(n)}\mathbf{e}_n^T \\ &= \mathbf{W}_n \widetilde{\mathbf{T}}_{n+1},\end{aligned} \tag{8}$$

where

$$\mathbf{W}_{n-1} = \left[\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(n-1)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}\right]_{n\times n}$$
$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 1]_{1\times n},$$
$$\mathbf{T}_n \equiv \begin{bmatrix} \alpha_0 & \beta_1 & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sigma_{n-1} & \alpha_{n-3} & \beta_{n-2} & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} \\ & & & & & \sigma_{n-1} & \alpha_{n-1} \end{bmatrix}_{n\times n},$$
$$\widetilde{\mathbf{T}}_{n+1} \equiv \begin{bmatrix} \alpha_0 & \beta_1 & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sigma_{n-3} & \alpha_{n-3} & \beta_{n-2} & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} \\ & & & & & \sigma_{n-1} & \alpha_{n-1} \\ & & & & & & \sigma_n \end{bmatrix}_{(n+1)\times n}. \tag{9}$$

Example 2. We illustrate Theorem 1 for the case $n = 3$.

From (2) and (3), we have

$$\begin{aligned}\sigma_1 \mathbf{w}^{(1)} &= \widetilde{\mathbf{w}}^{(1)} = \mathbf{A}\mathbf{w}^{(0)} - \alpha_0 \mathbf{w}^{(0)} - \beta_0 \mathbf{w}^{(-1)}, \\ \sigma_2 \mathbf{w}^{(2)} &= \widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1 \mathbf{w}^{(1)} - \beta_1 \mathbf{w}^{(0)}, \\ \sigma_3 \mathbf{w}^{(3)} &= \widetilde{\mathbf{w}}^{(3)} = \mathbf{A}\mathbf{w}^{(2)} - \alpha_2 \mathbf{w}^{(2)} - \beta_2 \mathbf{w}^{(1)}.\end{aligned} \tag{10}$$

Consequently, we obtain, since $\mathbf{w}^{(-1)} = 0$,

$$\begin{aligned}\mathbf{A}\mathbf{W}_2 &= \mathbf{A}\left[\mathbf{w}^{(0)} \quad \mathbf{w}^{(1)} \quad \mathbf{w}^{(2)}\right] \\ &= \left[\mathbf{A}\mathbf{w}^{(0)} \quad \mathbf{A}\mathbf{w}^{(1)} \quad \mathbf{A}\mathbf{w}^{(2)}\right] \\ &= \begin{bmatrix} \beta_0 \mathbf{w}^{(-1)} + \alpha_0 \mathbf{w}^{(0)} + \sigma_1 \mathbf{w}^{(1)} \\ \beta_1 \mathbf{w}^{(0)} + \alpha_1 \mathbf{w}^{(1)} + \sigma_2 \mathbf{w}^{(2)} \\ \beta_2 \mathbf{w}^{(1)} + \alpha_2 \mathbf{w}^{(2)} + \sigma_3 \mathbf{w}^{(3)} \end{bmatrix}^T \\ &= \left[\mathbf{w}^{(0)} \quad \mathbf{w}^{(1)} \quad \mathbf{w}^{(2)} \quad \mathbf{w}^{(3)}\right] \\ &\quad \times \begin{bmatrix} \alpha_0 & \beta_1 & 0 \\ \sigma_1 & \alpha_1 & \beta_2 \\ 0 & \sigma_2 & \alpha_2 \\ 0 & 0 & \sigma_3 \end{bmatrix}_{4\times 3} \\ &= \left[\mathbf{W}_2 \quad \mathbf{w}^{(3)}\right] \begin{bmatrix} & \mathbf{T}_3 & \\ 0 & 0 & \sigma_3 \end{bmatrix}.\end{aligned} \tag{11}$$

So we obtain

$$\begin{aligned}\mathbf{A}\mathbf{W}_2 &= \mathbf{W}_2 \mathbf{T}_3 + \sigma_3 \mathbf{w}^{(3)}\mathbf{e}_3^T \\ &= \mathbf{W}_3 \widetilde{\mathbf{T}}_4\end{aligned} \tag{12}$$

3. SYMMLQ Method

We choose $\mathbf{u}^{(n)}$, such that $\mathbf{u}^{(n)} - \mathbf{u}^{(0)} \in \mathscr{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$. Hence, we have

$$\begin{aligned}\mathbf{u}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)}, \\ \mathbf{r}^{(n)} &= \mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_{n-1}\mathbf{y}^{(n)}.\end{aligned} \tag{13}$$

Imposing the Galerkin condition $\mathbf{r}^{(n)} \perp \mathscr{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$, we obtain

$$\begin{aligned}\left(\mathbf{r}^{(n)}\right)^T \mathbf{W}_{n-1} &= \mathbf{0}, \\ \mathbf{W}_{n-1}^T \mathbf{r}^{(n)} &= \mathbf{0}, \\ \mathbf{W}_{n-1}^T \mathbf{r}^{(0)} &= \mathbf{W}_{n-1}^T \mathbf{A}\mathbf{W}_{n-1}\mathbf{y}^{(n)} \quad (\text{by } (13)).\end{aligned} \tag{14}$$

We obtain

$$\mathbf{T}_n \mathbf{y}^{(n)} = \sigma_0 \mathbf{e}_1 \tag{15}$$

because

$$\begin{aligned}\mathbf{W}_{n-1}^T \mathbf{A}\mathbf{W}_{n-1} &= \mathbf{T}_n, \\ \mathbf{W}_{n-1}^T \mathbf{r}^{(0)} &= \sigma_0 \mathbf{e}_1, \quad \text{where } \sigma_0 = \left\|\mathbf{r}^{(0)}\right\|.\end{aligned} \tag{16}$$

Instead of solving for $\mathbf{y}^{(n)}$ directly from the triangular linear system (15), Paige and Saunders [1] factorize the matrix $\mathbf{T}_n$ into a lower triangular matrix with bandwidth three (resulting in the SYMMLQ method). Also, we have

$$\mathbf{T}_n\mathbf{Q}_{n-1} = \widehat{\mathbf{L}}_n = \begin{bmatrix} \gamma_0 & & & & & & \\ \delta_1 & \gamma_1 & & & & & \\ \varepsilon_2 & \delta_2 & \gamma_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \varepsilon_{n-3} & \delta_{n-3} & \gamma_{n-3} & & \\ & & & \varepsilon_{n-2} & \delta_{n-2} & \gamma_{n-2} & \\ & & & & \varepsilon_{n-1} & \delta_{n-1} & \widehat{\gamma}_{n-1} \end{bmatrix}_{n\times n}, \quad (17)$$

where $\mathbf{Q}_{n-1} \equiv \mathbf{Q}_{1,2}\mathbf{Q}_{2,3}\cdots\mathbf{Q}_{n-1,n}$ is an orthogonal matrix, and

$$\mathbf{Q}_{i,i+1} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & c_i & s_i & & & \\ & & & & s_i & -c_i & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix}_{n\times n}, \quad (18)$$

where $c_i^2 + s_i^2 = 1$. Since $\mathbf{T}_n\mathbf{Q}_{n-1} = \widehat{\mathbf{L}}_n$, we have

$$\begin{aligned} \mathbf{T}_n\mathbf{Q}_{n-1}\mathbf{Q}_{n-1}^{-1}\mathbf{y}^{(n)} &= \widehat{\mathbf{L}}_n\mathbf{Q}_{n-1}^{-1}\mathbf{y}^{(n)} \\ &= \sigma_0\mathbf{e}_1. \end{aligned} \quad (19)$$

Letting

$$\widehat{\mathbf{z}}^{(n)} = \mathbf{Q}_{n-1}^{-1}\mathbf{y}^{(n)}, \quad (20)$$

then

$$\widehat{\mathbf{L}}_n\widehat{\mathbf{z}}^{(n)} = \sigma_0\mathbf{e}_1. \quad (21)$$

Next letting

$$\widehat{\mathbf{x}}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)}, \quad (22)$$

we have

$$\begin{aligned} \widehat{\mathbf{x}}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{Q}_{n-1}\mathbf{Q}_{n-1}^{-1}\mathbf{y}^{(n)} \\ &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{Q}_{n-1}\widehat{\mathbf{z}}^{(n)}. \end{aligned} \quad (23)$$

Defining

$$\begin{aligned} \widehat{\mathbf{V}}_n &= \mathbf{W}_{n-1}\mathbf{Q}_{n-1}, \\ \widehat{\mathbf{z}}_n &= \left[\zeta_0, \zeta_1, \ldots, \zeta_{n-2}, \widehat{\zeta}_{n-1}\right]_{1\times n}^T, \end{aligned} \quad (24)$$

we have

$$\widehat{x}^{(n)} = \mathbf{x}^{(0)} + \widehat{\mathbf{V}}_n\widehat{\mathbf{z}}_n, \quad (25)$$

where

$$\widehat{\mathbf{V}}_n = \left[\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n-2)}, \widehat{\mathbf{v}}^{(n-1)}\right]_{n\times n}. \quad (26)$$

We let

$$\begin{aligned} \mathbf{z}_n &= \left[\zeta_0, \zeta_1, \ldots, \zeta_{n-2}, \zeta_{n-1}\right]_{1\times n}^T, \\ \mathbf{V}_n &= \left[\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n-2)}, \mathbf{v}^{(n-1)}\right]_{n\times n}, \\ \mathbf{L}_n\mathbf{z}^{(n)} &= \sigma_0\mathbf{e}_1, \end{aligned} \quad (27)$$

where

$$\mathbf{L}_n = \begin{bmatrix} \gamma_0 & & & & & & \\ \delta_1 & \gamma_1 & & & & & \\ \varepsilon_2 & \delta_2 & \gamma_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \varepsilon_{n-3} & \delta_{n-3} & \gamma_{n-3} & & \\ & & & \varepsilon_{n-2} & \delta_{n-2} & \gamma_{n-2} & \\ & & & & \varepsilon_{n-1} & \delta_{n-1} & \gamma_{n-1} \end{bmatrix}_{n\times n}, \quad (28)$$

$$\gamma_{n-1} = \sqrt{\widehat{\gamma}_{n-1}^2 + \beta_{n-1}^2}, \qquad c_n = \frac{\widehat{\gamma}_{n-1}}{\gamma_{n-1}}, \qquad s_n = \frac{\beta_n}{\gamma_{n-1}}.$$

From (21) and (28), we have $\zeta_{n-1} = (\widehat{\zeta}_{n-1}/\zeta_{n-1})\widehat{\zeta}_{n-1} = c_n\widehat{\zeta}_{n-1}$. Since

$$\begin{aligned} \widehat{\mathbf{V}}_{n+1} &= \left[\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n-1)}, \widehat{\mathbf{v}}^{(n)}\right] \\ &= \mathbf{W}_n\mathbf{Q}_n \\ &= \left[\mathbf{W}_{n-1}, \mathbf{w}^{(n)}\right]\mathbf{Q}_{1,2}, \mathbf{Q}_{2,3}, \ldots, \mathbf{Q}_{n-1,n}\mathbf{Q}_{n,n+1} \\ &= \left[\widehat{\mathbf{V}}_n, \mathbf{w}^{(n)}\right]\mathbf{Q}_{n,n+1} \\ &= \left[\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \ldots, \widehat{\mathbf{v}}^{(n-1)}, \mathbf{w}^{(n)}\right]\mathbf{Q}_{n,n+1}, \end{aligned} \quad (29)$$

we have

$$\begin{aligned} \mathbf{v}^{(n-1)} &= c_n\widehat{\mathbf{v}}^{(n-1)} + s_n\mathbf{w}^{(n)}, \\ \widehat{\mathbf{v}}^{(n)} &= s_n\widehat{\mathbf{v}}^{(n-1)} - c_n\mathbf{w}^{(n)}. \end{aligned} \quad (30)$$

If $\beta_n \neq 0$, then $\mathbf{L}_n$ is nonsingular. We can find $\mathbf{z}^{(n)}$ by solving

$$\begin{aligned} \mathbf{L}_n\mathbf{z}^{(n)} &= \sigma_0\mathbf{e}_1, \\ \mathbf{u}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{V}_n\mathbf{z}^{(n)} = \mathbf{u}^{(n-1)} + \zeta_{n-1}\mathbf{v}^{(n-1)}. \end{aligned} \quad (31)$$

4. SYMMQR Method

We choose $\mathbf{u}^{(0)}$ such that $\mathbf{u}^{(n)} - \mathbf{u}^{(0)} \in \mathscr{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$. Hence, we have

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)}, \\ \mathbf{r}^{(n)} &= \mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_{n-1}\mathbf{y}^{(n)}. \end{aligned} \quad (32)$$

Imposing the Galerkin condition $\mathbf{r}^{(n)} \perp \mathcal{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$, as before, we obtain

$$\begin{gathered}
\left(\mathbf{r}^{(n)}\right)^T \mathbf{W}_{n-1} = 0, \\
\mathbf{W}_{n-1}^T \mathbf{r}^{(n)} = 0, \\
\mathbf{W}_{n-1}^T \mathbf{r}^{(0)} = \mathbf{W}_{n-1}^T \mathbf{A} \mathbf{W}_{n-1} \mathbf{y}^{(n)} \quad (\text{by } (32)).
\end{gathered} \tag{33}$$

Since

$$\begin{gathered}
\mathbf{W}_{n-1}^T \mathbf{A} \mathbf{W}_{n-1} = \mathbf{T}_n, \\
\mathbf{W}_{n-1}^T \mathbf{r}^{(0)} = \sigma_0 \mathbf{e}_1, \quad \text{where } \sigma_0 = \left\| \mathbf{r}^{(0)} \right\|,
\end{gathered} \tag{34}$$

we have

$$\mathbf{T}_n \mathbf{y}^{(n)} = \sigma_0 \mathbf{e}_1. \tag{35}$$

Instead of solving for $\mathbf{y}^{(n)}$ directly from the triangular system (35), Paige and Saunders [1] factorized the matrix $\mathbf{T}_n$ into a lower triangular matrix with bandwidth three.

We can use a different factorization of $\mathbf{T}_n$ to obtain a slightly different method, which is called the SYMMQR method. We multiply the matrix $\mathbf{T}_n$ by an orthogonal matrix on the left-hand side instead of the right-hand side. We have

$$\mathbf{Q}_{n-1} \mathbf{T}_n \widehat{\mathbf{y}}_n = \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_1, \tag{36}$$

where

$$\mathbf{Q}_{n-1} \mathbf{T}_n = \widehat{\mathbf{R}} = \begin{bmatrix}
\gamma_0 & \delta_1 & \varepsilon_2 & & & & \\
& \gamma_1 & \delta_2 & \varepsilon_3 & & & \\
& & \gamma_2 & \delta_3 & \varepsilon_4 & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\
& & & & & \gamma_{n-2} & \delta_{n-1} \\
& & & & & & \widehat{\gamma}_{n-1}
\end{bmatrix}_{n \times n}. \tag{37}$$

We obtain the matrix $\mathbf{Q}_{n-1} \equiv \mathbf{Q}_{n-1,n} \mathbf{Q}_{n-2,n-1} \cdots \mathbf{Q}_{1,2}$, where

$$\mathbf{Q}_{i,i+1} = \begin{bmatrix}
1 & & & & & & & & \\
& 1 & & & & & & & \\
& & \ddots & & & & & & \\
& & & 1 & & & & & \\
& & & & c_i & -s_i & & & \\
& & & & s_i & c_i & & & \\
& & & & & & 1 & & \\
& & & & & & & 1 & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 1
\end{bmatrix}_{n \times n}, \tag{38}$$

with $c_i^2 + s_i^2 = 1$ being the Givens rotation. Letting $\widehat{\mathbf{y}}_n$ be the solution of

$$\widehat{\mathbf{R}} \widehat{\mathbf{y}}_n = \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_1, \tag{39}$$

then we have

$$\begin{aligned}
\widehat{\mathbf{x}}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \widehat{\mathbf{y}}_n \\
&= \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \widehat{\mathbf{R}}_n^{-1} \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_1
\end{aligned} \tag{40}$$

which satisfies the Galerkin condition $\widehat{\mathbf{r}}_n \perp \mathcal{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$, where $\widehat{\mathbf{r}}_n = \mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}^{(n)}$. We note that $\widehat{\gamma}_{n-1}$ is *not* always nonzero and, thus, $\widehat{\mathbf{R}}_n$ might be singular. We assume that $\widehat{\mathbf{R}}_n$ is nonsingular and then we define

$$\widehat{\mathbf{P}}_{n-1} = \mathbf{W}_{n-1} \widehat{\mathbf{R}}_n^{-1}, \tag{41}$$

where

$$\begin{gathered}
\widehat{\mathbf{P}}_{n-1} = \left[\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(n-2)}, \widehat{\mathbf{p}}^{(n-1)}\right]_{n \times n}, \\
\widehat{\mathbf{z}}^{(n)} = \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_1 = \left[\zeta_0, \zeta_1, \ldots, \zeta_{n-2}, \widehat{\zeta}_{n-1}\right]^T.
\end{gathered} \tag{42}$$

We have

$$\widehat{\mathbf{x}}^{(n)} = \mathbf{u}^{(0)} + \widehat{\mathbf{P}}_{n-1} \widehat{\mathbf{z}}^{(n)}. \tag{43}$$

For the next iterate $\widehat{\mathbf{x}}^{(n+1)}$, we need to solve

$$\mathbf{T}_{n+1} \widehat{\mathbf{y}}^{(n+1)} = \sigma_0 \mathbf{e}_1, \tag{44}$$

where

$$\mathbf{T}_{n+1} = \begin{bmatrix}
\alpha_0 & \beta_1 & & & & & \\
\beta_1 & \alpha_1 & \beta_2 & & & & \\
& \beta_2 & \alpha_2 & \beta_3 & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \beta_{n-2} & \alpha_{n-2} & \beta_{n-1} & \\
& & & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\
& & & & & \beta_n & \alpha_n
\end{bmatrix}_{n+1 \times n}. \tag{45}$$

Applying the Givens rotation $\mathbf{Q}_{n-1}$ to both sides of (45), we have

$$\mathbf{Q}_{n-1} \mathbf{T}_{n+1} = \begin{bmatrix}
\gamma_0 & \delta_1 & \varepsilon_2 & & & & \\
& \gamma_1 & \delta_2 & \varepsilon_3 & & & \\
& & \gamma_2 & \delta_3 & \varepsilon_4 & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\
& & & & & \gamma_{n-2} & \delta_{n-1} & \varepsilon_n \\
& & & & & & \widehat{\gamma}_{n-1} & \psi \\
& & & & & & \beta_n & \alpha_n
\end{bmatrix}_{n+1 \times n}, \tag{46}$$

where $[0, 0, 0, \ldots, 0, 0, \varepsilon_n, \psi, \alpha_n]^T = \mathbf{Q}_{n-1,n}[0, 0, 0, \ldots, 0, 0, \beta_n, \alpha_n]^T$.

To eliminate β_n, we compute the nth Given rotation $\mathbf{Q}_{n,n+1}$ by

$$\begin{gathered}
\gamma_{n-1} = \sqrt{\widehat{\gamma}_{n-1}^2 + \beta_n^2}, \qquad c_n = \frac{\widehat{\gamma}_{n-1}}{\gamma_{n-1}}, \\
s_n = -\frac{\beta_n}{\gamma_{n-1}}.
\end{gathered} \tag{47}$$

By multiplying $\mathbf{Q}_{n,n+1}$ times $\mathbf{Q}_{n-1}\mathbf{T}_{n-1}$ and times $\mathbf{Q}_{n-1}\sigma_0\mathbf{e}_1$, we have

$$\mathbf{Q}_{n,n+1}\left(\mathbf{Q}_{n-1}\mathbf{T}_{n+1}\right) = \widehat{\mathbf{R}}_{n+1} = \begin{bmatrix} \gamma_0 & \delta_1 & \varepsilon_2 & & & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & & & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\ & & & & & \gamma_{n-2} & \delta_{n-1} & \varepsilon_n \\ & & & & & & \gamma_{n-1} & \delta_n \\ & & & & & & & \widehat{\gamma}_n \end{bmatrix}_{n+1\times n},$$
$$\mathbf{Q}_{n,n+1}\left(\mathbf{Q}_{n-1}\sigma_0\mathbf{e}_1\right) = \widehat{\mathbf{z}}^{(n+1)} = \left[\zeta_0, \zeta_1, \ldots, \zeta_{n-1}, \widehat{\zeta}_n\right]^T, \tag{48}$$

where

$$\left[0, 0, 0, \ldots, 0, 0, \varepsilon_n, \delta_n, \widehat{\gamma}_n\right]^T = \mathbf{Q}_{n,n+1}\left[0, 0, 0, \ldots, 0, 0, \varepsilon_n, \psi, \alpha_n\right]^T,$$
$$\zeta_{n-1} = c_n\widehat{\zeta}_{n-1},$$
$$\widehat{\zeta}_n = s_n\widehat{\zeta}_{n-1}. \tag{49}$$

Let

$$\mathbf{R}_n = \begin{bmatrix} \gamma_0 & \delta_1 & \varepsilon_2 & & & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & & & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\ & & & & & \gamma_{n-2} & \delta_{n-1} \\ & & & & & & \gamma_{n-1} \end{bmatrix}_{n\times n}. \tag{50}$$

We define $\mathbf{z}^{(n)} = [\zeta_0, \zeta_1, \ldots, \zeta_{n-2}, \zeta_{n-1}]^T$. Since $\beta_n = ||\mathbf{w}^{(n)}|| \neq 0$, then $\gamma_n \neq 0$ and $\mathbf{R}_n$ is nonsingular. We can solve for $\mathbf{y}^{(n)}$ from

$$\mathbf{R}_n\mathbf{y}^{(n)} = \mathbf{z}^{(n)}. \tag{51}$$

We discuss the case $\beta_n = \|\mathbf{w}^{(n)}\|$ later.

Consider solving the least square problem involving $y^{(n)}$ minimizing $\|\widetilde{\mathbf{T}}_{n+1}\mathbf{y}^{(n)} - \sigma_0\mathbf{e}_1\|$, where

$$\widetilde{\mathbf{T}}_{n+1} = \begin{bmatrix} \alpha_0 & \beta_1 & & & & \\ \beta_1 & \alpha_1 & \beta_2 & & & \\ & \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta_{n-2} & \alpha_{n-2} & \beta_{n-1} \\ & & & & \beta_{n-1} & \alpha_{n-1} \\ & & & & & \beta_n \end{bmatrix}_{n+1\times n}. \tag{52}$$

We have

$$\left\|\mathbf{Q}_{n,n+1}\mathbf{Q}_{n-1}\left(\widetilde{\mathbf{T}}_{n+1}\mathbf{y}^{(n)} - \sigma_0\mathbf{e}_1\right)\right\| = \left\| \begin{bmatrix} \mathbf{R}_n \\ 0, 0, \ldots, 0, 0 \end{bmatrix}_{n+1\times n} \mathbf{y}^{(n)} - \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_1 \\ \vdots \\ \zeta_{n-1} \\ \widehat{\zeta}_n \end{bmatrix} \right\|. \tag{53}$$

Hence, the solution $\mathbf{y}^{(n)}$ from $\mathbf{R}_n\mathbf{y}^{(n)} = \mathbf{z}^{(n)}$ minimizes $||\widetilde{\mathbf{T}}_{n+1}\mathbf{y}^{(n)} - \sigma_0\mathbf{e}_1||$ and $\widehat{\zeta}_n = ||\widetilde{\mathbf{T}}_{n+1}\mathbf{y}^{(n)} - \sigma_0\mathbf{e}_1||$.

Let

$$\mathbf{u}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{R}_n^{-1}\mathbf{z}^{(n)}, \tag{54}$$

where

$$\mathbf{P}_{n-1}\mathbf{R}_n = \mathbf{W}_{n-1},$$
$$\mathbf{P}_{n-1} = \left[\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(n-2)}, \mathbf{p}^{(n-1)}\right]. \tag{55}$$

We have

$$\mathbf{p}^{(n-1)} = \left(\frac{1}{\gamma_{n-1}}\right)\left[\mathbf{w}^{(n-1)} - \varepsilon_{n-1}\mathbf{p}^{(n-3)} - \delta_{n-1}\mathbf{p}^{(n-2)}\right],$$
$$\mathbf{u}^{(n)} = \mathbf{u}^{(0)} + \mathbf{P}_{n-1}\mathbf{z}^{(n)} = \mathbf{u}^{(n-1)} + \gamma_{n-1}\mathbf{p}^{(n-1)}. \tag{56}$$

Since

$$\widehat{\mathbf{x}}^{(n+1)} = \mathbf{u}^{(0)} + \mathbf{W}_n\widehat{\mathbf{y}}^{(n+1)}, \tag{57}$$

we obtain

$$\widehat{\mathbf{x}}^{(n+1)} = \mathbf{u}^{(0)} + \mathbf{W}_n\widehat{\mathbf{R}}_{n+1}^{-1}\widehat{\mathbf{z}}^{(n+1)} = \mathbf{u}^{(0)} + \widehat{\mathbf{P}}_n\widehat{\mathbf{z}}^{(n+1)} = \mathbf{u}^{(0)} + \mathbf{P}_{n-1}\mathbf{z}^{(n)} + \widehat{\zeta}_n\widehat{\mathbf{p}}^{(n)} = \mathbf{u}^{(n)} + \widehat{\zeta}_n\widehat{\mathbf{p}}^{(n)}. \tag{58}$$

We note that $\widehat{\mathbf{x}}^{(n)}$ is the estimated solution vector satisfying the Galerkin condition, while

$$\mathbf{u}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)} \tag{59}$$

with $\mathbf{y}^{(n)}$ minimizing $||\widetilde{\mathbf{T}}_{n+1}\mathbf{y}^{(n)} - \sigma_0\mathbf{e}_1||$.

5. Generalized SYMMLQ and SYMMQR Methods

Now, we generalize the SYMMLQ and SYMMQR methods.

Theorem 3. *Suppose that* $\mathbf{E}$ *is an* $n \times n$ *symmetric positive definite (SPD) matrix and* $\mathbf{EA}$ *is an* $n\times n$ *symmetric matrix. One can generate orthonormal vectors* $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}$ *using this short-term recurrence*

$$\widetilde{\mathbf{w}}^{(j+1)} \equiv \mathbf{A}\mathbf{w}^{(j)} - \alpha_j \mathbf{w}^{(j)} - \beta_j \mathbf{w}^{(j-1)} \quad (0 \le j \le n-2),$$
$$\widetilde{\mathbf{w}}^{(j+1)} = \left(\frac{1}{\gamma_{j+1}}\right)\widetilde{\mathbf{w}}^{(j+1)}, \quad \text{where } \gamma_{j+1} = \sqrt{\left|\left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(j+1)}, \widetilde{\mathbf{w}}^{(j+1)}\right\rangle\right|}, \tag{60}$$

where

$$\alpha_j = \left\langle \mathbf{EA}\mathbf{w}^{(j)}, \mathbf{w}^{(j)} \right\rangle$$
$$\beta_j = \left\langle \mathbf{EA}\mathbf{w}^{(j)}, \mathbf{w}^{(j-1)} \right\rangle. \tag{61}$$

Then the following properties hold, for $(0 \le i,\ j \le n-1)$:

$$\left\langle \mathbf{w}^{(j)}, \mathbf{E}\mathbf{w}^{(j)} \right\rangle = \delta_{ij}, \qquad \beta_j = \gamma_j. \tag{62}$$

Proof. We obtain

$$\begin{aligned}
\beta_j &= \left\langle \mathbf{EA}\mathbf{w}^{(j)}, \mathbf{w}^{(j-1)} \right\rangle \quad (\text{by } (61)) \\
&= \left\langle \mathbf{w}^{(j)}, \mathbf{EA}\mathbf{w}^{(j-1)} \right\rangle \quad \left((\mathbf{EA})^{\mathbf{T}} = \mathbf{EA}\right) \\
&= \left\langle \mathbf{w}^{(j)}, \mathbf{E}\left(\gamma_j \mathbf{w}^{(j)} + \alpha_{j-1}\mathbf{w}^{(j-1)} + \beta_{j-1}\mathbf{w}^{(j-2)}\right) \right\rangle \quad (\text{by } (60)) \\
&= \left\langle \mathbf{w}^{(j)}, \gamma_j \mathbf{E}\mathbf{w}^{(j)} \right\rangle \\
&= \gamma_j.
\end{aligned} \tag{63}$$

Since $\langle \mathbf{w}^{(i)}, \mathbf{E}\mathbf{w}^{(j)} \rangle = \delta_{ij}$. □

As before, we let

$$\mathbf{W}_n = \left[\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-1)}, \mathbf{w}^{(n)}\right]_{n\times n},$$
$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 1]_{1\times n}. \tag{64}$$

Moreover, we have

$$\mathbf{AW}_n = \mathbf{W}_n \mathbf{T}_n + \gamma_{n+1}\mathbf{w}^{(n+1)}\mathbf{e}_n^T, \tag{65}$$

where

$$\mathbf{T}_n = \begin{bmatrix} \alpha_1 & \beta_2 & & & & & \\ \sigma_2 & \alpha_2 & \beta_3 & & & & \\ & \sigma_3 & \alpha_3 & \beta_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} & \\ & & & & \sigma_{n-1} & \alpha_{n-1} & \beta_n \\ & & & & & \sigma_n & \alpha_n \end{bmatrix}_{n\times n}. \tag{66}$$

As before, we let

$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_n \mathbf{y}^{(n)},$$
$$\mathbf{r}^{(n)} = \mathbf{r}^{(0)} - \mathbf{AW}_n \mathbf{y}^{(n)}. \tag{67}$$

Imposing the Galerkin condition again, we have

$$\left(\mathbf{E}\mathbf{r}^{(n)}\right)^T \mathbf{W}_n = \mathbf{0},$$
$$\left(\mathbf{EW}_n\right)^T \mathbf{r}^{(n)} = \mathbf{0}, \tag{68}$$
$$\left(\mathbf{EW}_n\right)^T \mathbf{r}^{(0)} = \left(\mathbf{EW}_n\right)^T \mathbf{AW}_n \mathbf{y}^{(n)} \quad (\text{by } (67)).$$

We obtain

$$\mathbf{T}_n \mathbf{y}^{(n)} = \beta_1 \mathbf{e}_1 \quad \text{where } \beta_1 = \sqrt{\left|\left\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{r}^{(0)} \right\rangle\right|} \tag{69}$$

because

$$\left(\mathbf{EW}_n\right)^T \mathbf{AW}_n = \mathbf{T}_n,$$
$$\left(\mathbf{EW}_n\right)^T \mathbf{W}_n = \mathbf{I}, \tag{70}$$
$$\left(\mathbf{EW}_n\right)^T \mathbf{r}^{(0)} = \beta_1 \mathbf{e}_1.$$

Since $\mathbf{T}_n$ is symmetric, we can apply the same techniques as in the SYMMLQ method. Also, if $\mathbf{E} = \mathbf{I}$, the method reduces to the SYMMQR method.

6. Modified SYMMLQ Method

Next, we outline the modified SYMMLQ method.

Theorem 4. *Suppose that* $\mathbf{E}$ *is an* $n \times n$ *symmetric (not necessary positive definite) matrix and* $\mathbf{EA}$ *is an* $n \times n$ *symmetric matrix. One can generate orthonormal vectors* $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}$ *using this short-term recurrence*

$$\widetilde{\mathbf{w}}^{(j+1)} \equiv \mathbf{A}\mathbf{w}^{(j)} - \alpha_j \mathbf{w}^{(j)} - \beta_j \mathbf{w}^{(j-1)}, \quad (0 \le j \le n-2),$$
$$\mathbf{w}^{(j+1)} = \left(\frac{1}{\gamma_{j+1}}\right)\widetilde{\mathbf{w}}^{(j+1)}, \quad \text{where } \gamma_{i+1} = \sqrt{\left|\mathbf{E}\widetilde{\mathbf{w}}^{(j+1)}, \widetilde{\mathbf{w}}^{(j+1)}\right\rangle\right|}, \tag{71}$$

where

$$\alpha_j = \frac{\left\langle \mathbf{EA}\mathbf{w}^{(j)}, \mathbf{w}^{(j)} \right\rangle}{\left\langle \mathbf{E}\mathbf{w}^{(j)}, \mathbf{w}^{(j)} \right\rangle},$$
$$\beta_j = \frac{\left\langle \mathbf{EA}\mathbf{w}^{(j)}, \mathbf{w}^{(j-1)} \right\rangle}{\left\langle \mathbf{E}\mathbf{w}^{(j-1)}, \mathbf{w}^{(j-1)} \right\rangle}. \tag{72}$$

Then the following properties hold, for $(0 \le j \le n-2)$:

$$d_{j+1} = \left\langle \mathbf{E}\mathbf{w}^{(j+1)}, \mathbf{w}^{(j+1)} \right\rangle = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(j+1)}, \widetilde{\mathbf{w}}^{(j+1)} \right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(j+1)}, \widetilde{\mathbf{w}}^{(j+1)} \right\rangle < 0, \end{cases} \tag{73}$$

and, for $(0 \le i,\ j \le n-1)$,

$$\left\langle \mathbf{E}\mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle = \delta_{ij}. \tag{74}$$

From Theorem 4, in matrix form, we obtain

$$\mathbf{A}\mathbf{W}_n = \mathbf{W}_n\mathbf{T}_n + \gamma_{n+1}\mathbf{w}^{(n+1)}\mathbf{e}_n^T, \tag{75}$$

where

$$\mathbf{W}_n = \left[\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}, \mathbf{w}^{(n)}\right]_{n\times n},$$
$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 1]_{1\times n},$$
$$\mathbf{T}_n = \begin{bmatrix} \alpha_1 & \beta_2 & & & & & \\ \gamma_2 & \alpha_2 & \beta_3 & & & & \\ & \gamma_3 & \alpha_3 & \beta_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{n-2} & \alpha_n & \beta_{n-1} & \\ & & & & \gamma_{n-1} & \alpha_{n-1} & \beta_n \\ & & & & & \gamma_n & \alpha_n \end{bmatrix}_{n\times n}. \tag{76}$$

Moreover, from Theorem 4, we obtain

$$(\mathbf{E}\mathbf{W}_n)^T\mathbf{W}_n = \begin{bmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & d_3 & & & & \\ & & & \ddots & & & \\ & & & & d_{n-2} & & \\ & & & & & d_{n-1} & \\ & & & & & & d_n \end{bmatrix}_{n\times n} \equiv \mathbf{D}_n. \tag{77}$$

Then, we have

$$\begin{aligned}(\mathbf{E}\mathbf{W}_n)^T\mathbf{A}\mathbf{W}_n &= (\mathbf{E}\mathbf{W}_n)^T\mathbf{W}_n\mathbf{T}_n \\ &\quad + (\mathbf{E}\mathbf{W}_n)^T\left(\gamma_{n+1}\mathbf{w}^{(n+1)}\mathbf{e}_n^T\right) \\ &= \mathbf{D}_n\mathbf{T}_n + \mathbf{0}.\end{aligned} \tag{78}$$

Here the second term on the right-hand side is the zero matrix!

In addition, we have

$$\begin{aligned}\mathbf{x}^{(n)} &= \mathbf{x}^{(0)} + \mathbf{W}_n\mathbf{y}^{(n)}, \\ \mathbf{r}^{(n)} &= \mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_n\mathbf{y}^{(n)}.\end{aligned} \tag{79}$$

Imposing the Galerkin condition, $\mathbf{r}^{(n)} \perp \mathcal{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$, as we did before, we obtain

$$\begin{aligned}\left(\mathbf{E}\mathbf{r}^{(n)}\right)^T\mathbf{W}_n &= \mathbf{0}, \\ (\mathbf{E}\mathbf{W}_n)^T\mathbf{r}^{(n)} &= \mathbf{0}, \\ (\mathbf{E}\mathbf{W}_n)^T\mathbf{r}^{(0)} &= (\mathbf{E}\mathbf{W}_n)^T\mathbf{A}\mathbf{W}_n\mathbf{y}^{(n)} \quad (\text{by } (79)), \\ (\mathbf{E}\mathbf{W}_n)^T\mathbf{r}^{(0)} &= \mathbf{D}_n\mathbf{T}_n\mathbf{y}^{(n)} \quad (\text{by } (78)).\end{aligned} \tag{80}$$

In other words, we use

$$\begin{aligned}(\mathbf{E}\mathbf{W}_n)^T\mathbf{A}\mathbf{W}_n &= \mathbf{D}_n\mathbf{T}_n, \\ (\mathbf{E}\mathbf{W}_n)^T\mathbf{W}_n &= \mathbf{D}_n.\end{aligned} \tag{81}$$

We obtain

$$(\mathbf{E}\mathbf{W}_n)^T\mathbf{r}^{(0)} = \beta_1\mathbf{e}_1 \tag{82}$$

because

$$\mathbf{D}_n\mathbf{T}_n\mathbf{y}^{(n)} = \beta_1\mathbf{e}_1, \quad \text{where } \beta_1 = \sqrt{\left|\left\langle\mathbf{E}\mathbf{r}^{(0)}, \mathbf{r}^{(0)}\right\rangle\right|}. \tag{83}$$

Here

$$\mathbf{D}_n\mathbf{T}_n = \begin{bmatrix} d_1\alpha_1 & d_1\beta_2 & & & & & \\ d_2\gamma_2 & d_2\alpha_2 & d_2\beta_3 & & & & \\ & d_3\gamma_3 & d_3\alpha_3 & d_3\alpha_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & d_{n-2}\gamma_{n-2} & d_{n-2}\alpha_{n-2} & d_{n-2}\beta_{n-1} & \\ & & & & d_{n-1}\gamma_{n-1} & d_{n-1}\alpha_{n-1} & d_{n-1}\beta_n \\ & & & & & d_n\gamma_n & d_n\alpha_n \end{bmatrix}_{n\times n}. \tag{84}$$

We note that $\mathbf{D}_n\mathbf{T}_n$ is symmetric, for $(1 \le i \le n-1)$:

$$\begin{aligned}d_i\beta_{i+1} &= d_i\frac{\left\langle\mathbf{E}\mathbf{A}\mathbf{w}^{(i+1)}, \mathbf{w}^{(i)}\right\rangle}{\left\langle\mathbf{E}\mathbf{w}^{(i)}, \mathbf{w}^{(i)}\right\rangle} \quad (\text{by } (72)) \\ &= \frac{d_i\left\langle\mathbf{w}^{(i+1)}, \mathbf{E}\mathbf{A}\mathbf{w}^{(i)}\right\rangle}{\left\langle\mathbf{E}\mathbf{w}^{(i)}, \mathbf{w}^{(i)}\right\rangle} \quad \left((\mathbf{E}\mathbf{A})^T = \mathbf{E}\mathbf{A}\right) \\ &= d_i\frac{\left\langle\mathbf{w}^{(i+1)}, \mathbf{E}\left(\gamma_{i+1}\mathbf{w}^{(i+1)} + \alpha_i\mathbf{w}^{(i)} + \beta_i\mathbf{w}^{(i-1)}\right)\right\rangle}{\left\langle\mathbf{E}\mathbf{w}^{(i)}, \mathbf{w}^{(i)}\right\rangle} \\ &\qquad (\text{by } (71)) \\ &= d_{i+1}\gamma_{j+1}.\end{aligned} \tag{85}$$

7. Lanczos/MSYMMLQ Method

Next, we consider this $2n \times 2n$ double linear system:

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \hat{\mathbf{b}} \end{bmatrix}. \quad (86)$$

We obtain the block symmetric matrices $\mathscr{A}$, $\mathscr{E}$, and $\mathscr{E}\mathscr{A}$, where

$$\mathscr{A} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T \end{bmatrix}, \quad \mathscr{E} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathscr{E}\mathscr{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}. \quad (87)$$

For example, the modified SYMMLQ method and the modified SYMMQR method can be applied to the double linear system (86). This leads us to the LAN/MSYMMLQ method and the LAN/MSYMMQR method. The pseudocodes for these methods are given in the following sections. For additional details, see Li [3]. See the books by Golub and Van Loan [4] and Saad [5], as well as the papers by Lanczos [6] and Kincaid et al. [7], among others.

8. MSYMMLQ Pseudocode

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)},$$

$$\beta_1 = \sqrt{\left|\mathbf{E}\mathbf{r}^{(0)}, \mathbf{r}^{(0)}\right|},$$

$$\mathbf{w}^{(1)} = \left(\frac{1}{\beta_1}\right)\mathbf{r}^{(0)},$$

$$d_1 = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{E}\mathbf{r}^{(0)} \right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{E}\mathbf{r}^{(0)} \right\rangle < 0, \end{cases}$$

$$\varepsilon_1 = \varepsilon_2 = 0, \quad s_0 = 0, \quad c_0 = -1,$$

$$\alpha_1 = \left(\frac{1}{d_1}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(1)}, \mathbf{w}^{(1)} \right\rangle,$$

$$\widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1 \mathbf{w}^{(1)},$$

$$\beta_2 = \sqrt{\left|\left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle\right|},$$

$$\mathbf{w}^{(2)} = \left(\frac{1}{\beta_2}\right)\widetilde{\mathbf{w}}^{(2)},$$

$$d_2 = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle < 0, \end{cases}$$

$$\hat{\zeta}_1 = \frac{\beta_1}{\alpha_1},$$

$$\hat{\mathbf{v}}^{(1)} = \mathbf{w}^{(1)},$$

$$\hat{\mathbf{x}}^{(1)} = \mathbf{x}^{(0)} + \hat{\zeta}_1 \mathbf{v}^{(1)},$$

$$c_1 = \frac{d_1\alpha_1}{\sqrt{\alpha_1^2 + \beta_2^2}}, \quad s_1 = \frac{d_1\beta_2}{\sqrt{\alpha_1^2 + \beta_2^2}},$$

$$\gamma_1 = (d_1\alpha_1)\, c_1 + (d_1\beta_2)\, s_1,$$

$$\zeta_1 = \left(\frac{\beta_1}{\gamma_1}\right),$$

$$\mathbf{v}^{(1)} = c_i \hat{\mathbf{v}}^{(1)} + s_1 \mathbf{w}^{(2)},$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \zeta_1 \mathbf{v}^{(1)}, \quad (88)$$

for $i = 2, 3, \ldots, N$,

$$\alpha_i = \left(\frac{1}{d_i}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(i)}, \mathbf{w}^{(i)} \right\rangle,$$

$$\beta_i = \left(\frac{1}{d_{i-1}}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(i)}, \mathbf{w}^{(i-1)} \right\rangle,$$

$$\widetilde{\mathbf{w}}^{(i+1)} = \mathbf{A}\mathbf{w}^{(i)} - \alpha_i \mathbf{w}^{(i)} - \beta_i \mathbf{w}^{(i-1)},$$

$$\beta_{i+1} = \sqrt{\left|\left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)} \right\rangle\right|},$$

$$\mathbf{w}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right)\widetilde{\mathbf{w}}^{(i+1)},$$

$$d_{i+1} = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)} \right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)} \right\rangle < 0, \end{cases}$$

$$\varepsilon_i = (d_{i-1}\beta_i)\, s_{i-2}, \quad \delta_i = -(d_{i-1}\beta_i)\, c_{i-2}, \quad hh = \delta_i,$$

$$\delta_i = (hh)\, c_{i-1} + (d_i\alpha_i)\, s_{i-1},$$

$$\hat{\gamma}_i = (hh)\, s_{i-1} - (d_i\alpha_i)\, c_{i-1},$$

$$\hat{\zeta}_i = \left(\frac{1}{\hat{\gamma}_i}\right)(-\varepsilon_i\zeta_{i-2} - \delta_i\zeta_{i-1}),$$

$$\hat{\mathbf{v}}^{(i)} = s_{i-1}\hat{\mathbf{v}}^{(i-1)} - c_{i-1}\mathbf{w}^{(i)},$$

$$\hat{\mathbf{x}}^{(i)} = \mathbf{x}^{(i-1)} + \hat{\zeta}_i \hat{\mathbf{v}}^{(i)},$$

$$\gamma_i = \sqrt{\alpha_i^2 + \beta_{i+1}^2},$$

$$c_i = d_i\left(\frac{\alpha_i}{\gamma_i}\right), \quad s_i = d_i\left(\frac{\beta_{i+1}}{\gamma_i}\right),$$

$$\zeta_i = \left(\frac{1}{\gamma_1}\right)(-\varepsilon_i\zeta_{i-2} - \delta_i\zeta_{i-1}),$$

$$\mathbf{v}^{(i)} = c_1 \hat{\mathbf{v}}^{(i)} + s_i \mathbf{w}^{(i+1)},$$

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i-1)} + \zeta_i \mathbf{v}^{(i)}$$

end for (89)

9. MSYMMQR Pseudocode

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)},$$
$$\beta_1 = \sqrt{|\mathbf{E}\mathbf{r}^{(0)}, \mathbf{r}^{(0)}|},$$
$$\mathbf{w}^{(1)} = \left(\frac{1}{\beta_1}\right)\mathbf{r}^{(0)},$$
$$d_1 = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{E}\mathbf{r}^{(0)}\right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{E}\mathbf{r}^{(0)}\right\rangle < 0, \end{cases}$$
$$\varepsilon_1 = \varepsilon_2 = 0, \quad s_0 = 0, \quad c_0 = 1,$$
$$\alpha_1 = \left(\frac{1}{d_1}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(1)}, \mathbf{w}^{(1)}\right\rangle,$$
$$\widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1 \mathbf{w}^{(1)},$$
$$\beta_2 = \sqrt{|\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)}\rangle|},$$
$$\mathbf{w}^{(2)} = \left(\frac{1}{\beta_2}\right)\widetilde{\mathbf{w}}^{(2)},$$
$$d_2 = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)}\right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(2)}, \widetilde{\mathbf{w}}^{(2)}\right\rangle < 0, \end{cases}$$
$$\widehat{\zeta}_1 = \beta_1 d_1,$$
$$\widehat{\mathbf{p}}^{(1)} = \left(\frac{1}{\alpha_1 d_1}\right)\mathbf{w}^{(1)},$$
$$\widehat{\mathbf{x}}^{(1)} = \mathbf{x}^{(0)} + \widehat{\zeta}_1 \widehat{\mathbf{p}}^{(1)},$$
$$c_1 = \frac{d_1\alpha_1}{\sqrt{\alpha_1^2 + \beta_2^2}}, \quad s_1 = -\frac{d_1\beta_2}{\sqrt{\alpha_1^2 + \beta_2^2}},$$
$$\gamma_1 = (d_1\alpha_1)c_1 - (d_1\beta_2)s_1, \quad \zeta_1 = c_1\widehat{\zeta}_1,$$
$$\mathbf{p}^{(1)} = \left(\frac{1}{\gamma_1}\right)\mathbf{w}^{(1)},$$
$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \zeta_1 \mathbf{p}^{(1)}, \tag{90}$$

for $i = 2, 3, \ldots, N$,

$$\alpha_i = \left(\frac{1}{d_i}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(i)}, \mathbf{w}^{(i)}\right\rangle,$$
$$\beta_i = \left(\frac{1}{d_{i-1}}\right)\left\langle \mathbf{E}\mathbf{A}\mathbf{w}^{(i)}, \mathbf{w}^{(i-1)}\right\rangle,$$
$$\widetilde{\mathbf{w}}^{(i+1)} = \mathbf{A}\mathbf{w}^{(i)} - \alpha_i \mathbf{w}^{(i)} - \beta_i \mathbf{w}^{(i-1)},$$
$$\beta_{i+1} = \sqrt{|\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)}\rangle|},$$
$$\mathbf{w}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right)\widetilde{\mathbf{w}}^{(i+1)},$$
$$d_{i+1} = \begin{cases} 1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)}\right\rangle > 0, \\ -1, & \text{if } \left\langle \mathbf{E}\widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{i+1}\right\rangle < 0, \end{cases}$$
$$\varepsilon_i = -(d_{i-1}\beta_i)s_{i-2}, \quad \delta_i = -(d_{i-1}\beta_i)c_{i-2},$$
$$hh = \delta_i,$$
$$\delta_i = (hh)c_{i-1} - (d_i\alpha_i)s_{i-1},$$
$$\widehat{\gamma}_i = (hh)s_{i-1} + (d_i\alpha_i)c_{i-1}, \quad \widehat{\zeta}_i = s_{i-1}\widehat{\zeta}_{i-1},$$
$$\widehat{\mathbf{p}}^{(i)} = \left(\frac{1}{\widehat{\gamma}_i}\right)\left[\mathbf{w}^{(i)} - \varepsilon_i \mathbf{p}^{(i-2)} - \delta_i \mathbf{p}^{(i-1)}\right],$$
$$\widehat{\mathbf{x}}^{(i)} = \mathbf{x}^{(i-1)} + \widehat{\zeta}_i \widehat{\mathbf{p}}^{(i)},$$
$$\gamma_i = \sqrt{\widehat{\zeta}_i^2 + \beta_{i+1}^2},$$
$$c_i = \frac{\widehat{\gamma}_i}{\gamma_i}, \quad s_i = -\left(\frac{1}{\gamma_i}\right)(d_i\beta_{i+1}), \quad \zeta_i = c_{i-1}\widehat{\zeta}_i,$$
$$\mathbf{p}^{(i)} = \left(\frac{1}{\gamma_i}\right)\left[\mathbf{w}^{(i)} - \varepsilon_i \mathbf{p}^{(i-2)} - \delta_i \mathbf{p}^{(i-1)}\right],$$
$$\mathbf{x}^{(i)} = \mathbf{x}^{(i-1)} + \zeta_i \mathbf{p}^{(i)}$$

end for

(91)

10. LAN/MSYMMLQ Pseudocode

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)},$$
$$\widehat{\mathbf{r}}^{(0)} = \widehat{\mathbf{b}} - \mathbf{A}^T\mathbf{x}^{(0)},$$
$$\beta_1 = \sqrt{|2\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)}\rangle|},$$
$$\mathbf{w}^{(1)} = \left(\frac{1}{\beta_1}\right)\mathbf{r}^{(0)},$$
$$\widehat{\mathbf{w}}^{(1)} = \left(\frac{1}{\beta_1}\right)\widehat{\mathbf{r}}^{(0)},$$
$$d_1 = \begin{cases} 1, & \text{if } 2\left\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)}\right\rangle > 0, \\ -1, & \text{if } 2\left\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)}\right\rangle < 0, \end{cases}$$
$$\varepsilon_1 = \varepsilon_2 = 0, \quad s_0 = 0, \quad c_0 = -1,$$
$$\alpha_1 = \left(\frac{2}{d_1}\right)\left\langle \widehat{\mathbf{w}}^{(1)}, \mathbf{w}^{(1)}\right\rangle,$$
$$\widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1 \mathbf{w}^{(1)},$$
$$\widetilde{\widehat{\mathbf{w}}}^{(2)} = \mathbf{A}^T\mathbf{w}^{(1)} - \alpha_1 \widehat{\mathbf{w}}^{(1)},$$
$$\beta_2 = \sqrt{\left|2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)}\right\rangle\right|},$$

$$\mathbf{w}_2 = \left(\frac{1}{\beta_2}\right)\widetilde{\mathbf{w}}^{(2)},$$

$$\widehat{\mathbf{w}}^{(2)} = \left(\frac{1}{\beta_2}\right)\widetilde{\widehat{\mathbf{w}}}^{(2)},$$

$$d_2 = \begin{cases} 1, & \text{if } 2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle > 0, \\ -1, & \text{if } 2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle < 0, \end{cases},$$

$$\widehat{\zeta}_1 = \frac{\beta_1}{\alpha_1},$$

$$\widehat{\mathbf{v}}^{(1)} = \mathbf{w}^{(1)},$$

$$\widehat{\mathbf{x}}^{(1)} = \mathbf{x}^{(0)} + \widehat{\zeta}_1 \widehat{\mathbf{v}}^{(1)},$$

$$c_1 = \frac{d_1\alpha_1}{\sqrt{\alpha_1^2 + \beta_2^2}}, \quad s_1 = \frac{d_1\beta_2}{\sqrt{\alpha_1^2 + \beta_2^2}},$$

$$\gamma_1 = (d_1\alpha_1)\, c_1 + (d_1\beta_2)\, s_1, \quad \zeta_1 = \frac{\beta_1}{\gamma_1},$$

$$\mathbf{v}^{(1)} = c_1\widehat{\mathbf{v}}^{(1)} + s_1\mathbf{w}^{(2)},$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \zeta_1\mathbf{v}^{(1)},$$

(92)

for $i = 2, 3, \ldots, N,$

$$\alpha_i = \left(\frac{2}{d_i}\right)\left\langle \mathbf{A}\mathbf{w}^{(i)}, \widehat{\mathbf{w}}^{(i)} \right\rangle,$$

$$\beta_i = \left(\frac{1}{d_{i-1}}\right)\left[\left\langle \mathbf{A}\mathbf{w}^{(i)}, \widehat{\mathbf{w}}^{(i-1)} \right\rangle + \left\langle \widehat{\mathbf{w}}^{(i)}, \mathbf{A}\mathbf{w}^{(i-1)} \right\rangle\right],$$

$$\widetilde{\mathbf{w}}^{(i+1)} = \mathbf{A}\mathbf{w}^{(i)} - \alpha_i\mathbf{w}^{(i)} - \beta_i\mathbf{w}^{(i-1)},$$

$$\widetilde{\widehat{\mathbf{w}}}^{(i+1)} = \mathbf{A}^T\widehat{\mathbf{w}}^{(i)} - \alpha_i\widehat{\mathbf{w}}^{(i)} - \beta_i\widehat{\mathbf{w}}^{(i-1)},$$

$$\beta_{i+1} = \sqrt{\left|2\left\langle \widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)} \right\rangle\right|},$$

$$\mathbf{w}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right)\widetilde{\mathbf{w}}^{(i+1)},$$

$$\widehat{\mathbf{w}}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right)\widetilde{\widehat{\mathbf{w}}}^{(i+1)},$$

$$d_{i+1} = \begin{cases} 1, & \text{if } 2\left\langle \widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}^{(i+1)} \right\rangle > 0, \\ -1, & \text{if } 2\left\langle \widetilde{\mathbf{w}}^{(i+1)}, \widetilde{\mathbf{w}}_{i+1} \right\rangle < 0, \end{cases}$$

$$\varepsilon_i = d_{i-1}\beta_i s_{i-2}, \quad \delta_i = -(d_{i-1}\beta_i)\, c_{i-2}, \quad hh = \delta_i,$$

$$\delta_i = (hh)\, c_{i-1} + (d_i\alpha_i)\, s_{i-1},$$

$$\widehat{\gamma}_i = (hh)\, s_{i-1} - (d_i\alpha_i)\, c_{i-1},$$

$$\widehat{\zeta}_i = \left(\frac{1}{\widehat{\gamma}_i}\right)(-\varepsilon_i\zeta_{i-2} - \delta_i\zeta_{i-1}),$$

$$\widehat{\mathbf{v}}^{(i)} = s_{i-1}\widehat{\mathbf{v}}^{(i-1)} - c_{i-1}\mathbf{w}^{(i)},$$

$$\widehat{\mathbf{x}}^{(i)} = \mathbf{x}^{(i-1)} + \widehat{\zeta}_i\widehat{\mathbf{v}}^{(i)},$$

$$\gamma_i = \sqrt{\alpha_i^2 + \beta_{i+1}^2}, \quad c_i = d_i\left(\frac{\alpha_i}{\gamma_i}\right), \quad s_i = d_i\left(\frac{\beta_{i+1}}{\gamma_i}\right),$$

$$\zeta_i = \left(\frac{1}{\gamma_i}\right)(-\varepsilon_i\zeta_{i-2} - \delta_i\zeta_{i-1}),$$

$$\mathbf{v}^{(i)} = c_i\widehat{\mathbf{v}}^{(i)} + s_i\mathbf{w}^{(i+1)},$$

$$\mathbf{x}^{(i)} = \mathbf{x}^{(i-1)} + \zeta_i\mathbf{v}^{(i)}$$

end for

(93)

11. LAN/MSYMMQR Pseudocode

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)},$$

$$\widehat{\mathbf{r}}^{(0)} = \widehat{\mathbf{b}} - \mathbf{A}^T\mathbf{x}^{(0)},$$

$$\beta_1 = \sqrt{\left|2\left\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)} \right\rangle\right|},$$

$$\mathbf{w}^{(1)} = \left(\frac{1}{\beta_1}\right)\mathbf{r}^{(0)},$$

$$\widehat{\mathbf{w}}^{(1)} = \left(\frac{1}{\beta_1}\right)\widehat{\mathbf{r}}^{(0)},$$

$$d_1 = \begin{cases} 1, & \text{if } 2\left\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)} \right\rangle > 0, \\ -1, & \text{if } 2\left\langle \widehat{\mathbf{r}}^{(0)}, \mathbf{r}^{(0)} \right\rangle < 0, \end{cases}$$

$$\varepsilon_1 = \varepsilon_2 = 0, \quad s_0 = 0, \quad c_0 = 1,$$

$$\alpha_1 = \left(\frac{2}{d_1}\right)\left\langle \widehat{\mathbf{w}}^{(1)}, \mathbf{w}^{(1)} \right\rangle,$$

$$\widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1\mathbf{w}^{(1)},$$

$$\widetilde{\widehat{\mathbf{w}}}^{(2)} = \mathbf{A}^T\mathbf{w}^{(1)} - \alpha_1\widehat{\mathbf{w}}^{(2)},$$

$$\beta_2 = \sqrt{\left|2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle\right|},$$

$$\mathbf{w}^{(2)} = \left(\frac{1}{\beta_2}\right)\widetilde{\mathbf{w}}^{(2)},$$

$$\widehat{\mathbf{w}}^{(2)} = \left(\frac{1}{\beta_2}\right)\widetilde{\widehat{\mathbf{w}}}^{(2)},$$

$$d_2 = \begin{cases} 1, & \text{if } 2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle > 0 \\ -1, & \text{if } 2\left\langle \widetilde{\widehat{\mathbf{w}}}^{(2)}, \widetilde{\mathbf{w}}^{(2)} \right\rangle < 0 \end{cases},$$

$$\widehat{\mathbf{p}}^{(1)} = \left(\frac{1}{\alpha_i d_i}\right)\mathbf{w}^{(1)},$$

$$
\begin{aligned}
&\hat{\zeta}_1 = \frac{\beta_1}{\alpha_1}, \\
&\hat{\mathbf{x}}^{(1)} = \mathbf{x}^{(0)} + \hat{\zeta}_1 \hat{\mathbf{p}}^{(1)}, \\
&c_1 = \frac{d_1\alpha_1}{\sqrt{\alpha_1^2 + \beta_2^2}}, \qquad s_1 = -\frac{d_1\beta_2}{\sqrt{\alpha_1^2 + \beta_2^2}}, \\
&\gamma_1 = (d_1\alpha_1)\, c_1 - (d_1\beta_2)\, s_1, \\
&\zeta_1 = \beta_1 \hat{\gamma}_1, \\
&\mathbf{p}^{(1)} = \left(\frac{1}{\gamma_1}\right) \mathbf{w}^{(1)}, \\
&\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \zeta_1 \mathbf{p}^{(1)},
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
&\textbf{for} \quad i = 2, 3, \ldots, N, \\
&\qquad \alpha_i = \left(\frac{2}{d_i}\right) \left\langle \mathbf{A}\mathbf{w}^{(i)}, \hat{\mathbf{w}}^{(i)} \right\rangle, \\
&\qquad \beta_i = \left(\frac{1}{d_{i-1}}\right) \left[\left\langle \mathbf{A}\mathbf{w}^{(i)}, \hat{\mathbf{w}}^{(i-1)} \right\rangle + \left\langle \hat{\mathbf{w}}^{(i)}, \mathbf{A}\mathbf{w}^{(i-1)} \right\rangle \right], \\
&\qquad \tilde{\mathbf{w}}^{(i+1)} = \mathbf{A}\mathbf{w}^{(i)} - \alpha_i \mathbf{w}^{(i)} - \beta_i \mathbf{w}^{(i-1)}, \\
&\qquad \tilde{\tilde{\mathbf{w}}}^{(i+1)} = \mathbf{A}^T \hat{\mathbf{w}}^{(i)} - \alpha_i \hat{\mathbf{w}}^{(i)} - \beta_i \hat{\mathbf{w}}^{(i-1)}, \\
&\qquad \beta_{i+1} = \sqrt{\left| 2 \left\langle \tilde{\mathbf{w}}^{(i+1)}, \tilde{\mathbf{w}}^{(i+1)} \right\rangle \right|}, \\
&\qquad \mathbf{w}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right) \tilde{\mathbf{w}}^{(i+1)}, \\
&\qquad \hat{\mathbf{w}}^{(i+1)} = \left(\frac{1}{\beta_{i+1}}\right) \tilde{\tilde{\mathbf{w}}}^{(i+1)}, \\
&\qquad d_{i+1} = \begin{cases} 1, & \text{if } 2\left\langle \tilde{\mathbf{w}}^{(i+1)}, \tilde{\mathbf{w}}^{(i+1)} \right\rangle > 0, \\ -1, & \text{if } 2\left\langle \tilde{\mathbf{w}}^{(i+1)}, \tilde{\mathbf{w}}^{(i+1)} \right\rangle < 0, \end{cases} \\
&\qquad \varepsilon_i = (d_{i-1}\beta_i)\, s_{i-2}, \quad \delta_i = (d_{i-1}\beta_i)\, c_{i-2}, \quad hh = \delta_i, \\
&\qquad \delta_i = (hh)\, c_{i-1} - d_i (\alpha_i s_{i-1}), \\
&\qquad \hat{\gamma}_i = (hh)\, sc_{i-1} + d_i (\alpha_i c_{i-1}), \\
&\qquad \hat{\zeta}_i = \hat{\hat{\zeta}}_{i-1} s_{i-1}, \\
&\qquad \hat{\mathbf{p}}^{(i)} = \left(\frac{1}{\hat{\gamma}_i}\right) \left[\mathbf{w}^{(i)} - \varepsilon_i \mathbf{p}^{(i)} - \delta_i \mathbf{p}^{(i-1)} \right], \\
&\qquad \hat{\mathbf{x}}^{(i)} = \mathbf{x}^{(i-1)} + \hat{\zeta}_i \hat{\mathbf{p}}^{(i)}, \\
&\qquad \gamma_i = \sqrt{\hat{\gamma}^2 + \beta_{i+1}^2}, \\
&\qquad c_i = \left(\frac{\hat{\gamma}_i}{\gamma_i}\right), \qquad s_i = -d_i \left(\frac{\beta_{i+1}}{\gamma_i}\right), \\
&\qquad \zeta_i = c_i \hat{\zeta}_i, \\
&\qquad \mathbf{p}^{(i)} = \left(\frac{1}{\gamma_i}\right) \left[\mathbf{w}^{(i)} - \varepsilon_i \mathbf{p}^{(i-2)} - \delta_i \mathbf{p}^{(i-1)} \right], \\
&\qquad \mathbf{x}^{(i)} = \mathbf{x}^{(i-1)} + \zeta_i \mathbf{p}^{(i)} \\
&\textbf{end for}
\end{aligned}
\tag{95}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] C. C. Paige and M. A. Saunders, "Solutions of sparse indefinite systems of linear equations," *SIAM Journal on Numerical Analysis*, vol. 12, no. 4, pp. 617–629, 1975.

[2] Y. Saad and M. H. Schultz, "GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems," *SIAM Journal on Scientific and Statistical Computing*, vol. 7, no. 3, pp. 856–869, 1986.

[3] Y. Li, *The Modified MINRES methods for solving large sparse nonsymmetric linear systems [M.S. thesis]*, Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, Taiwan, 2007.

[4] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, Md, USA, 3rd edition, 1996.

[5] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, 2nd edition, 2003.

[6] C. Lanczos, "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," *Journal of Research of the National Bureau of Standards*, vol. 45, no. 4, pp. 255–282, 1950.

[7] D. R. Kincaid, D. M. Young, and J.-Y. Chen, "An overview of MGMRES and LAN/MGMRES methods for solving nonsymmetric linear systems," *Taiwanese Journal of Mathematics*, vol. 4, no. 3, pp. 385–396, 2000.