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Research Article

Strong Convergence of a Unified General Iteration for *k*-Strictly Pseudononspreading Mapping in Hilbert Spaces

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We introduce a unified general iterative method to approximate a fixed point of k-strictly pseudononspreading mapping. Under some suitable conditions, we prove that the iterative sequence generated by the proposed method converges strongly to a fixed point of a k-strictly pseudononspreading mapping with an idea of mean convergence, which also solves a class of variational inequalities as an optimality condition for a minimization problem. The results presented in this paper may be viewed as a refinement and as important generalizations of the previously known results announced by many other authors.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Recall that a mapping $T : C \to C$ is said to be k-strict pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1}$$

If $k = 0$, T is said to be nonexpansive mapping; that is,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

The set of fixed points of T is denoted by $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$. Recall also that a mapping $T : C \to C$ is said to be nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \tag{3}$$

It is shown in the study by Iemoto and Takahashi [1] that (3) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{4}$$

Observe that every nonspreading mapping is quasinonexpansive; that is, $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and all $p \in F(T)$. Following the terminology of Browder and Petryshyn [2], a mapping $T : C \to C$ is called k-strictly pseudononspreading if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{5}$$

Clearly, every nonspreading mapping is k-strictly pseudononspreading, but the converse is not true. This shows that the class of k-strictly pseudononspreading mappings is more general than the class of nonspreading mappings. Moreover, we remark also that the class of k-strictly pseudononspreading mappings is independent of the class of k-strict pseudocontractions.

Fixed point problem of nonlinear mappings recently becomes an attractive subject because of its application in solving variational inequalities and equilibrium problems arising in various fields of pure and applied sciences. Moreover, various iterative schemes and methods have been developed for finding fixed points of nonlinear mappings. It is worth mentioning that iterative methods for nonexpansive and nonspreading mappings have been extensively investigated. However, iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [2] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction. This case is aggravated by adding another inner product to the right-hand side of (5) for k-strictly pseudononspreading mapping; see, for example, [3–13] and the references therein. On the other hand, k-strictly pseudononspreading mappings have more powerful applications than nonexpansive mappings do in solving mean ergodic problems; see, for example, [14, 15]. Therefore, it is interesting to develop the effective numerical methods for approximating fixed point of k-strictly pseudononspreading mapping.

In 2006, Marino and Xu [10] introduced a general iterative method and proved that, for a given $x_0 \in H$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) T x_n, \quad \forall n \geq 1, \tag{6}$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself, $\{\alpha_n\} \subseteq (0,1)$ satisfies certain conditions, and B is a strongly positive bounded linear operator on H, converges strongly to $x^* \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f) x^*, x^* - w \rangle \leq 0, \quad \forall w \in F(T), \tag{7}$$

and is also the optimality condition of problem $\min_{x\in C}(1/2)\langle Bx, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x), \forall x \in H$). Thereafter, the general iterative method is used to find a common element of the set of fixed point problems and the set of solutions of variational inequalities and equilibrium problems (see, e.g., [11–13]).

Recently, Kurokawa and Takahashi [14] obtained a weak mean ergodic theorem for nonspreading mappings in Hilbert spaces. Furthermore, they proved a strong convergence theorem using an idea of mean convergence. In 2011, Osilike and Isiogugu [15] introduced a more general k-strictly pseudononspreading mapping and considered the following iterative schemes:

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1 - \alpha_n) z_n, \\ z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T_\beta^k x_n, \quad n \geq 1, \end{aligned} \tag{8}$$

where auxiliary mapping $T_\beta = \beta I + (1 - \beta) T$. They proved that the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)} u$, which is the metric projection of H onto $F(T)$. Moreover, they considered the following Halpern type iterative scheme:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\beta x_n, \quad n \geq 1. \tag{9}$$

They also proved that $\{x_n\}$ generated by (9) converges strongly to $q \in F(T)$ under some suitable conditions and hence resolved in the affirmative the open problem raised by Kurokawa and Takahashi [14] in their final remark for the case where the mapping T is averaged.

In 2013, Kangtunyakarn [16] further studied variational inequalities and fixed point problem of k-strictly pseudononspreading mapping T by modifying the auxiliary mapping with projection technique. To be more precise, he introduced the following iterative scheme:

$$x_{n+1} = \alpha_n u + \beta_n P_C [I - \lambda_n (I - T)] x_n + \gamma_n S x_n, \quad n \geq 1, \tag{10}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\beta_n \in [c, d] \subset (0,1)$ and S is a nonexpansive mapping generated by a finite family of defining operators, whose fixed point problems are equivalent to variational inequalities. Moreover, under some suitable conditions, he proved that the sequence $\{x_n\}$ converges strongly to $P_\Omega u$, where Ω is the intersection of the set of fixed point problems and the set of solutions for variational inequalities.

Inspired and motivated by research going on in this area, we introduce a modified general iterative method for k-strictly pseudononspreading mapping, which is defined in the following way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n B] T_{\lambda_n} x_n, \quad n \geq 1, \tag{11}$$

where $T_{\lambda_n} = P_C[I - \lambda_n(I - T)]$ with $\lambda_n \in (0,1)$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$. Note that, if $\beta_n = 0$, scheme (11) reduces to general iterative method (6), which is mainly due to Marino and Xu [10]. If $\beta_n = 0$, $\gamma = 1$, and $B = I$, scheme (11) reduces to viscosity approximate method introduced by Moudafi [17] and developed by Inchan [18], which also extends the Halpern type results of [19, 20] with an idea of mean convergence for k-strictly pseudononspreading mapping.

Our purpose is not only to modify the general iterative method (6) and projection method (10) to the case of a k-strictly pseudononspreading mapping, but also to establish a new strong convergence theorem with an idea of mean convergence for a k-strictly pseudononspreading mapping, which also solves a class of variational inequalities as an optimality condition for a minimization problem. Our main results presented in this paper improve and extend the corresponding results of [10, 14–17] and many others.

2. Preliminaries

Let C be a nonempty closed convex subset of real Hilbert H space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_C, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{12}$$

Then P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and the following inequality holds:

$$\langle x - u, u - y \rangle \geq 0, \quad \forall y \in C, \tag{13}$$

if and only if $u = P_C x$ for given $x \in H$ and $u \in C$.

Let A be a mapping from C into H. The normal variational inequality problem is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{14}$$

The set of all solutions of the variational inequality is denoted by $VI(C, A)$. Note that $u \in VI(C, A)$ if and only if $u = P_C(I - \lambda A)u$ for some $\lambda > 0$.

Recall that an operator B is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{15}$$

Recall also that an operator $f : C \to C$ is a contraction, if there exists a constant $\rho \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{16}$$

In order to prove our main results, we need the following lemmas and propositions.

Lemma 1. *Let H be a real Hilbert space. There hold the following well-known results:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle$, $\forall x, y \in H$;

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $t \in [0, 1]$, $\forall x, y \in H$.

Lemma 2 (see [6]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ and*

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0, \quad \forall n \geq 0. \tag{17}$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 3 (see [10]). *Let B be a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \leq 1 - \varrho\overline{\gamma}$.*

Lemma 4 (see [10]). *Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0, 1)$ and B is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}/\rho$,*

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in H. \tag{18}$$

That is, $B - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\rho$.

Lemma 5 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping. Then $I - T$ is demiclosed at zero.*

Lemma 6 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 7 (see [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, (I - T))$.*

Lemma 8 (see [21]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0, \tag{19}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Theorem 9. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strictly pseudononspreading mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0, 1)$ and let B be a strongly positive bounded linear operator with $\overline{\gamma} > 0$. For a given point $x_0 \in C$ and $0 < \gamma < \overline{\gamma}/\rho$, assume that $\alpha_n, \beta_n, \lambda_n \in [0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lambda_n \in (0, 1 - k)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then the sequence $\{x_n\}$ generated by (11) converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w \rangle \leq 0, \quad \forall w \in F(T). \tag{20}$$

Proof. First, we show that sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Indeed, from the property of k-strictly pseudononspreading mapping defined on T and $p \in F(T)$, we have

$$\begin{aligned}
&\|Tx_n - Tp\|^2 \\
&\quad = \|(x_n - p) - [(I - T)x_n - (I - T)p]\|^2 \\
&\quad = \|x_n - p\|^2 - 2\langle x_n - p, (I - T)x_n \rangle + \|(I - T)x_n\|^2 \\
&\quad \leq \|x_n - p\|^2 + k\|(I - T)x_n - (I - T)p\|^2 \\
&\qquad + 2\langle (I - T)x_n, (I - T)p \rangle \\
&\quad = \|x_n - p\|^2 + k\|(I - T)x_n\|^2,
\end{aligned} \tag{21}$$

which implies that

$$(1-k)\left\|(I-T)x_n\right\|^2 \le 2\left\langle x_n-p,(I-T)x_n\right\rangle. \tag{22}$$

From $T_{\lambda_n} = P_C[I-\lambda_n(I-T)]$ and (22), we obtain

$$\begin{aligned}
&\left\|T_{\lambda_n}x_n-p\right\|^2\\
&\quad\le \left\|(x_n-p)-\lambda_n[(I-T)x_n-(I-T)p]\right\|^2\\
&\quad= \left\|x_n-p\right\|^2-2\lambda_n\left\langle x_n-p,(I-T)x_n\right\rangle+\lambda_n^2\left\|(I-T)x_n\right\|^2\\
&\quad\le \left\|x_n-p\right\|^2-\lambda_n(1-k)\left\|(I-T)x_n\right\|^2+\lambda_n^2\left\|(I-T)x_n\right\|^2\\
&\quad= \left\|x_n-p\right\|^2-\lambda_n\left[(1-k)-\lambda_n\right]\left\|(I-T)x_n\right\|^2 \le \left\|x_n-p\right\|^2.
\end{aligned} \tag{23}$$

By (i) and Lemma 3, we have that $(1-\beta_n)I-\alpha_nB$ is positive and $\|(1-\beta_n)I-\alpha_nB\| \le 1-\beta_n-\alpha_n\overline{\gamma}$ for all $n \ge 1$ (see, i.e., [8]). It follows from (11) and (23) that

$$\begin{aligned}
&\left\|x_{n+1}-p\right\|\\
&\quad= \Big\|\alpha_n\left(\gamma f(x_n)-Bp\right)+\beta_n\left(x_n-p\right)\\
&\qquad+\left[(1-\beta_n)I-\alpha_nB\right]\left(T_{\lambda_n}x_n-p\right)\Big\|\\
&\quad\le \alpha_n\left\|\gamma f(x_n)-Bp\right\|\\
&\qquad+\beta_n\left\|x_n-p\right\|+\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|T_{\lambda_n}x_n-p\right\|\\
&\quad\le \alpha_n\gamma\left\|f(x_n)-f(p)\right\|+\alpha_n\left\|\gamma f(p)-Bp\right\|\\
&\qquad+\beta_n\left\|x_n-p\right\|+\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|x_n-p\right\|\\
&\quad\le \left[1-\alpha_n\left(\overline{\gamma}-\gamma\rho\right)\right]\left\|x_n-p\right\|+\alpha_n\left\|\gamma f(p)-Bp\right\|.
\end{aligned} \tag{24}$$

By induction, we have that

$$\left\|x_n-p\right\| \le \max\left\{\left\|x_0-p\right\|,\frac{1}{\overline{\gamma}-\gamma\rho}\left\|\gamma f(p)-Bp\right\|\right\}. \tag{25}$$

Therefore, $\{x_n\}$ is bounded and so is $\{T_{\lambda_n}x_n\}$. On the other hand, we estimate

$$\begin{aligned}
&\left\|Tx_n-p\right\|^2\\
&\quad\le \left\|x_n-p\right\|^2+k\left\|(I-T)x_n-(I-T)p\right\|^2\\
&\qquad+2\left\langle x_n-Tx_n,p-Tp\right\rangle\\
&\quad= \left\|x_n-p\right\|^2+k\left\|(x_n-p)-(Tx_n-p)\right\|^2\\
&\quad= \left\|x_n-p\right\|^2+k\Big(\left\|x_n-p\right\|^2-2\langle x_n-p,Tx_n-p\rangle\\
&\qquad+\left\|Tx_n-p\right\|^2\Big),
\end{aligned} \tag{26}$$

which implies that

$$\begin{aligned}
(1-k)\left\|Tx_n-p\right\|^2 &\le (1+k)\left\|x_n-p\right\|^2\\
&\quad+2k\left\|x_n-p\right\|\left\|Tx_n-p\right\|.
\end{aligned} \tag{27}$$

From (27), we can obtain

$$\begin{aligned}
0 &\ge (1-k)\left\|Tx_n-p\right\|^2\\
&\quad-(1+k)\left\|x_n-p\right\|^2-2k\left\|x_n-p\right\|\left\|Tx_n-p\right\|\\
&= (1-k)\left(\left\|Tx_n-p\right\|^2+\left\|x_n-p\right\|\left\|Tx_n-p\right\|\right)\\
&\quad-(1+k)\left(\left\|x_n-p\right\|^2+\left\|x_n-p\right\|\left\|Tx_n-p\right\|\right)\\
&= (1-k)\left\|Tx_n-p\right\|\left(\left\|Tx_n-p\right\|+\left\|x_n-p\right\|\right)\\
&\quad-(1+k)\left\|x_n-p\right\|\left(\left\|x_n-p\right\|+\left\|Tx_n-p\right\|\right).
\end{aligned} \tag{28}$$

It follows that

$$\left\|Tx_n-p\right\| \le \frac{1+k}{1-k}\left\|x_n-p\right\|. \tag{29}$$

Combining (25) and (29), we conclude that $\{Tx_n\}$ is bounded.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. To do this, define a sequence $\{z_n\}$ by

$$x_{n+1} = \beta_nx_n+\left(1-\beta_n\right)z_n, \quad n \ge 1. \tag{30}$$

Observe that

$$\begin{aligned}
&z_{n+1}-z_n\\
&\quad= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\\
&\quad= \frac{\alpha_{n+1}\gamma f(x_{n+1})+\left[(1-\beta_{n+1})I-\alpha_{n+1}B\right]w_{n+1}}{1-\beta_{n+1}}\\
&\qquad-\frac{\alpha_n\gamma f(x_n)+\left[(1-\beta_n)I-\alpha_nB\right]w_n}{1-\beta_n}\\
&\quad= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\gamma f(x_{n+1})-Bw_{n+1}\right]+\left(w_{n+1}-w_n\right)\\
&\qquad-\frac{\alpha_n}{1-\beta_n}\left[\gamma f(x_n)-Bw_n\right],
\end{aligned} \tag{31}$$

where $w_n = T_{\lambda_n}x_n$, and

$$\begin{aligned}
&\left\|w_{n+1}-w_n\right\|\\
&\quad\le \left\|\left(I-\lambda_{n+1}(I-T)\right)x_{n+1}-\left(I-\lambda_n(I-T)\right)x_n\right\|\\
&\quad= \left\|x_{n+1}-x_n-\lambda_{n+1}(I-T)x_{n+1}+\lambda_n(I-T)x_n\right\|\\
&\quad\le \left\|x_{n+1}-x_n\right\|+\lambda_{n+1}\left\|(I-T)x_{n+1}-(I-T)x_n\right\|\\
&\qquad+\left|\lambda_{n+1}-\lambda_n\right|\left\|(I-T)x_n\right\|.
\end{aligned} \tag{32}$$

From (31) and (32), we obtain

$$\begin{aligned}
&\left\|z_{n+1}-z_n\right\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f\left(x_{n+1}\right)-B w_{n+1}\right\| \\
&\qquad +\left\|w_{n+1}-w_n\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f\left(x_n\right)-B w_n\right\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f\left(x_{n+1}\right)-B w_{n+1}\right\| \\
&\qquad +\left\|x_{n+1}-x_n\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f\left(x_n\right)-B w_n\right\| \\
&\qquad +\lambda_{n+1}\left\|(I-T) x_{n+1}-(I-T) x_n\right\| \\
&\qquad +\left|\lambda_{n+1}-\lambda_n\right|\left\|(I-T) x_n\right\| .
\end{aligned} \tag{33}$$

It follows from conditions (i)–(iii) and Lemma 2 that

$$\lim_{n \rightarrow \infty}\left\|z_n-x_n\right\|=0. \tag{34}$$

From (30) and (34) and condition (ii), we have

$$\lim_{n \rightarrow \infty}\left\|x_{n+1}-x_n\right\|=\lim_{n \rightarrow \infty}\left(1-\beta_n\right)\left\|z_n-x_n\right\|=0. \tag{35}$$

Moreover, note that $w_n=T_{\lambda_n} x_n$ and

$$\begin{aligned}
&\left\|x_n-w_n\right\| \\
&\quad \leq\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-w_n\right\| \\
&\quad =\left\|x_n-x_{n+1}\right\| \\
&\qquad +\left\|\alpha_n \gamma f\left(x_n\right)+\beta_n x_n+\left[\left(1-\beta_n\right) I-\alpha_n B\right] w_n-w_n\right\| \\
&\quad \leq\left\|x_n-x_{n+1}\right\|+\alpha_n\left\|\gamma f\left(x_n\right)-B w_n\right\|+\beta_n\left\|x_n-w_n\right\|,
\end{aligned} \tag{36}$$

which implies that

$$\left\|x_n-w_n\right\| \leq \frac{1}{1-\beta_n}\left\|x_n-x_{n+1}\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f\left(x_n\right)-B w_n\right\| . \tag{37}$$

Combining conditions (i)-(ii) and (35), we obtain

$$\lim_{n \rightarrow \infty}\left\|x_n-w_n\right\|=\lim_{n \rightarrow \infty}\left\|x_n-T_{\lambda_n} x_n\right\|=0. \tag{38}$$

That is,

$$\lim_{n \rightarrow \infty}\left\|x_n-P_C\left[I-\lambda_n(I-T)\right] x_n\right\|=0. \tag{39}$$

Next, we will prove that $\limsup_{n \rightarrow \infty}\langle\gamma f(q)-B q, x_n-q\rangle \leq 0$, where $q=P_{F(T)}(I-B+\gamma f) q$. To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n \rightarrow \infty}\left\langle\gamma f(q)-B q, x_n-q\right\rangle \\
&\quad =\lim_{j \rightarrow \infty}\left\langle\gamma f(q)-B q, x_{n_j}-q\right\rangle .
\end{aligned} \tag{40}$$

Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to w; that is, $x_{n_j} \rightharpoonup w$ as $j \rightarrow \infty$, where $w \in C$. We will show that $w \in F(T)$. From Lemmas 5 and 7, we have $F(T)=F(T_{\lambda_{n_j}})=F(P_C[I-\lambda_{n_j}(I-T)])$. Assume that $w \neq P_C[I-\lambda_{n_j}(I-T)] w$. By condition (iii), (38), and Opial's property, we obtain

$$\begin{aligned}
&\liminf_{j \rightarrow \infty}\left\|x_{n_j}-w\right\| \\
&\quad <\liminf_{j \rightarrow \infty}\left\|x_{n_j}-P_C\left[I-\lambda_{n_j}(I-T)\right] w\right\| \\
&\quad \leq \liminf_{j \rightarrow \infty}\left(\left\|x_{n_j}-T_{\lambda_{n_j}} x_{n_j}\right\|\right. \\
&\qquad\qquad +\left\|P_C\left[I-\lambda_{n_j}(I-T)\right] x_{n_j}\right. \\
&\qquad\qquad\qquad \left.\left.-P_C\left[I-\lambda_{n_j}(I-T)\right] w\right\|\right) \\
&\quad \leq \liminf_{j \rightarrow \infty}\left(\left\|x_{n_j}-T_{\lambda_{n_j}} x_{n_j}\right\|\right. \\
&\qquad\qquad +\left\|x_{n_j}-w\right\| \\
&\qquad\qquad \left.+\lambda_{n_j}\left\|(I-T) x_{n_j}-(I-T) w\right\|\right) \\
&\quad \leq \liminf_{j \rightarrow \infty}\left\|x_{n_j}-w\right\| .
\end{aligned} \tag{41}$$

This is a contradiction. Then $w \in F(T)$. Since $x_{n_j} \rightharpoonup w$ as $j \rightarrow \infty$, we have

$$\begin{aligned}
&\limsup_{n \rightarrow \infty}\left\langle\gamma f(q)-B q, x_n-q\right\rangle \\
&\quad =\lim_{j \rightarrow \infty}\left\langle\gamma f(q)-B q, x_{n_j}-q\right\rangle \\
&\quad =\langle\gamma f(q)-B q, w-q\rangle \leq 0 .
\end{aligned} \tag{42}$$

On the other hand, we will show the uniqueness of a solution of the variational inequality

$$\langle(B-\gamma f) x, x-w\rangle \leq 0, \quad w \in F(T). \tag{43}$$

Suppose $q \in F(T)$ and $\widehat{q} \in F(T)$ both are solutions to (43); then

$$\begin{aligned}
&\langle(B-\gamma f) q, q-\widehat{q}\rangle \leq 0, \\
&\langle(B-\gamma f) \widehat{q}, \widehat{q}-q\rangle \leq 0 .
\end{aligned} \tag{44}$$

Adding up (44), we get

$$\langle(B-\gamma f) q-(B-\gamma f) \widehat{q}, q-\widehat{q}\rangle \leq 0. \tag{45}$$

From Lemma 4, the strong monotonicity of $B-\gamma f$, we obtain $q=\widehat{q}$ and the uniqueness is proved.

Finally, we show that $\{x_n\}$ converges strongly to q as $n \to \infty$. From (11), (23), and Lemma 1, we have (note that $w_n = T_{\lambda_n} x_n$)

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \langle \alpha_n \gamma f(x_n) + \beta_n x_n \\
&\quad + [(1-\beta_n) I - \alpha_n B] w_n - q, x_{n+1} - q \rangle \\
&= \alpha_n \langle \gamma f(x_n) - Bq, x_{n+1} - q \rangle \\
&\quad + \langle [(1-\beta_n) I - \alpha_n B](w_n - q), x_{n+1} - q \rangle \\
&\quad + \beta_n \langle x_n - q, x_{n+1} - q \rangle \\
&\le \alpha_n \gamma \langle f(x_n) - f(q), x_{n+1} - q \rangle \\
&\quad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\quad + \beta_n \|x_n - q\| \|x_{n+1} - q\| \\
&\quad + (1 - \beta_n - \alpha_n \overline{\gamma}) \|w_n - q\| \|x_{n+1} - q\| \\
&\le \alpha_n \gamma \rho \|x_n - q\| \|x_{n+1} - q\| \\
&\quad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\quad + (1 - \alpha_n \overline{\gamma}) \|x_n - q\| \|x_{n+1} - q\| \\
&= [1 - (\overline{\gamma} - \gamma\rho)\alpha_n] \|x_n - q\| \|x_{n+1} - q\| \\
&\quad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2} \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\quad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2} \|x_n - q\|^2 + \frac{1}{2} \|x_{n+1} - q\|^2 \\
&\quad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle .
\end{aligned} \tag{46}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 \le{}& [1 - (\overline{\gamma} - \gamma\rho)\alpha_n] \|x_n - q\|^2 \\
&+ 2\alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle .
\end{aligned} \tag{47}$$

Together with $0 < \gamma < \overline{\gamma}/\rho$, condition (i), and (42), we can arrive at the desired conclusion $\lim_{n\to\infty} \|x_n - q\| = 0$ by Lemma 8. This completes the proof. □

Theorem 10. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strictly pseudononspreading mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0,1)$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ in the following manner:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) T_{\lambda_n} x_n, \quad n \ge 1, \tag{48}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lambda_n \in (0, 1-k)$ *and* $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f) q, q - w \rangle \le 0, \quad \forall\, w \in F(T). \tag{49}$$

Proof. Putting $B = I$ and $\gamma = 1$, general iterative scheme (11) reduces to viscosity iteration (48). The desired conclusion follows immediately from Theorem 9. This completes the proof. □

Theorem 11. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonspreading mapping (or quasinonexpansive) such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0,1)$ and let B be a strongly positive bounded linear operator with $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\rho$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ in the following manner:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n) I - \alpha_n B] T_{\lambda_n} x_n, \quad n \ge 1, \tag{50}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lambda_n \in (0, 1)$ *and* $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f) q, q - w \rangle \le 0, \quad \forall\, w \in F(T). \tag{51}$$

Proof. Clearly, every nonspreading mapping T is 0-strictly pseudononspreading, which is also quasinonexpansive. Therefore, the desired conclusion follows immediately from Theorem 9. This completes the proof. □

Remark 12. Theorems 9 and 10 extend the Halpern type methods of [14, 15] and viscosity methods of Moudafi [17] to more general unified general iterative methods for k-strictly pseudononspreading mapping, which also solves a class of variational inequalities related to an optimality problem.

Remark 13. Theorems 9 and 10 improve and extend the main results of Kangtunyakarn [16] for k-strictly pseudononspreading mapping in different directions.

Remark 14. The auxiliary mapping T_β of Osilike and Isiogugu [15] is generalized to the averaged mapping T_{λ_n} presented in scheme (11) with variable coefficient and projection operator based on the equivalence between variational inequality and fixed point problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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