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Research Article

An Illusion: "A Suzuki Type Coupled Fixed Point Theorem"

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We admonish to be careful on studying coupled fixed point theorems since most of the reported fixed point results can be easily derived from the existing corresponding theorems in the literature. In particular, we notice that the recent paper [Semwal and Dimri (2014)] has gaps and the announced result is false. The authors claimed that their result generalized the main result in [Đoric and Lazović (2011)] but, in fact, the contrary case is true. Finally, we present a fixed point theorem for Suzuki type (α, r)-admissible contractions.

1. Introduction and Preliminaries

Throughout this note, we follow the notions and notations given in [1, 2]. Let $\phi : [0,1) \to (0,1]$ be the mapping defined, for all $r \in [0,1)$, by

$$\phi(r) = \begin{cases} 1, & \text{if } 0 \le r < \dfrac{1}{2}, \\ 1 - r, & \text{if } \dfrac{1}{2} \le r < 1. \end{cases} \tag{1}$$

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ (or by $\mathrm{CB}(X, d)$ when it is convenient to clarify the involved metric) the class of all nonempty closed and bounded subsets of (X, d). For every $A, B \in \mathrm{CB}(X)$, let

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}, \tag{2}$$

where $d(a, B) = \inf_{y \in B} d(x, y)$ for all $a \in X$ and all $B \subseteq X$. It is well known that H is a metric on $\mathrm{CB}(X)$.

Very recently, Semwal and Dimri [1] announced the following result.

Theorem 1 (Semwal and Dimri [1], Theorem 2.1). *Let (X, d) be a complete metric space and let T be mapping from $X \times X$ into $\mathrm{CB}(X)$. Assume that there exists $r \in [0,1)$ such that*

$$\phi(r)\left(d(x, T(x, y)) + d(u, T(u, v))\right) \le d(x, u) + d(y, v) \tag{3}$$

implies

$$\begin{aligned} H&(T(x, y), T(u, v)) \\ &\le \frac{r}{2} \max\Bigg(d(x, u) + d(y, v), \\ &\qquad d(x, T(x, y)) + d(y, T(y, x)), \\ &\qquad d(u, T(u, v)) + d(v, T(v, u)), \\ &\qquad \frac{d(x, T(u, v)) + d(y, T(v, u))}{2}, \\ &\qquad \frac{d(u, T(x, y)) + d(v, T(y, x))}{2} \Bigg), \end{aligned} \tag{4}$$

for all $x, y, u, v \in X$. Then there exist $z, w \in X$ such that $z \in T(z, w)$ and $w \in T(w, z)$.

Semwal and Dimri [1] claimed that it was a generalization of Đoric and Lazović's recent result (see [2]), which is the following theorem.

Theorem 2 (Đoric and Lazović [2], Theorem 2.1). *Let (X, d) be a complete metric space and let T be mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$\phi(r)\, d(x, Tx) \le d(x, y) \tag{5}$$

implies

$$H(Tx, Ty) \le r \max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2} \right), \tag{6}$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.

This note is devoted to the following three aims. (1) We will show that the proof of the main result of Semwal and Dimri [1] is incorrect; in fact, it is possible to fix the glitch of the given proof in [1]. (2) By modifying its contractivity condition, we obtain a correct version of Theorem 1 but, in such a case, we realize that the obtained result is a simple consequence of Theorem 2. (3) Finally, we present a generalization of Theorem 2 for Suzuki type α-admissible contractions.

2. Main Gaps

Let us review the lines of their proof. First at all, notice that, in general, if (X, d) is a metric space, we know that, for all A, B, $C \in \mathrm{CB}(X)$, all $a \in A$ and all $b \in B$:

$$\begin{gathered} d(a, B) \le d(a, b), \qquad d(a, B) \le H(A, B), \\ d(a, C) \le d(a, B) + H(B, C). \end{gathered} \tag{7}$$

The contrary inequalities can be false.

The authors took $x_1, y_1 \in X$ arbitrarily and, later, they chose

$$\begin{gathered} x_2 \in F(x_1, y_1), \qquad y_2 \in F(y_1, x_1), \\ x_3 \in F(x_2, y_2), \qquad y_3 \in F(y_2, x_2). \end{gathered} \tag{8}$$

Taking into account that $\phi(r) \le 1$ and using (7), the authors wrote the following (see [1], page 3, line 17):

$$\begin{aligned} \phi(r)&\left(d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2))\right) \\ &\le d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2)) \\ &\le d(x_2, x_3) + d(y_2, y_3). \end{aligned} \tag{9}$$

However, this inequality is not strong enough to apply the contractivity condition given in Theorem 1 because in the antecedent condition

$$\phi(r)\left(d(x, F(x, y)) + d(u, F(u, v))\right) \le d(x, u) + d(y, v) \tag{10}$$

cannot appear the points $u = x_3$ and $v = y_3$ in the second member if they are not in the first member. Therefore, the contractivity condition that we found in Theorem 1 is not applicable.

Furthermore, assume that we would have been able to apply the mentioned contractivity condition. In this case, the authors wrote the following (see [1], page 3, lines 19–22):

$$\begin{aligned} d&(x_2, T(x_2, y_2)) \\ &\le H(T(x_1, y_1), T(x_2, y_2)) \\ &\le \frac{r}{2} \max\Bigg(d(x_1, x_2) + d(y_1, y_2), \\ &\qquad d(x_1, T(x_1, y_1)) + d(y_1, T(y_1, x_1)), \\ &\qquad d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2)), \\ &\qquad \frac{d(x_1, T(x_2, y_2)) + d(y_1, T(y_2, x_2))}{2}, \\ &\qquad \frac{d(x_2, T(x_1, y_1)) + d(y_2, T(y_1, x_1))}{2} \Bigg). \end{aligned} \tag{11}$$

Immediately, they deduced that

$$\begin{aligned} d&(x_2, x_3) \\ &\le \frac{r}{2} \max\Bigg(d(x_1, x_2) + d(y_1, y_2),\ d(x_1, x_2) + d(y_1, y_2), \\ &\qquad d(x_2, x_3) + d(y_2, y_3), \frac{d(x_1, x_3) + d(y_1, y_3)}{2}, \\ &\qquad \frac{d(x_2, x_2) + d(y_2, y_2)}{2} \Bigg). \end{aligned} \tag{12}$$

However, this inequality is based on $d(x_2, x_3) \le H(T(x_1, y_1), T(x_2, y_2))$ which, in general, is false. Then, this argument is not correct.

The same mistake occurred when the authors tried to upper bound the terms $d(y_2, T(y_2, x_2))$ and $d(y_2, y_3)$ (see [1], page 4).

3. A Correct Version of Theorem 1

If we want to modify the contractivity condition given in Theorem 1 in order that (9) can be applied, then antecedent condition (3) must be replaced by the following one:

$$\phi(r)\left(d(x, T(x, y)) + d(y, T(y, x))\right) \le d(x, u) + d(y, v), \tag{13}$$

for all $x, y, u, v \in X$. In such a case, we obtain the following result.

Theorem 3. *Let (X, d) be a complete metric space and let $F : X \times X \to CB(X)$ be a mapping. Assume that there exists $r \in [0, 1)$ such that*

$$\phi(r)\left(d\left(x, T\left(x, y\right)\right) + d\left(y, T\left(y, x\right)\right)\right) \leq d\left(x, u\right) + d\left(y, v\right) \tag{14}$$

implies

$$\begin{aligned} H&\left(F\left(x, y\right), F\left(u, v\right)\right) \\ &\leq \frac{r}{2}\max\Big(d\left(x, u\right) + d\left(y, v\right), \\ &\qquad d\left(x, F\left(x, y\right)\right) + d\left(y, F\left(y, x\right)\right), \\ &\qquad d\left(u, F\left(u, v\right)\right) + d\left(v, F\left(v, u\right)\right), \\ &\qquad \left(d\left(x, F\left(u, v\right)\right) + d\left(y, F\left(v, u\right)\right)\right. \\ &\qquad \left. + d\left(u, F\left(x, y\right)\right) + d\left(v, F\left(y, x\right)\right)\right) \\ &\qquad \times (2)^{-1}\Big), \end{aligned} \tag{15}$$

for all x, y, u, $v \in X$. Then there exist $z, w \in X$ such that $z \in F(z, w)$ and $w \in F(w, z)$.

However, we claim that this result is not a proper generalization of Theorem 2, but it is an immediate consequence of such theorem. To prove it, we need some preliminaries.

Lemma 4 (see, e.g., [3, 4]). *Given a metric d on X, define $d_2^s : X^2 \times X^2 \to [0, \infty)$, for all $(x, y), (u, v) \in X^2$, by*

$$d_2^s\left(\left(x, y\right), \left(u, v\right)\right) = d\left(x, u\right) + d\left(y, v\right). \tag{16}$$

Then d_2^s is a metric on X^2. In addition to this, if d is complete and then d_2^s is also complete.

Remark 5. Notice that $CB(X, d)^2 \subseteq CB(X^2, d_2^s)$.

Given a mapping $F : X^2 \to CB(X)$, denote by $T_F^2 : X^2 \to CB(X, d)^2 \subseteq CB(X^2, d_2^s)$ the mapping

$$T_F^2\left(x, y\right) = \left(F\left(x, y\right), F\left(y, x\right)\right) \quad \forall \left(x, y\right) \in X^2. \tag{17}$$

If there exists a point $(z, w) \in X^2$ such that $(z, w) \in T_F^2(z, w) = (F(z, w), F(w, z))$, then there exist two points $z, w \in X$ such that $z \in F(z, w)$ and $w \in F(w, z)$. This is precisely the thesis of Theorem 3. Therefore, we only have to prove that T_F^2 has a fixed point (z, w).

Notice that antecedent condition (14) can be written as

$$\phi\left(r\right) d_2^s\left(\left(x, y\right), T_F^2\left(x, y\right)\right) \leq d_2^s\left(\left(x, y\right), \left(u, v\right)\right). \tag{18}$$

Moreover, the second member in (15) is

$$\begin{aligned} &\frac{r}{2}\max\Big(d\left(x, u\right) + d\left(y, v\right), d\left(x, F\left(x, y\right)\right) + d\left(y, F\left(y, x\right)\right), \\ &\qquad d\left(u, F\left(u, v\right)\right) + d\left(v, F\left(v, u\right)\right), \\ &\qquad \left(d\left(x, F\left(u, v\right)\right) + d\left(y, F\left(v, u\right)\right)\right. \\ &\qquad \left. + d\left(u, F\left(x, y\right)\right) + d\left(v, F\left(y, x\right)\right)\right) \\ &\qquad \times (2)^{-1}\Big) \\ &= \frac{r}{2}\max\Bigg(d_2^s\left(\left(x, y\right), \left(u, v\right)\right), d_2^s\left(\left(x, y\right), T_F^2\left(x, y\right)\right), \\ &\qquad d_2^s\left(\left(u, v\right), T_F^2\left(u, v\right)\right), \\ &\qquad \frac{d_2^s\left(\left(x, y\right), T_F^2\left(u, v\right)\right) + d_2^s\left(\left(u, v\right), T_F^2\left(x, y\right)\right)}{2}\Bigg). \end{aligned} \tag{19}$$

Notice that, associated to the metric d_2^s, we can also consider H_2^s given, for all $(A, B), (C, E) \in CB(X, d)^2 \subseteq CB(X^2, d_2^s)$, by

$$\begin{aligned} H_2^s&\left(\left(A, B\right), \left(C, E\right)\right) \\ &= \max\Bigg(\sup_{(c,e)\in(C,E)} d_2^s\left(\left(c, e\right), \left(A, B\right)\right), \\ &\qquad \sup_{(a,b)\in(A,B)} d_2^s\left(\left(a, b\right), \left(C, E\right)\right)\Bigg) \\ &= \max\Bigg(\sup_{(c,e)\in(C,E)} \left[d\left(c, A\right) + d\left(e, B\right)\right], \\ &\qquad \sup_{(a,b)\in(A,B)} \left[d\left(a, C\right) + d\left(b, E\right)\right]\Bigg) \\ &= \max\Bigg(\sup_{c\in C} d\left(c, A\right) + \sup_{e\in E} d\left(e, B\right), \\ &\qquad \sup_{a\in A} d\left(a, C\right) + \sup_{b\in B} d\left(b, E\right)\Bigg). \end{aligned} \tag{20}$$

In such a case,

$$\begin{aligned} &H\left(A, C\right) + H\left(B, E\right) \\ &= \max\left(\sup_{c\in C} d\left(c, A\right), \sup_{a\in A} d\left(a, C\right)\right) \\ &\quad + \max\left(\sup_{e\in E} d\left(e, B\right), \sup_{b\in B} d\left(b, E\right)\right) \\ &\geq \max\left(\sup_{c\in C} d\left(c, A\right) + \sup_{e\in E} d\left(e, B\right), \sup_{a\in A} d\left(a, C\right) + \sup_{b\in B} d\left(b, E\right)\right) \\ &= H_2^s\left(\left(A, B\right), \left(C, E\right)\right). \end{aligned} \tag{21}$$

Theorem 6. *Theorem 3 is a consequence of Theorem 2.*

Proof. It is evident from (17) that (14) and (15) are equivalent to (5) and (6). Regarding Lemma 4, Remark 5, and the observations above, we conclude that, under the conditions of Theorem 3, all hypotheses of Theorem 2 are satisfied. □

4. A Fixed Point Theorem for Suzuki Type (α,r)-Admissible Contractions

In this section, we introduce a generalization of Theorem 2 using a slightly different kind of contractivity condition. We use the following preliminaries. Let (X,d) be a metric space. Given a mapping $\alpha : X\times X \to [0,\infty)$, let $\alpha_* : \mathrm{CB}(X)\times \mathrm{CB}(X) \to [0,\infty)$ be the mapping

$$\alpha_*(A,B)=\inf(\{\alpha(a,b): a\in A, b\in B\}), \tag{22}$$

for all $A,B\in \mathrm{CB}(X)$. We say that the mapping α is transitive if $\alpha(x,y)\ge 1$ and $\alpha(y,z)\ge 1$ implies that $\alpha(x,z)\ge 1$.

Definition 7 (see [5]). Let (X,d) be a metric space and let $\alpha : X\times X\to[0,\infty)$ be a mapping. We say that a mapping $T : X\to \mathrm{CB}(X)$ is α_*-admissible if $\alpha_*(Tx,Ty)\ge 1$ for all x, $y\in X$ such that $\alpha(x,y)\ge 1$.

We say that the metric space (X,d) is α-regular if $\alpha(x_n,z)\ge 1$ for all $n\in\mathbb{N}$ provided that $\{x_n\}\subseteq X$ is a sequence such that $\{x_n\}\xrightarrow{d} z\in X$ and $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}$.

Remark 8. If $\alpha(x,y)\ge 1$ for all $x,y\in X$, then α is transitive, any metric space (X,d) is α-regular and any mapping $T : X\to\mathrm{CB}(X)$ is α_*-admissible.

Definition 9. Let (X,d) be a metric space, let $T : X\to\mathrm{CB}(X)$ be a multivalued mapping, let $\alpha : X\times X\to[0,\infty)$ be a mapping, and let $r\in[0,1)$. One says that T is a Suzuki type (α,r)-admissible contraction if

$$\phi(r)\,d(x,Tx)\le d(x,y) \tag{23}$$

implies

$$\begin{aligned}\alpha_*(Tx,Ty)\,H(Tx,Ty) &\le r\max\Bigg(d(x,y),d(x,Tx),d(y,Ty),\\ &\qquad \frac{d(y,Tx)+d(x,Ty)}{2}\Bigg),\end{aligned} \tag{24}$$

for all $x,y\in X$.

In the following theorem, we will use the following condition, which can be verified for $r\in[0,1)$.

$(P_{T,\alpha,r})$: if $\{x_n\}\subseteq X$ is a sequence such that $\{x_n\}\xrightarrow{d} z\in X$ verifying $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}$ and $z\notin Tz$, then there exist $a\in Tz$ and $n_0\in\mathbb{N}$ such that

$$\begin{gathered}\alpha(z,a)\ge 1,\qquad rd(a,z)<d(z,Tz),\\ \alpha(x_n,a)\ge 1\quad \forall n\ge n_0.\end{gathered} \tag{25}$$

We must clarify that this condition is always satisfied when $\alpha\ge 1$.

Lemma 10. *If (X,d) is a metric space and $\alpha : X\times X\to[0,\infty)$ verifies $\alpha(x,y)\ge 1$ for all $x,y\in X$, then condition $(P_{T,\alpha,r})$ holds for all $r\in[0,1)$ and all $T : X\to CB(X)$.*

Proof. Since $z\notin Tz$ and Tz is closed, then $d(z,Tz)>0$. If $r=0$, there is nothing to prove. Assume that $r>0$ and let ε be any positive real number in the interval

$$\left]0,\frac{1-r}{r}d(z,Tz)\right[. \tag{26}$$

As $d(z,Tz)$ is an infimum, there exists $a\in Tz$ such that $d(z,a)\le d(z,Tz)+\varepsilon$. Therefore

$$\begin{aligned}rd(z,a)&\le r(d(z,Tz)+\varepsilon)\\ &=rd(z,Tz)+r\varepsilon<rd(z,Tz)+r\frac{1-r}{r}d(z,Tz)\\ &=rd(z,Tz)+(1-r)d(z,Tz)=d(z,Tz).\end{aligned} \tag{27}$$

□

Theorem 11. *Let (X,d) be a complete metric space and let T be a Suzuki type (α,r)-admissible multi-valued contraction from X into $CB(X)$. Suppose also that*

(i) *T is α_*-admissible;*
(ii) *there exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\ge 1$;*
(iii) *at least, one of the following properties holds:*
  (iii.1) *T is continuous, or*
  (iii.2) *(X,d) is α-regular and*

$$\begin{aligned}&(a)\ \text{if } 0\le r<\frac{1}{2},\quad \text{then condition } (P_{T,\alpha,2r}) \text{ holds};\\ &(b)\ \text{if } \frac{1}{2}\le r<1,\quad \text{then } \alpha \text{ is transitive.}\end{aligned} \tag{28}$$

Then T has, at least, a fixed point; that is, there exists $z\in X$ such that $z\in Tz$.

Taking into account Remark 8, this result admits Theorem 2 as a particularization to the case in which $\alpha(x,y)=1$ for all $x,y\in X$. Notice that the following proof is a slightly modified version of the proof of Theorem 2.1 in [2] using α.

Proof. We follow the lines of the proof of Theorem 1.2 in [2], doing slight changes due to mapping α. Let $s\in]r,1[$ be arbitrary.

Step 1. There exists a sequence $\{x_n\}\subseteq X$ such that, for all $n\ge 0$,

$$\begin{gathered}x_{n+1}\in Tx_n,\qquad d(x_{n+1},x_{n+2})\le sd(x_n,x_{n+1}),\\ d(x_{n+1},Tx_{n+1})\le rd(x_n,x_{n+1}),\qquad \alpha(x_n,x_{n+1})\ge 1.\end{gathered} \tag{29}$$

Starting from $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\ge 1$, we notice that $\alpha_*(Tx_0,Tx_1)\ge 1$ because T is α_*-admissible. If

$x_0 = x_1$, then $x_0 = x_1 \in Fx_0$, so x_0 is a fixed point of T and the proof is finished. On the contrary, assume that $d(x_0, x_1) > 0$. As

$$\phi(r)\, d(x_0, Tx_0) \le d(x_0, Tx_0) \le d(x_0, x_1), \tag{30}$$

we can apply contractivity condition (24) and we deduce that

$$\begin{aligned} d(x_1, Tx_1) &\le H(Tx_0, Tx_1) \le \alpha_*(Tx_0, Tx_1) H(Tx_0, Tx_1) \\ &\le r \max\Bigg(d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \\ &\qquad \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2} \Bigg). \end{aligned} \tag{31}$$

As $x_1 \in Tx_0$, $d(x_0, Tx_0) \le d(x_0, x_1)$ and $d(x_1, Tx_0) = 0$. Moreover,

$$\begin{aligned} \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2} &= \frac{d(x_0, Tx_1)}{2} \\ &\le \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2}. \end{aligned} \tag{32}$$

Hence,

$$\begin{aligned} d(x_1, Tx_1) &\le r \max\Bigg(d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \\ &\qquad \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2} \Bigg) \\ &\le r \max\Bigg(d(x_0, x_1), d(x_1, Tx_1), \\ &\qquad \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2} \Bigg) \\ &= r \max(d(x_0, x_1), d(x_1, Tx_1)). \end{aligned} \tag{33}$$

If $d(x_0, x_1) \le d(x_1, Tx_1)$, then the maximum is $d(x_1, Tx_1)$, and we have $d(x_1, Tx_1) \le r d(x_1, Tx_1)$. As $r < 1$, we deduce $d(x_1, Tx_1) = 0$. Therefore, $x_1 \in \overline{Tx_1}$, and as Tx_1 is closed, we conclude $x_1 \in Tx_1$ and the proof is finished. Suppose, on the contrary, that $d(x_1, Tx_1) < d(x_0, x_1)$. In such a case, (33) means that

$$d(x_1, Tx_1) \le r d(x_0, x_1). \tag{34}$$

Since $d(x_1, Tx_1)$ is an infimum and $r < s$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \le s d(x_0, x_1). \tag{35}$$

Furthermore,

$$\begin{aligned} \alpha(x_1, x_2) &\ge \inf(\{\alpha(a, b) : a \in Tx_0, b \in Tx_1\}) \\ &= \alpha_*(Tx_0, Tx_1) \ge 1, \\ \phi(r)\, d(x_1, Tx_1) &\le d(x_1, Tx_1) \le d(x_1, x_2). \end{aligned} \tag{36}$$

Repeating this argument, either there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ (in this case, $x_{n_0} \in Tx_{n_0}$ and the proof is finished) or there exists a sequence $\{x_n\} \subseteq X$ verifying (29).

Step 2. There exists $z \in X$ such that $\{x_n\} \to z$. This fact is a consequence of

$$d(x_{n+1}, x_{n+2}) \le s d(x_n, x_{n+1}) \quad \forall n \ge 0, \tag{37}$$

being $s \in]0, 1[$. Following a classical argument, it is easy to prove that $\{x_n\}$ is Cauchy in (X, d) and, therefore, by the completeness, it is convergent.

Step 3. Assume that T is continuous. In such a case, we have that $\{Tx_n\} \xrightarrow{H} Tz$; that is, $\{H(Tx_n, Tz)\} \to 0$. By (7), it follows that $d(x_{n+1}, Tz) \le H(Tx_n, Tz)$ for all $n \in \mathbb{N}$ (because $x_{n+1} \in Tx_n$), and, taking limit as $n \to \infty$, we deduce that $d(z, Tz) = 0$. Therefore, $z \in \overline{Tz}$, and as Tz is closed, we conclude $z \in Tz$ and the proof is finished.

Step 4. Assume that (X, d) is α-regular and condition (28) holds.

In this case, using that (X, d) is α-regular, we have that

$$\alpha(x_n, z) \ge 1 \quad \text{for all } n \ge 0, \tag{38}$$

and taking into account that T is α_*-admissible, we also have that

$$\alpha_*(Tx_n, Tz) \ge 1 \quad \text{for all } n \ge 0. \tag{39}$$

If $z \in Tz$, the proof is also finished in this case. Therefore, we assume that

$$d(z, Tz) > 0, \tag{40}$$

and we will get a contradiction.

Next, we are going to show the following claim:

$$\begin{aligned} &\forall y \in X \setminus \{z\} \text{ such that } \alpha(x_n, y) \ge 1\, \forall n, \\ &\text{we have that } d(z, Ty) \le r \max(d(z, y), d(y, Ty)). \end{aligned} \tag{41}$$

Let $y \in X \setminus \{z\}$ be such that $\alpha(x_n, y) \ge 1$ for all n. As T is α_*-admissible,

$$\alpha_*(Tx_n, Ty) \ge 1 \quad \forall n \ge 0. \tag{42}$$

As $y \ne z$ and $\{x_n\} \to z$, there exists $n_0 \in \mathbb{N}$ such that

$$d(z, x_n) \le \frac{d(z, y)}{3} \quad \text{for all } n \ge n_0. \tag{43}$$

Therefore, for all $n \ge n_0$,

$$\begin{aligned} \phi(r)\, d(x_n, Tx_n) &\le d(x_n, Tx_n) \le d(x_n, x_{n+1}) \\ &\le d(x_n, z) + d(z, x_{n+1}) \\ &\le \frac{2}{3} d(z, y) = d(z, y) - \frac{1}{3} d(z, y) \\ &\le d(z, y) - d(x_n, z) \le d(x_n, y). \end{aligned} \tag{44}$$

As we can apply contractivity condition (24), we obtain that, for all $n \geq n_0$,

$$\begin{aligned} d(x_{n+1}, Ty) &\leq H(Tx_n, Ty) \\ &\leq \alpha_*(Tx_n, Ty) H(Tx_n, Ty) \\ &\leq r \max\left(d(x_n, z), d(x_n, x_{n+1}), d(y, Ty), \frac{d(x_n, Ty) + d(y, Tx_n)}{2} \right) \\ &\leq r \max\left(d(x_n, z), d(x_n, x_{n+1}), d(y, Ty), \frac{d(x_n, Ty) + d(y, x_{n+1})}{2} \right). \end{aligned} \tag{45}$$

Letting $n \to \infty$, we deduce that

$$d(z, Ty) \leq r \max\left(d(z, y), d(y, Ty), \frac{d(z, Ty) + d(y, z)}{2} \right). \tag{46}$$

Assume that the maximum value is the last term. In that case,

$$\begin{aligned} &\max\left(d(z, y), d(y, Ty), \frac{d(z, Ty) + d(y, z)}{2} \right) \\ &= \frac{d(z, Ty) + d(y, z)}{2} \\ &\Longrightarrow d(z, Ty) \leq r \frac{d(z, Ty) + d(y, z)}{2} \\ &\Longleftrightarrow \left(1 - \frac{r}{2}\right) d(z, Ty) \leq \frac{r}{2} d(y, z) \\ &\Longrightarrow d(z, Ty) \leq \frac{r/2}{1 - r/2} d(y, z) \\ &= \frac{r}{2 - r} d(y, z) \leq r d(y, z). \end{aligned} \tag{47}$$

This proves that if the maximum in (46) is $(d(z, Ty) + d(y, z))/2$, then it is also true that $d(y, z) \leq rd(y, z)$. Therefore, in any case, we have that

$$d(z, Ty) \leq r \max(d(z, y), d(y, Ty)), \tag{48}$$

and it follows that (41) holds.

Next we distinguish between the cases $0 \leq r < 1/2$ and $1/2 \leq r < 1$.

Case 4.1. Assume that $0 \leq r < 1/2$ and condition $(P_{T,\alpha,2r})$ holds. Then, there exists $a \in Tz$ and $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} \alpha(z, a) \geq 1, \qquad 2rd(a, z) < d(z, Tz), \\ \alpha(x_n, a) \geq 1 \quad \forall n \geq n_0. \end{gathered} \tag{49}$$

If $a \in Ta$, the proof is finished. On the contrary, assume that $a \notin Ta$; that is, $d(a, Ta) > 0$. As $a \in X \setminus \{z\}$ and $\alpha(x_n, a) \geq 1$ for all $n \geq n_0$, property (41) guarantees that

$$d(z, Ta) \leq r \max(d(z, a), d(a, Ta)). \tag{50}$$

Since $a \in Tz$, we have that $\phi(r) d(z, Tz) \leq d(z, Tz) \leq d(z, a)$, and we can use contractivity condition (24). Notice that, as $\alpha(z, a) \geq 1$, $\alpha_*(Tz, Ta) \geq 1$, and therefore

$$\begin{aligned} d(a, Ta) &\leq H(Tz, Ta) \\ &\leq \alpha_*(Tz, Ta) H(Tz, Ta) \\ &\leq r \max\left(d(z, a), d(z, Tz), d(a, Ta), \frac{d(z, Ta) + d(a, Tz)}{2} \right). \end{aligned} \tag{51}$$

As $d(a, Tz) = 0$ and $d(z, Tz) \leq d(z, a)$,

$$\begin{aligned} d(a, Ta) &\leq H(Tz, Ta) \\ &\leq r \max\left(d(z, a), d(a, Ta), \frac{d(z, Ta)}{2} \right) \\ &\leq r \max\left(d(z, a), d(a, Ta), \frac{d(z, a) + d(a, Ta)}{2} \right) \\ &= r \max(d(z, a), d(a, Ta)). \end{aligned} \tag{52}$$

If we suppose that $d(z, a) \leq d(a, Ta)$, then $\max(d(z, a), d(a, Ta)) = d(a, Ta)$, and the previous inequality leads to $d(a, Ta) = 0$, so $a \in \overline{Ta} = Ta$, which is false. Then

$$\begin{gathered} d(a, Ta) < d(z, a), \\ d(a, Ta) \leq H(Tz, Ta) \leq rd(z, a) < d(z, a), \end{gathered} \tag{53}$$

and (50) implies that

$$d(z, Ta) \leq rd(z, a). \tag{54}$$

However, in this case, using the last two inequalities, (7) and (49),

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq rd(z, a) + rd(z, a) \\ &= 2rd(z, a) < d(z, Tz), \end{aligned} \tag{55}$$

which is a contradiction. Then, we must admit that $z \in Tz$ and the proof is finished in this case.

Case 4.2. Assume that $1/2 \leq r < 1$ and α (or α_*) is transitive. We claim that, in this case, for all $x \in X$ such that $\alpha(z, x) \geq 1$, we have that

$$H(Tz, Tx) \leq r \max\left(d(z, x), d(z, Tz), d(x, Tx), \frac{d(z, Tx) + d(x, Tz)}{2} \right). \tag{56}$$

If $x = z$, there is nothing to prove. Assume that $x \neq z$. As T is α_*-admissible,

$$\alpha(z, x) \geq 1 \Longrightarrow \alpha_*(Tz, Tx) \geq 1. \quad (57)$$

Using (38) and the transitivity of α,

$$\left.\begin{array}{l}\alpha(x_n, z) \geq 1 \quad \forall n \geq 0 \\ \alpha(z, x) \geq 1\end{array}\right\} \Longrightarrow \alpha(x_n, x) \geq 1 \quad \forall n \geq 0, \quad (58)$$

and, as T is α_*-admissible, then

$$\alpha_*(Tx_n, Tx) \geq 1 \quad \forall n \geq 0. \quad (59)$$

(The same conclusion is valid if α_* is transitive.) By (41) and (58), we have that

$$d(z, Tx) \leq r \max(d(z, x), d(x, Tx)). \quad (60)$$

Given $m \in \mathbb{N}$, as $\varepsilon_m = d(x, z)/m > 0$ and $d(z, Tx)$ is an infimum, there exists a sequence $\{y_m\} \subseteq Tx$ such that

$$d(z, y_m) \leq d(z, Tx) + \frac{d(x, z)}{m} \quad \forall m \geq 1. \quad (61)$$

Therefore, by (60) and (61),

$$\begin{aligned} d(x, Tx) &\leq d(x, y_m) \leq d(x, z) + d(z, y_m) \\ &\leq d(x, z) + d(z, Tx) + \frac{d(x, z)}{m} \\ &\leq \left(1 + \frac{1}{m}\right) d(x, z) + r \max(d(z, x), d(x, Tx)). \end{aligned} \quad (62)$$

Letting $m \to \infty$, we deduce that

$$d(x, Tx) \leq d(x, z) + r \max(d(z, x), d(x, Tx)). \quad (63)$$

Two cases can be considered. If $d(z, x) \geq d(x, Tx)$, then the previous inequality means that

$$d(x, Tx) \leq d(x, z) + rd(z, x) = (1 + r) d(x, z). \quad (64)$$

Therefore

$$\begin{aligned} \phi(r) d(x, Tx) &= (1 - r) d(x, Tx) \\ &= \frac{1 - r^2}{1 + r} d(x, Tx) \\ &\leq \frac{1}{1 + r} d(x, Tx) \leq d(x, z). \end{aligned} \quad (65)$$

Hence, we can use contractivity condition (24), which means, using (57), that

$$\begin{aligned} H(Tz, Tx) &\leq \alpha_*(Tz, Tx) H(Tz, Tz) \\ &\leq r \max\Bigg(d(z, x), d(z, Tz), d(x, Tx), \\ &\qquad \frac{d(z, Tx) + d(x, Tz)}{2}\Bigg), \end{aligned} \quad (66)$$

which guarantees that (56) holds.

On the contrary, assume that $d(z, x) < d(x, Tx)$. Hence, inequality (63) yields

$$d(x, Tx) \leq d(x, z) + rd(x, Tx); \quad (67)$$

that is,

$$\phi(r) d(x, Tx) = (1 - r) d(x, Tx) \leq d(x, z). \quad (68)$$

As we can also apply contractivity condition (24), reasoning as in (66), we deduce that, in this case, (56) holds.

In any case, using (56), we deduce that

$$\begin{aligned} d(z, Tz) &= \lim_{n \to \infty} d(x_{n+1}, Tz) \\ &\leq \lim_{n \to \infty} d(Tx_n, Tz) \\ &\leq \lim_{n \to \infty} r \max\Bigg(d(z, x_n), d(z, Tz), d(x_n, Tx_n), \\ &\qquad \frac{d(z, Tx_n) + d(x_n, Tz)}{2}\Bigg) \\ &\leq \lim_{n \to \infty} r \max\Bigg(d(z, x_n), d(z, Tz), d(x_n, x_{n+1}), \\ &\qquad \frac{d(z, x_{n+1}) + d(x_n, Tz)}{2}\Bigg) \\ &= r \max\left(0, d(z, Tz), 0, \frac{d(z, Tz)}{2}\right) \\ &= rd(z, Tz). \end{aligned} \quad (69)$$

As $r < 1$, it follows that $d(z, Tz) = 0$, so $z \in \overline{Tz} = Tz$, which is the desired contradiction. □

If we take $\alpha(x, y) = 1$ for all $x, y \in X$, then we have the following result (using Remark 8).

Corollary 12. *Theorem 2 follows from Theorem 11.*

In the next example we show that Theorem 11 improves Theorem 2.

Example 13. Let $X = [0, \infty)$ be provided with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to \mathrm{CB}(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$\begin{aligned} Tx &= \begin{cases} \left\{\dfrac{x}{2}\right\}, & \text{if } 0 < x < 2, \\ \{5x - 9\}, & \text{if } x \geq 2; \end{cases} \\ \alpha(x, y) &= \begin{cases} 1, & \text{if } 0 \leq x < y < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (70)$$

Clearly, T is continuous and, if we identify $\{x\} \equiv x$, then

$$\alpha_*(Tx, Ty) = \alpha(Tx, Ty) \quad \forall x, y \in X. \quad (71)$$

Let $x = 3$ and $y = 10$. Then

$$d(x,y) = 7, \qquad d(x,Tx) = 6, \qquad d(y,Ty) = 31,$$
$$\frac{d(y,Tx)+d(x,Ty)}{2} = 19.5, \tag{72}$$
$$H(Tx,Ty) = d(6,41) = 35.$$

Any number $r \in [0,1)$ verifies $\phi(r)d(x,Tx) \le d(x,Tx) = 6 < 7 = d(x,y)$. However, condition

$$H(Tx,Ty) \le r\max\left(d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx)+d(x,Ty)}{2}\right) \tag{73}$$

is false. Therefore, Theorem 2 cannot be applied to deduce that T has a fixed point because T is not contractive in the sense of Theorem 2. However, let $r = 0.5$. We will show that Theorem 11 is applicable.

Indeed, let $x, y \in X$ be such that $\phi(0.5)d(x,Tx) \le d(x,y)$. If $\alpha_*(Tx,Ty) = 0$, then there is nothing to prove. Assume that $\alpha(Tx,Ty) = \alpha_*(Tx,Ty) = 1$. In this case, $0 \le Tx < Ty < 1$, which means that $0 \le x < y < 2$, $Tx = x/2$, and $Ty = y/2$. Taking into account that

$$H(Tx,Ty) = d\left(\frac{x}{2},\frac{y}{2}\right) = \frac{y-x}{2} = \frac{1}{2}d(x,y) = rd(x,y), \tag{74}$$

then condition

$$H(Tx,Ty) \le r\max\left(d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Tx)+d(x,Ty)}{2}\right) \tag{75}$$

is obvious. Therefore, Theorem 11 guarantees that T has a fixed point.

Conflict of Interests

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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