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Research Article

Convergence Theorem for Equilibrium and Variational Inequality Problems and a Family of Infinitely Nonexpansive Mappings in Hilbert Space

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We introduce a hybrid iterative scheme for finding a common element of the set of common fixed points for a family of infinitely nonexpansive mappings, the set of solutions of the varitional inequality problem and the equilibrium problem in Hilbert space. Under suitable conditions, some strong convergence theorems are obtained. Our results improve and extend the corresponding results in (Chang et al. (2009), Min and Chang (2012), Plubtieng and Punpaeng (2007), S. Takahashi and W. Takahashi (2007), Tada and Takahashi (2007), Gang and Changsong (2009), Ying (2013), Y. Yao and J. C. Yao (2007), and Yong-Cho and Kang (2012)).

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and P_C the metric projection of H onto C. Let $\phi : C \times C \to R$ be a bifunction. We consider the equilibrium problem EP which is to find $z \in C$ such that

$$\phi(z, y) \geq 0, \quad \forall y \in C. \quad (1)$$

Let EP(ϕ) be the set of solutions. Some methods have been proposed to solve the equilibrium problem.

A mapping A is said to be α-inverse strongly monotone if there exists a real number $\alpha > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$, for all $x, y \in C$.

The classical variational inequality problem is to find an element $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (2)$$

The solution set of inequality (2) is denoted by VI(C, A). For given elements $z \in H$ and $u \in C$, we have the following inequality:

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \quad (3)$$

if and only if $u = P_C z$. It is known that the projection operator P_C is nonexpansive. One can see that the variational inequality problem (2) is equivalent to a fixed point problem. Since an element $u \in C$ is the solution of variational inequality (2) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$. Recently, many researchers studied various iterative algorithms for finding an element of $\mathrm{VI}(C, A) \bigcap F(S)$. Takahashi and Toyoda [1] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (I - \lambda_n B) x_n, \quad \forall n \geq 0. \quad (4)$$

They proved that the sequence $\{x_n\}$ converges weakly to a point $q \in \mathrm{VI}(C, B) \bigcap F(S)$. Y. Yao and J. C. Yao [2] introduced the following iterative scheme:

$$\begin{aligned} x_1 &= u \in C, \\ y_n &= P_C (I - \lambda_n A) x_n, \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C (I - \lambda_n A) y_n. \end{aligned} \quad (5)$$

Chang et al. [3] introduced the following iterative scheme:

$$\begin{aligned} &\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C, \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \\ &k_n = P_C(I - \lambda_n B) y_n, \\ &y_n = P_C(I - \lambda_n B) u_n, \end{aligned} \tag{6}$$

and obtained some strong convergence theorems.

In this paper, we will introduce a new hybrid iterative scheme for finding a common element of the set of common fixed points for a family of infinitely nonexpansive mappings, the set of solutions of the variational inequality problem, and the equilibrium problem. Further, we obtain some strong convergence theorems and extend the results in [2–10].

2. Preliminaries

Let $x_n \rightharpoonup x$ and $x_n \to x$ be the weak convergence and strong convergence of the sequence $\{x_n\}$ in H. Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,1]$. For $n \geq 1$, we define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= \lambda_n S_n U_{n,n+1} + (1-\lambda_n) I \\ U_{n,n-1} &= \lambda_{n-1} S_{n-1} U_{n,n} + (1-\lambda_{n-1}) I \\ &\vdots \\ U_{n,k} &= \lambda_k S_k U_{n,k+1} + (1-\lambda_k) I \\ U_{n,k-1} &= \lambda_{k-1} S_{k-1} U_{n,k} + (1-\lambda_{k-1}) I \\ &\vdots \\ U_{n,2} &= \lambda_2 S_2 U_{n,3} + (1-\lambda_2) I \\ W_n = U_{n,1} &= \lambda_1 S_1 U_{n,2} + (1-\lambda_1) I. \end{aligned} \tag{7}$$

W_n is the W-mapping of C into itself which is generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In order to prove our main results, the following Lemmas are needed.

Lemma 1 (see [11]). *Let C be a nonempty closed convex subset of a Banach space E, let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. For any $n \geq 1$, let $\{W_n\}$ be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$.*

Lemma 2 (see [4]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then for every $x \in C$ and $k \geq 1 \lim_{k\to\infty} U_{n,k}x$ exists.*

Using Lemma 2, we can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in C. \tag{8}$$

Such a W is called the W-mapping generated by the sequence $\{S_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Throughout this paper, we always assume that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for an element $b \in (0,1)$.

Lemma 3 (see [4]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, W is a nonexpansive mapping and $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.*

Lemma 4 (see [4]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. If K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

Lemma 5 (see [10]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 6 (see [10]). *Assume that a_n is a sequence of nonnegative real numbers, such that*

$$a_{n+1} \leq (1-\gamma_n) a_n + \delta_n, \quad \forall n \geq n_0, \tag{9}$$

where n_0 is some nonnegative integer, $\gamma_n \in (0,1)$, and δ_n are sequences satisfying

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| = \infty$; *then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 7 (see [10]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_m : 1 \leq m \leq r\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{m=1}^{r} F(T_m)$ is nonempty. Let $\{\lambda_m\}$ be a sequence of positive numbers with $\sum_{m=1}^{r} \lambda_m = 1$. Then, a mapping S on C defined by $Sx = \sum_{m=1}^{r} \lambda_m T_m x$ for all $x \in C$ is well defined and nonexpansive and $F(S) = \bigcap_{m=1}^{r} F(T_m)$ holds.*

For solving the equilibrium problem for bifunction $F : C \times C \to R$, assume that F satisfies the following conditions:

(A_1) $F(x,x) = 0$ for all $x \in C$;

(A_2) F is monotone; that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A_3) $\lim_{t\to 0} F(tz+(1-t)x, y) \leq F(x, y)$, for any $x, y, z \in C$;

(A_4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $F : C \times C \rightarrow R$ satisfies conditions (A_1)–(A_4), then we have the following two important results.

Lemma 8 (see [4]). *Let C be a nonempty closed convex subset of a Hilbert space H and let F be an equilibrium bifunction $F : C \times C \rightarrow R$ that satisfies conditions (A_1)–(A_4). Let $r > 0$ and $x \in C$; then, there exists $y \in C$ such that $F(y, z)+(1/r)\langle z-y, y-x\rangle \geq 0$, for all $z \in C$.*

Lemma 9 (see [4]). *Let F be an equilibrium bifunction $F : C\times C \rightarrow R$ that satisfies conditions (A_1)–(A_4). For given $r > 0$ and $x \in C$, define a mapping $V_r : H \rightarrow C$ as follows:*

$$V_r(x) = \left\{y \in C : F(y,z) + \frac{1}{r}\langle z-y, y-x\rangle \geq 0,\ \forall z \in C\right\}. \tag{10}$$

Then, the following conclusions hold:

(1) *V_r is single-valued;*

(2) *V_r is firmly nonexpansive; that is, for any $x, y \in H$, $\|V_r x - V_r y\|^2 \leq \langle V_r x - V_r y, x - y\rangle$;*

(3) *$F(V_r) = \mathrm{EP}(F)$*

(4) *$\mathrm{EP}(F)$ is a closed and convex set.*

3. The Main Results

Theorem 10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_m : C \rightarrow H$ be a l_m-inverse strongly monotone mapping for each $1 \leq m \leq r$, where r is some positive integer. Let $D : C \rightarrow H$ be a α-inverse strongly monotone mapping. Let F be an equilibrium bifunction $F : C \times C \rightarrow R$ that satisfies conditions (A_1)–(A_4). Let $\{T_n\}_{n=1}^{\infty} : C \rightarrow C$ be a family of infinite k_n-strict pseudocontractive mappings with $0 \leq k_n < 1$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. $\{S_n\}_{n=1}^{\infty} : C \rightarrow C$ is a family of infinitely nonexpansive mappings such that $\mathcal{F} = F(W)\bigcap \mathrm{VI}(C, B_m)\bigcap \mathrm{EP} \neq \emptyset$, where $F(W) := \bigcap_{n=1}^{\infty} F(S_n)$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ and let $f : H \rightarrow H$ be a contraction with contraction constant h $(0 < h < 1)$ and $0 < \gamma < (\overline{\gamma}/h)$. Let $\{x_n\}, \{y_n\}, \{\rho_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{gathered} F(y_n, \eta) + \langle Dy_n, \eta - y_n\rangle + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \\ \forall \eta \in C, \\ \rho_n = \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m) y_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n \rho_n, \end{gathered} \tag{11}$$

where $\mu_m \in (0, 2l_m)$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, and $\{r_n\} \subset [0, \infty]$. If the following conditions are satisfied:

(C_1) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(C_2) *$\lim_{n\to\infty} \eta_n^m = \eta^m \in (0, 1)$;*

(C_3) *$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;*

(C_4) *$\liminf_{n\to\infty} r_n > 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;*

(C_5) *$\sum_{n=1}^{\infty} \eta_n^m = 1$, for all $n \geq 1$,*

then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}(\gamma f + (I - A))q$.

Proof. We define a bifunction $\phi : C \times C \rightarrow R$ by $\phi(z, y) = F(z, y) + \langle Dz, y - z\rangle$, for all $y, z \in C$, so the equilibrium problem is equivalent to the following equilibrium problem: find an element $z \in C$ such that $\phi(z, y) \geq 0$, for all $y \in C$ and (11) can be written as

$$\begin{gathered} \phi(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \\ \rho_n = \sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m) y_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n \rho_n. \end{gathered} \tag{12}$$

□

Step 1. First, we prove the sequences $\{x_n\}$, $\{y_n\}$, $\{\rho_n\}$ are bounded.

Let $p \in \mathcal{F}$; as $y_n = V_{r_n} x_n$, we have $\|y_n - p\| = \|V_{r_n} x_n - p\| \leq \|x_n - p\|$. Next we show that the mapping $I - \mu_m B_m$ is nonexpansive for each m. Consider

$$\begin{aligned} &\|(I - \mu_m B_m)x - (I - \mu_m B_m)y\|^2 \\ &\quad = \|(x - y) - \mu_m(B_m x - B_m y)\|^2 \\ &\quad = \|x - y\|^2 + \mu_m^2\|B_m x - B_m y\|^2 \\ &\qquad - 2\mu_m\langle B_m x - B_m y, x - y\rangle \\ &\quad \leq \|x - y\|^2 + \mu_m^2\|B_m x - B_m y\|^2 - 2\mu_m l_m\|B_m x - B_m y\|^2 \\ &\quad = \|x - y\|^2 - \mu_m(2l_m - \mu_m)\|B_m x - B_m y\|^2 \\ &\quad \leq \|x - y\|^2, \\ &\|\rho_n - p\| = \left\|\sum_{m=1}^{r} \eta_n^m P_C(I - \mu_m B_m) y_n - p\right\| \\ &\qquad \leq \sum_{m=1}^{r} \eta_n^m \|y_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{13}$$

Since A is a strongly positive linear bounded operator, then $\|A\| = \sup\{|\langle Au,u\rangle| : u \in H, \|u\| = 1\}$, $\langle((1-\beta_n)I - \alpha_n A)u,u\rangle = 1-\beta_n-\alpha_n\langle Au,u\rangle \geq 1-\beta_n-\alpha_n\|A\| \geq 0$, so

$$\begin{aligned}
&\|(1-\beta_n)I-\alpha_n A\| \\
&\quad = \sup\{|\langle((1-\beta_n)I-\alpha_n A)u,u\rangle| : u\in H, \|u\|=1\} \\
&\quad \leq 1-\beta_n-\alpha_n\overline{\gamma},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\|(1-\beta_n)I-\alpha_n A\| \\
&\quad = \sup\{|\langle((1-\beta_n)I-\alpha_n A)u,u\rangle| : u\in H, \|u\|=1\} \\
&\quad \leq 1-\beta_n-\alpha_n\overline{\gamma} \\
&\|x_{n+1}-p\| \\
&\quad = \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n\rho_n - p\| \\
&\quad = \|\alpha_n\gamma(f(x_n)-f(p))+\beta_n(x_n-p) \\
&\qquad + ((1-\beta_n)I-\alpha_n A)(W_n\rho_n-p) \\
&\qquad +\alpha_n(\gamma f(p)-Ap)\| \\
&\quad \leq \alpha_n\gamma h\|x_n-p\|+\beta_n\|x_n-p\| \\
&\qquad + ((1-\beta_n)I-\alpha_n A)\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\| \\
&\quad \leq \alpha_n\gamma h\|x_n-p\|+\beta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\| \\
&\qquad - \alpha_n\overline{\gamma}\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\| \\
&\quad = (1-\alpha_n(\overline{\gamma}-\gamma h))\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\| \\
&\quad \leq \max\left\{\|x_n-p\|, \frac{1}{\overline{\gamma}-\gamma h}\|\gamma f(p)-Ap\|\right\} \\
&\quad \vdots \\
&\quad \leq \max\left\{\|x_1-p\|, \frac{1}{\overline{\gamma}-\gamma h}\|\gamma f(p)-Ap\|\right\}.
\end{aligned} \tag{15}$$

This implies that $\{x_n\}$ is bounded sequence in H. Therefore $\{y_n\}$, $\{\rho_n\}$, $\{\gamma f(x_n)\}$, $\{W_n\rho_n\}$ are all bounded.

Step 2. Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|\rho_{n+1}-\rho_n\| = 0$.

In fact, let us define a sequence $\{z_n\}$ by $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, for all $n \geq 1$; then, we have

$$\begin{aligned}
&z_{n+1}-z_n \\
&\quad = \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\
&\quad = \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}\rho_{n+1}}{1-\beta_{n+1}} \\
&\qquad - \frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)W_n\rho_n}{1-\beta_n} \\
&\quad = \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1})-AW_{n+1}\rho_{n+1}] \\
&\qquad - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n)-AW_n\rho_n]+W_{n+1}\rho_{n+1}-W_n\rho_n \\
&\quad = \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1})-AW_{n+1}\rho_{n+1}] \\
&\qquad - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n)-AW_n\rho_n] \\
&\qquad + (W_{n+1}\rho_{n+1}-W_{n+1}\rho_n)+(W_{n+1}\rho_n-W_n\rho_n), \\
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}\rho_{n+1}\|) \\
&\qquad + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|AW_n\rho_n\|) \\
&\qquad + \|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|-\|x_{n+1}-x_n\|.
\end{aligned} \tag{16}$$

Because $\rho_n = \sum_{m=1}^{r}\eta_n^m P_C(I-\mu_m B_m)y_n$, we have

$$\begin{aligned}
&\|\rho_{n+1}-\rho_n\| \\
&\quad = \left\|\sum_{m=1}^{r}\eta_{n+1}^m P_C(I-\mu_m B_m)y_{n+1} - \sum_{m=1}^{r}\eta_n^m P_C(I-\mu_m B_m)y_n\right\| \\
&\quad = \left\|\sum_{m=1}^{r}\eta_{n+1}^m P_C(I-\mu_m B_m)y_{n+1} - \sum_{m=1}^{r}\eta_{n+1}^m P_C(I-\mu_m B_m)y_n\right. \\
&\qquad \left. +\sum_{m=1}^{r}\eta_{n+1}^m P_C(I-\mu_m B_m)y_n - \sum_{m=1}^{r}\eta_n^m P_C(I-\mu_m B_m)y_n\right\| \\
&\quad \leq \|y_{n+1}-y_n\|+M\sum_{m=1}^{r}|\eta_{n+1}^m-\eta_n^m|,
\end{aligned} \tag{17}$$

where $M = \max\{\sup\{\|P_C(I-\mu_m B_m)y_n\| : n\geq 1\} : 1\leq m\leq r\}$, so

$$\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}\rho_{n+1}\|) \\
&\qquad + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|AW_n\rho_n\|) \\
&\qquad + \|y_{n+1}-y_n\|+M\sum_{m=1}^{r}|\eta_{n+1}^m-\eta_n^m| \\
&\qquad + \|W_{n+1}\rho_n-W_n\rho_n\|-\|x_{n+1}-x_n\|.
\end{aligned} \tag{18}$$

Observing $y_n = V_{r_n} x_n$, $y_{n+1} = V_{r_{n+1}} x_{n+1}$, we have

$$\phi(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \tag{19}$$

$$\phi(y_{n+1}, \eta) + \frac{1}{r_{n+1}}\langle \eta - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall \eta \in C. \tag{20}$$

Putting $\eta = y_{n+1}$ in (19), $\eta = y_n$ in (20), adding up these two inequalities, and using condition (A_2) to simplify, we have

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{21}$$

By condition (C_4), without loss of generality, we can assume that there exists a real number m such that $r_n > m > 0$, so

$$\begin{aligned} \|y_{n+1} - y_n\|^2 &\leq \|y_{n+1} - y_n\| \times \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|(y_{n+1} - x_{n+1})\right) \\ \|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \frac{M_1}{m}|r_{n+1} - r_n|, \end{aligned} \tag{22}$$

where $M_1 = \sup_{n\geq 1}\|y_n - x_n\|$.

Since $S_i, U_{n,i}$ are nonexpansive, so $\|W_{n+1}\rho_n - W_n\rho_n\| \leq \prod_{i=1}^{n+1}\lambda_i\|S_{n+1}\rho_n - \rho_n\| \leq L(\prod_{i=1}^{n+1}\lambda_i)$, where $L = \sup_{n\geq 1}\|S_{n+1}\rho_n - \rho_n\|$. Consider

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\rho_{n+1}\|) \\ &\qquad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(x_n)\| + \|AW_n\rho_n\|) \\ &\qquad + \frac{M_1}{m}|r_{n+1} - r_n| + L\left(\prod_{i=1}^{n+1}\lambda_i\right). \end{aligned} \tag{23}$$

Using $0 < \lambda_i \leq b < 1$ $(i \geq 1)$ and the conditions (C_1)–(C_3), $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. By Lemma 5, we conclude that $\lim_{n\to\infty}\|z_n - x_n\| = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|z_n - x_n\| = 0$. Consider

$$\begin{aligned} \|\rho_{n+1} - \rho_n\| &\leq \|y_{n+1} - y_n\| + M\sum_{m=1}^{r}|\eta_{n+1}^m - \eta_n^m| \\ &\leq \|x_{n+1} - x_n\|\frac{M_1}{m}|r_{n+1} - r_n| + M\sum_{m=1}^{r}|\eta_{n+1}^m - \eta_n^m|. \end{aligned} \tag{24}$$

So $\lim_{n\to\infty}\|\rho_{n+1} - \rho_n\| = 0$.

Step 3. Consider

$$\begin{aligned} \|x_n - W_n\rho_n\| &= \|x_n - x_{n+1} + x_{n+1} - W_n\rho_n\| \\ &\leq \|x_n - x_{n+1}\| \\ &\quad + \|\alpha_n\gamma f(x_n) + \beta_n x_n \\ &\qquad + ((1 - \beta_n)I - \alpha_n A)W_n\rho_n - W_n\rho_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - AW_n\rho_n\| \\ &\quad + \beta_n\|x_n - W_n\rho_n\| \\ \|x_n - W_n\rho_n\| &\leq \frac{1}{1 - \beta_n}\|x_{n+1} - x_n\| \\ &\quad + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - AW_n\rho_n\|. \end{aligned} \tag{25}$$

So we have $\lim_{n\to\infty}\|x_n - W_n\rho_n\| = 0$.

Step 4. For any given $p \in \mathcal{F}$,

$$\begin{aligned} \|y_n - p\|^2 &= \|V_{r_n}x_n - p\|^2 = \langle V_{r_n}x_n - V_{r_n}p, x_n - p\rangle \\ &= \langle y_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2), \end{aligned} \tag{26}$$

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\quad= \|\alpha_n\gamma f(x_n) + \beta_n x_n \\ &\qquad + ((1 - \beta_n)I - \alpha_n A)W_n\rho_n - p\|^2 \\ &\quad= \|[(I - \alpha_n A)(W_n\rho_n - p) + \beta_n(x_n - W_n\rho_n)] \\ &\qquad + \alpha_n(\gamma f(x_n) - Ap)\| \\ &\quad\leq \|(I - \alpha_n A)(W_n\rho_n - p) + \beta_n(x_n - W_n\rho_n)\|^2 \\ &\qquad + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\ &\quad\leq [(1 - \alpha_n\overline{\gamma})\|\rho_n - p\| + \beta_n\|x_n - W_n\rho_n\|]^2 \\ &\qquad + 2\alpha_n\|\gamma f(x_n) - Ap\| \cdot \|x_{n+1} - p\| \\ &\quad= (1 - \alpha_n\overline{\gamma})^2\|\rho_n - p\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2 \\ &\qquad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - W_n\rho_n\| \\ &\qquad + 2\alpha_n\|\gamma f(x_n) - Ap\| \cdot \|x_{n+1} - p\| \\ &\quad\leq (1 - \alpha_n\overline{\gamma})^2[\|x_n - p\|^2 - \|x_n - y_n\|^2] \\ &\qquad + \beta_n^2\|x_n - W_n\rho_n\|^2 \\ &\qquad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - W_n\rho_n\| \\ &\qquad + 2\alpha_n\|\gamma f(x_n) - Ap\| \cdot \|x_{n+1} - p\|. \end{aligned} \tag{27}$$

Simplifying it, we have

$$\begin{aligned}
&(1-\alpha_n\overline{\gamma})^2\|x_n-y_n\|^2\\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+\alpha_n^2\overline{\gamma}^2\|x_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|\\
&\quad\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\qquad+\alpha_n^2\overline{\gamma}^2\|x_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|.
\end{aligned}\tag{28}$$

So $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Step 5. Consider

$$\begin{aligned}
&\|\rho_n-p\|^2\\
&\quad=\left\|\sum_{m=1}^{r}\eta_n^m P_C(I-\mu_m B_m)y_n-p\right\|^2\\
&\quad\le\|(y_n-p)-\mu_m(B_m y_n-B_m p)\|^2\\
&\quad=\|y_n-p\|^2+\mu_m^2\|B_m y_n-B_m p\|^2\\
&\qquad-2\mu_m\langle y_n-p,B_m y_n-B_m p\rangle\\
&\quad\le\|y_n-p\|^2+\mu_m^2\|B_m y_n-B_m p\|^2\\
&\qquad-2\mu_m l_m\|B_m y_n-B_m p\|^2\\
&\quad\le\|x_n-p\|^2+\mu_m(\mu_m-2l_m)\|B_m y_n-B_m p\|^2\\
&\|x_{n+1}-p\|^2\\
&\quad\le(1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2\\
&\qquad+(1-\alpha_n\overline{\gamma})^2\mu_m(\mu_m-2l_m)\|B_m y_n-B_m p\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|\\
&(2l_m-\mu_m)(1-\alpha_n\overline{\gamma})^2\mu_m\|B_m y_n-B_m p\|^2\\
&\quad\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\qquad+\alpha_n^2\overline{\gamma}^2\|x_n-p\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|.
\end{aligned}\tag{29}$$

So $\lim_{n\to\infty}\|B_m y_n-B_m p\|=0$.

Step 6. Consider

$$\begin{aligned}
\|\rho_n-p\|^2&=\left\|\sum_{m=1}^{r}\eta_n^m P_C(I-\mu_m B_m)y_n-p\right\|^2\\
&\le\langle(y_n-\mu_m B_m y_n)-(p-\mu_m B_m p),\rho_n-p\rangle\\
&=\frac{1}{2}\Big\{\|I-\mu_m B_m y_n-I-\mu_m B_m p\|^2+\|\rho_n-p\|^2\\
&\qquad-\|(y_n-\rho_n)-\mu_m(B_m y_n-B_m p)\|^2\Big\}\\
&\le\frac{1}{2}\Big\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|y_n-\rho_n\|^2\\
&\qquad-\mu_m^2\|B_m y_n-B_m p\|^2\\
&\qquad+2\mu_m\langle y_n-\rho_n,B_m y_n-B_m p\rangle\Big\}
\end{aligned}$$

$$\begin{aligned}
\|\rho_n-p\|^2&\le\|y_n-p\|^2-\|y_n-\rho_n\|^2-\mu_m^2\|B_m y_n-B_m p\|^2\\
&\quad+2\mu_m\langle y_n-\rho_n,B_m y_n-B_m p\rangle
\end{aligned}$$

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad\le(1-\alpha_n\overline{\gamma})^2\|\rho_n-p\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|\\
&\quad\le(1-\alpha_n\overline{\gamma})^2\Big[\|y_n-p\|^2-\|y_n-\rho_n\|^2\\
&\qquad\qquad-\mu_m^2\|B_m y_n-B_m p\|^2\\
&\qquad\qquad+2\mu_m\langle y_n-\rho_n,B_m y_n-B_m p\rangle\Big]\\
&\qquad+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|\\
&\quad\le\|x_n-p\|^2+\alpha_n\overline{\gamma}^2\|x_n-p\|^2\\
&\qquad+2\mu_m(1-\alpha_n\overline{\gamma})^2\|y_n-\rho_n\|\cdot\|B_m y_n-B_m p\|\\
&\qquad-(1-\alpha_n\overline{\gamma})^2\|y_n-\rho_n\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\
&\qquad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|\\
&\qquad+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|\\
&(1-\alpha_n\overline{\gamma})^2\|y_n-\rho_n\|^2\\
&\quad\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\qquad+2\mu_m(1-\alpha_n\overline{\gamma})^2\|y_n-\rho_n\|\cdot\|B_m y_n-B_m p\|\\
&\qquad+\beta_n^2\|x_n-W_n\rho_n\|^2
\end{aligned}$$

$$+2\left(1-\alpha_n\overline{\gamma}\right)\beta_n\left\|\rho_n-p\right\|\cdot\left\|x_n-W_n\rho_n\right\| + 2\alpha_n\left\|\gamma f\left(x_n\right)-Ap\right\|\cdot\left\|x_{n+1}-p\right\|. \quad (30)$$

So $\lim_{n\to\infty}\|y_n-\rho_n\|=0$.

Step 7. Consider

$$\begin{aligned}\left\|y_n-W_ny_n\right\| &\le \left\|W_ny_n-W_n\rho_n\right\| + \left\|W_n\rho_n-x_n\right\|+\left\|x_n-y_n\right\| \\ &\le \left\|y_n-\rho_n\right\|+\left\|W_n\rho_n-x_n\right\|+\left\|x_n-y_n\right\|.\end{aligned} \quad (31)$$

So $\lim_{n\to\infty}\|y_n-W_ny_n\|=0$.

Step 8. Next, we prove that $\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle\le 0$, where $q=P_{\mathcal{F}}(\gamma f+(I-Aq))$.

To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle = \lim_{i\to\infty}\left\langle\gamma f(q)-Aq,x_{n_i}-q\right\rangle. \quad (32)$$

Since $\{y_{n_i}\}$ is bounded, so there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup\omega$; then, $\phi(y_n,\eta)+(1/r_n)\langle\eta-y_n,y_n-x_n\rangle\ge 0$, for all $\eta\in C$, $\langle\eta-y_n,(y_n-x_n)/r_n\rangle\ge\phi(\eta,y_n)$, $\langle\eta-y_{n_i},(y_{n_i}-x_{n_i})/r_{n_i}\rangle\ge\phi(\eta,y_{n_i})$. So $\phi(\eta,\omega)\le 0$.

For any t with $0\le t\le 1$ and $\eta\in C$, let $\eta_t=t\eta+(1-t)\omega$. Since $\eta\in C$ and $\phi(\eta_t,\omega)\le 0$, from conditions (A_1) and (A_4), we have

$$0=\phi\left(\eta_t,\eta_t\right)\le t\phi\left(\eta_t,\eta\right)+(1-t)\phi\left(\eta_t,\omega\right)\le t\phi\left(\eta_t,\eta\right). \quad (33)$$

This implies that $\phi(\eta_t,\eta)\ge 0$. From condition (A_3), we have $\phi(\omega,\eta)\ge 0$. So $\omega\in \mathrm{EP}(\phi)$.

Define a mapping $J:C\to C$ by $Jx=\sum_{m=1}^{r}\eta_n^mP_C(I-\mu_mB_m)x$, for all $x\in C$, where $\lim_{n\to\infty}\eta_n^m=\eta^m$. From Lemma 7 we see that J is nonexpansive such that

$$F(J)=\bigcap_{m=1}^{r}F\left(P_C\left(I-\mu_mB_m\right)\right)=\bigcap_{m=1}^{r}\mathrm{VI}\left(C,B_m\right).$$

$$\begin{aligned}\left\|y_n-Jy_n\right\| &\le \left\|y_n-\rho_n\right\|+\left\|\rho_n-Jy_n\right\| \\ &= \left\|y_n-\rho_n\right\|+\left\|\sum_{m=1}^{r}\eta_n^mP_C\left(I-\mu_mB_m\right)y_n - \sum_{m=1}^{r}\eta^mP_C\left(I-\mu_mB_m\right)y_n\right\| \\ &\le \left\|y_n-\rho_n\right\|+M\sum_{m=1}^{r}\left|\eta_n^m-\eta^m\right|\longrightarrow 0 \quad (n\longrightarrow\infty);\end{aligned} \quad (34)$$

Since every nonexpansive mapping is strictly pseudocontractive, so $\omega\in F(J)=\bigcap_{m=1}^{r}\mathrm{VI}(C,B_m)$.

Now we prove that $\omega\in F(\omega)$, and if not, we have $\omega\ne W(\omega)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\left\|y_{n_i}-\omega\right\| &< \liminf_{i\to\infty}\left\|y_{n_i}-W\omega\right\| \\ &\le \liminf_{i\to\infty}\left(\left\|y_{n_i}-Wy_{n_i}\right\|+\left\|Wy_{n_i}-W\omega\right\|\right) \\ &\le \liminf_{i\to\infty}\left(\left\|y_{n_i}-Wy_{n_i}\right\|+\left\|y_{n_i}-\omega\right\|\right), \\ \lim_{i\to\infty}\left\|Wy_{n_i}-y_{n_i}\right\| &\le \lim_{i\to\infty}\left(\left\|Wy_{n_i}-W_{n_i}y_{n_i}\right\|+\left\|W_{n_i}y_{n_i}-y_{n_i}\right\|\right) \\ &\le \lim_{i\to\infty}\left\{\sup_{x\in C}\left\|W\omega-W_{n_i}\omega\right\|\right\} + \lim_{i\to\infty}\left\|W_{n_i}y_{n_i}-y_{n_i}\right\|=0.\end{aligned} \quad (35)$$

Therefore, $\liminf_{i\to\infty}\|y_{n_i}-\omega\| < \liminf_{i\to\infty}\|y_{n_i}-\omega\|$, so $\omega\in F(\omega)$.

Step 9. Finally, we prove that $x_n\to q=P_{\mathcal{F}}(\gamma f+(I-A)q)$. Since $q=P_{\mathcal{F}}(\gamma f+(I-A)q)$, so

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle &= \lim_{i\to\infty}\left\langle\gamma f(q)-Aq,x_{n_i}-q\right\rangle \\ &= \lim_{i\to\infty}\left\langle\gamma f(q)-Aq,x_{n_i}-y_{n_i}+y_{n_i}-q\right\rangle \\ &= \langle\gamma f(q)-Aq,\omega-q\rangle\le 0,\end{aligned}$$

$$\begin{aligned}\left\|x_{n+1}-q\right\|^2 &= \left\|\alpha_n\left(\gamma f\left(x_n\right)-Aq\right) + \left[\left(1-\beta_nI-\alpha_nA\right)\left(W_n\rho_n-q\right)+\beta_n\left(x_n-q\right)\right]\right\|^2 \\ &\le \left\|\left(1-\beta_nI-\alpha_nA\right)\left(W_n\rho_n-q\right)+\beta_n\left(x_n-q\right)\right]\right\|^2 + 2\alpha_n\gamma\left\langle f\left(x_n\right)-f\left(q\right),x_{n+1}-q\right\rangle + 2\alpha_n\left\langle\gamma f(q)-Aq,x_{n+1}-q\right\rangle \\ &\le \left[\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|\rho_n-p\right\|+\beta_n\left\|x_n-q\right\|\right]^2 + 2\alpha_n\gamma h\left\|x_n-q\right\|\cdot\left\|x_{n+1}-q\right\| + 2\alpha_n\left\langle\gamma f(q)-Aq,x_{n+1}-q\right\rangle \\ &\le \left(1-\alpha_n\overline{\gamma}\right)^2\left\|x_n-q\right\|^2 + \alpha_n\gamma h\left(\left\|x_n-q\right\|^2+\left\|x_{n+1}-q\right\|^2\right) + 2\alpha_n\left\langle\gamma f(q)-Aq,x_{n+1}-q\right\rangle.\end{aligned} \quad (36)$$

This implies that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \left[1-\frac{2(\overline{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}\right]\|x_n-q\|^2 \\ &\quad+\frac{2(\overline{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}\left\{\frac{\alpha_n\gamma^2}{2(\overline{\gamma}-\gamma h)}\|x_n-q\|^2\right. \\ &\quad\left.+\frac{1}{\overline{\gamma}-\gamma h}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\right\}.\end{aligned} \tag{37}$$

From Lemma 6, $\{x_n\}$ converges strongly to q.

Taking $F(x,y)=0, D=0$ for all $x,y\in H$, $r_n=1, m=1$ in Theorem 10, then Theorem 10 is reduced to the following.

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B: C \to H$ be a λ-inverse strongly monotone mapping and $D: C \to H$ a α-inverse strongly monotone mapping. Let $\{T_n\}_{n=1}^{\infty}: C \to C$ be a family of infinite k_n-strict pseudocontractive mappings with $0 \leq k_n < 1$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. $\{S_n\}_{n=1}^{\infty}: C \to C$ is a family of infinitely nonexpansive mappings such that $\mathcal{F}=F(W)\bigcap \mathrm{VI}(C,B)\neq\emptyset$, where $F(W):=\bigcap_{n=1}^{\infty}F(S_n)$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma}>0$ and let $f: H\to H$ be a contraction with contraction constant h $(0<h<1)$ and $0<\gamma<(\overline{\gamma}/h)$. Let $\{x_n\}$ and $\{y_n\}$ be generated in sequence by $x_1\in H$ and*

$$\begin{aligned} y_n &= P_C(I-\lambda B)x_n, \\ x_{n+1} &= \alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n, \end{aligned} \tag{38}$$

where $\lambda\in(0,2\alpha], \{\alpha_n\}, \{\beta_n\}\subset[0,1]$. If the following conditions are satisfied:

(C_1) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C_2) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,

then $\{x_n\}$ converges strongly to $q\in\mathcal{F}$, where $q=P_{\mathcal{F}}(\gamma f+(I-A))q$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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