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Research Article

Super-Hamiltonian Structures and Conservation Laws of a New Six-Component Super-Ablowitz-Kaup-Newell-Segur Hierarchy

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A six-component super-Ablowitz-Kaup-Newell-Segur (-AKNS) hierarchy is proposed by the zero curvature equation associated with Lie superalgebras. Supertrace identity is used to furnish the super-Hamiltonian structures for the resulting nonlinear superintegrable hierarchy. Furthermore, we derive the infinite conservation laws of the first two nonlinear super-AKNS equations in the hierarchy by utilizing spectral parameter expansions. PACS: 02.30.Ik; 02.30.Jr; 02.20.Sv.

1. Introduction

It is well known that many physically important integrable partial differential equations belong to the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [1–3], such as the KdV equation, the mKdV equation, the nonlinear Schrödinger equation, the Sin-Gordon equation, and the mixed KdV-mKdV equation. The AKNS hierarchy is based on the Zakharov and Shabat [4] spectral problem

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{1}$$

It possesses Lax representation, Hamiltonian structures, and infinitely many conserved quantities and can be solved by the method of inverse scattering, Hirota method, Darboux transform, and others.

The superintegrable systems had already aroused strong interest in theoretical physics [5, 6], where the fermion fields are added and equally treated with the boson fields. Many classical integrable equations have been extended to the super ones by adding fermion fields, such as the super-AKNS [6–10], the super-KdV [5], the super-Dirac [9, 11, 12], and the super-Kadomtsev Petviashvili (KP) [13–15]. The super-AKNS hierarchy was first proposed in [6] based on the superalgebra sl(2,R). Extension of this work to other and higher dimensional superalgebras is given in [16]. The super-AKNS matrix superspectral problem is

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}_x = \begin{pmatrix}-\lambda & q & \alpha\\ r & \lambda & \beta\\ -\beta & \alpha & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{2}$$

where ψ_3, α, and β are fermion fields. It reduces to the spectral AKNS's system as $\alpha = \beta = 0$.

In this paper, we consider a new 3×3 matrix superspectral problem which generates a six-component super-AKNS hierarchy. As we will show this spectral problem takes the spectral AKNS's system and super-AKNS's system as special cases.

The paper is organized as follows. In Section 2, we will construct a six-component super-AKNS hierarchy related to the 3×3 matrix superspectral problem. In Section 3, we present the super-Hamiltonian structures for the six-component super-AKNS hierarchy with the help of the supertrace identity. In Section 4, we consider some special reductions of the superintegrable hierarchy. In Section 5, we derive the infinite conservation laws for the associated hierarchy. The last section contains concluding remarks.

2. A Six-Component Super-AKNS Hierarchy

In this section, we will derive a new six-component super-AKNS hierarchy associated with the 3×3 matrix spectral problem. Let $\mathscr{G}$ be a commutative superalgebra over $\mathbb{R}$ and $\widetilde{\mathscr{G}}$ a matrix loop superalgebra over $\mathscr{G}$ with the nondegenerate Killing form. We take a matrix superspectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad U = \begin{pmatrix} -\lambda & q & \alpha_1 \\ r & \lambda & \beta_1 \\ \beta_2 & \alpha_2 & 0 \end{pmatrix}, \tag{3}$$

where ϕ_x and ϕ_t denote the partial derivatives with respect to x and t, $u = (q, r, \alpha_1, \alpha_2, \beta_1, \beta_2)^T \in \mathscr{G}^6$ is a potential consisting of commuting and anticommuting variables, $q, r, \lambda, \phi_1, \phi_2$ are the commuting variables, which can be indicated by the degree p as $p(q) = p(r) = p(\lambda) = p(\phi_1) = p(\phi_2) = 0$, and $\alpha_1, \alpha_2, \beta_1, \beta_2, \phi_3$ are the anticommuting variables, which can be indicated by the degree p as $p(\alpha_1) = p(\alpha_2) = p(\beta_1) = p(\beta_2) = p(\phi_3) = 1$. Here λ is assumed to be a constant spectral parameter (i.e., $\lambda_t = 0$).

Let us find the following temporal evolution equation associated with (3):

$$\phi_t = V\phi, \quad V = \begin{pmatrix} \frac{A+G}{2} & B & \rho \\ C & \frac{-A+G}{2} & \delta \\ \varepsilon & \tau & G \end{pmatrix}, \tag{4}$$

where A, B, C, and G are commuting fields and $\rho, \delta, \varepsilon$, and τ are anticommuting fields. From the stationary zero curvature equation

$$V_x = [U, V], \tag{5}$$

it gives rise to

$$\begin{aligned} A_x &= 2qC - 2rB + \alpha_1\varepsilon - \beta_1\tau - \alpha_2\delta + \beta_2\rho, \\ B_x &= -2\lambda B - qA + \alpha_1\tau + \alpha_2\rho, \\ C_x &= 2\lambda C + rA + \beta_1\varepsilon + \beta_2\delta, \\ G_x &= \alpha_1\varepsilon + \beta_1\tau + \alpha_2\delta + \beta_2\rho, \\ \rho_x &= -\lambda\rho + q\delta - \frac{1}{2}\alpha_1 A + \frac{1}{2}\alpha_1 G - \beta_1 B, \\ \delta_x &= \lambda\delta + r\rho - \alpha_1 C + \frac{1}{2}\beta_1 A + \frac{1}{2}\beta_1 G, \\ \tau_x &= -\lambda\tau - q\varepsilon - \frac{1}{2}\alpha_2 A - \frac{1}{2}\alpha_2 G + \beta_2 B, \\ \varepsilon_x &= \varepsilon - r\tau + \alpha_2 C + \frac{1}{2}\beta_2 A - \frac{1}{2}\beta_2 G. \end{aligned} \tag{6}$$

We put $A, B, C, G, \rho, \delta, \varepsilon$, and τ to be polynomial of λ:

$$\begin{aligned} A &= \sum_{m\ge0}^{\infty} A_m\lambda^{-m}, \qquad B = \sum_{m\ge0}^{\infty} B_m\lambda^{-m}, \\ C &= \sum_{m\ge0}^{\infty} C_m\lambda^{-m}, \qquad G = \sum_{m\ge0}^{\infty} G_m\lambda^{-m}, \\ \rho &= \sum_{m\ge0}^{\infty} \rho_m\lambda^{-m}, \qquad \varepsilon = \sum_{m\ge0}^{\infty} \varepsilon_m\lambda^{-m}, \\ \delta &= \sum_{m\ge0}^{\infty} \delta_m\lambda^{-m}, \qquad \tau = \sum_{m\ge0}^{\infty} \tau_m\lambda^{-m}, \end{aligned} \tag{7}$$

and substituting (7) into (6) and equating the coefficients of λ, we obtain

$$\begin{aligned} A_{m,x} &= 2qC_m - 2rB_m + \alpha_1\varepsilon_m - \beta_1\tau_m - \alpha_2\delta_m + \beta_2\rho_m, \\ B_{m,x} &= -2B_{m+1} - qA_m + \alpha_1\tau_m + \alpha_2\rho_m, \\ C_{m,x} &= 2C_{m+1} + rA_m + \beta_1\varepsilon_m + \beta_2\delta_m, \\ G_{m,x} &= \alpha_1\varepsilon_m + \beta_1\tau_m + \alpha_2\delta_m + \beta_2\rho_m, \\ \rho_{m,x} &= -\rho_{m+1} + q\delta_m - \frac{1}{2}\alpha_1 A_m + \frac{1}{2}\alpha_1 G_m - \beta_1 B_m, \\ \delta_{m,x} &= \delta_{m+1} + r\rho_m - \alpha_1 C_m + \frac{1}{2}\beta_1 A_m + \frac{1}{2}\beta_1 G_m, \\ \tau_{m,x} &= -\tau_{m+1} - q\varepsilon_m - \frac{1}{2}\alpha_2 A_m - \frac{1}{2}\alpha_2 G_m + \beta_2 B_m, \\ \varepsilon_{m,x} &= \varepsilon_{m+1} - r\tau_m + \alpha_2 C_m + \frac{1}{2}\beta_2 A_m - \frac{1}{2}\beta_2 G_m. \end{aligned} \tag{8}$$

Upon choosing the initial data

$$\begin{gathered} A_0 = -2, \qquad B_0 = C_0 = \rho_0 = \delta_0 = \varepsilon_0 = \tau_0 = 0, \\ G_0 = -g_0 = \text{constant}, \end{gathered} \tag{9}$$

then the recursion relations in (8) uniquely define a series of sets of differential polynomial functions in u with respect to x. The first two sets are as follows:

$$\begin{gathered} B_1 = q, \qquad C_1 = r, \qquad A_1 = 0, \\ G_1 = 0, \qquad \rho_1 = -\frac{1}{2}\alpha_1(g_0 - 2), \qquad \tau_1 = \frac{1}{2}\alpha_2(g_0 + 2), \\ \delta_1 = \frac{1}{2}\beta_1(g_0 + 2), \qquad \varepsilon_1 = -\frac{1}{2}\beta_2(g_0 - 2), \\ B_2 = -\frac{1}{2}(q_x - g_0\alpha_1\alpha_2), \qquad C_2 = \frac{1}{2}(r_x + g_0\beta_1\beta_2), \\ \rho_2 = -\alpha_{1,x} + \frac{1}{2}g_0(\alpha_{1,x} + q\beta_1), \\ \tau_2 = -\alpha_{2,x} - \frac{1}{2}g_0(\alpha_{2,x} - q\beta_2), \end{gathered}$$

$$\delta_2 = \beta_{1,x} + \frac{1}{2}g_0(\beta_{1,x} + r\alpha_1),$$
$$A_2 = qr + \alpha_1\beta_2 - \alpha_2\beta_1 - \frac{1}{2}g_0(\alpha_1\beta_2 + \alpha_2\beta_1),$$
$$\varepsilon_2 = \beta_{2,x} - \frac{1}{2}g_0(\beta_{2,x} - r\alpha_2),$$
$$G_2 = \alpha_1\beta_2 + \alpha_2\beta_1 - \frac{1}{2}g_0(\alpha_1\beta_2 - \alpha_2\beta_1). \tag{10}$$

From the recursion relations in (8), we can obtain the hereditary recursion operator L which satisfies that

$$(C_{m+1}, B_{m+1}, -\varepsilon_{m+1}, -\tau_{m+1}, \delta_{m+1}, \rho_{m+1})^T = L(C_m, B_m, -\varepsilon_m, -\tau_m, \delta_m, \rho_m)^T, \tag{11}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix},$$
$$L_{11} = \begin{pmatrix} -\frac{1}{2}\partial + r\partial^{-1}q & -r\partial^{-1}r \\ q\partial^{-1}q - \frac{1}{2}\alpha_1 & \frac{1}{2}\partial - q\partial^{-1}r \end{pmatrix},$$
$$L_{12} = \begin{pmatrix} -\frac{r}{2}\partial^{-1}\alpha_1 & \frac{r}{2}\partial^{-1}\beta_1 \\ -\partial^{-1}\alpha_1 & \partial^{-1}\beta_1 \end{pmatrix},$$
$$L_{13} = \begin{pmatrix} -\frac{r}{2}\partial^{-1}\alpha_2 & \frac{r}{2}\partial^{-1}\beta_2 \\ -\partial^{-1}\alpha_2 & \partial^{-1}\beta_2 - \frac{1}{2}\alpha_2 \end{pmatrix},$$
$$L_{21} = \begin{pmatrix} -\alpha_2 - \beta_2\partial^{-1}q & \beta_2\partial^{-1}r \\ -\alpha_2\partial^{-1}q & \alpha_2\partial^{-1}r + \beta_2 \end{pmatrix}, \tag{12}$$
$$L_{22} = \begin{pmatrix} -\partial - \partial^{-1}\alpha_1 - \frac{1}{2}\beta_2\partial^{-1}\alpha_1 & -r - \beta_2\partial^{-1}\beta_1 \\ q + \alpha_2\partial^{-1}\alpha_1 & \partial \end{pmatrix},$$
$$L_{23} = \begin{pmatrix} \beta_2\partial^{-1}\alpha_2 & 0 \\ 0 & -\alpha_2\partial^{-1}\beta_2 \end{pmatrix},$$
$$L_{31} = \begin{pmatrix} -\alpha_1 + \beta_1\partial^{-1}q & -\beta_1\partial^{-1}r \\ \alpha_1\partial^{-1}q & -\alpha_1\partial^{-1}r + \beta_1 \end{pmatrix},$$
$$L_{32} = \begin{pmatrix} -\beta_1\partial^{-1}\alpha_1 & 0 \\ 0 & \alpha_1\partial^{-1}\beta_1 \end{pmatrix},$$
$$L_{33} = \begin{pmatrix} -\partial & r + \beta_1\partial^{-1}\beta_2 \\ -q - \alpha_1\partial^{-1}\alpha_2 & \partial + \partial^{-1}\beta_2 - \frac{1}{2}\alpha_1\partial^{-1}\beta_2 \end{pmatrix}.$$

Taking

$$V^{(n)} = (\lambda^n V)_+ = \sum_{j=0}^{n} \begin{pmatrix} \frac{A_j + G_j}{2} & B_j & \rho_j \\ C_j & \frac{-A_j + G_j}{2} & \delta_j \\ \varepsilon_j & \tau_j & G_j \end{pmatrix} \lambda^{n-j}, \tag{13}$$

here $(\lambda^n V)_+$ denotes the polynomial part of $\lambda^n V$.

The compatibility conditions (i.e., zero curvature equation)

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{14}$$

of the matrix superspectral problems

$$\phi_x = U\phi, \quad \phi_{t_n} = V^{(n)}\phi, \quad n \geq 0, \tag{15}$$

determine a new six-component super-AKNS integrable soliton hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}_{t_n} = K_n(u) = \begin{pmatrix} -2B_{n+1} \\ 2C_{n+1} \\ -\rho_{n+1} \\ \delta_{n+1} \\ -\tau_{n+1} \\ \varepsilon_{n+1} \end{pmatrix}, \quad n \geq 0. \tag{16}$$

3. The Super-Hamiltonian Structures

In this section, we will establish the super-Hamiltonian structure of the six-component super-AKNS hierarchy by supertrace identity [9, 17]

$$\frac{\delta}{\delta u}\int \mathrm{Str}\left(V\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{Str}\left(V\frac{\partial U}{\partial u}\right), \tag{17}$$

where the constant γ is determined by

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{Str}(VV)|. \tag{18}$$

Through direct calculations, we have

$$\mathrm{Str}\left(V\frac{\partial U}{\partial \lambda}\right) = -A, \qquad \mathrm{Str}\left(V\frac{\partial U}{\partial q}\right) = C,$$
$$\mathrm{Str}\left(V\frac{\partial U}{\partial r}\right) = B, \qquad \mathrm{Str}\left(V\frac{\partial U}{\partial \alpha_1}\right) = -\varepsilon,$$
$$\mathrm{Str}\left(V\frac{\partial U}{\partial \beta_1}\right) = -\tau, \qquad \mathrm{Str}\left(V\frac{\partial U}{\partial \alpha_2}\right) = \delta, \tag{19}$$
$$\mathrm{Str}\left(V\frac{\partial U}{\partial \beta_2}\right) = \rho.$$

Substituting the above results into the supertrace identity (17) yields that

$$\frac{\delta}{\delta u}\int -A\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(C, B, -\varepsilon, -\tau, \delta, \rho)^T. \tag{20}$$

Comparing the coefficients of λ^{-m-2} on both sides of (20) gives rise to

$$\frac{\delta}{\delta u}\int -A_{m+2}dx = (\gamma - m - 1)\begin{pmatrix} C_{m+1} \\ B_{m+1} \\ -\varepsilon_{m+1} \\ -\tau_{m+1} \\ \delta_{m+1} \\ \rho_{m+1} \end{pmatrix}, \quad m \geq 0. \tag{21}$$

By employing the computing formula (18) on the constant γ, we obtain $\gamma = 0$. Thus we have

$$\frac{\delta \mathscr{H}_m}{\delta u} = \begin{pmatrix} C_{m+1} \\ B_{m+1} \\ -\varepsilon_{m+1} \\ -\tau_{m+1} \\ \delta_{m+1} \\ \rho_{m+1} \end{pmatrix}, \quad \mathscr{H}_m = \int \frac{2}{m+1}A_{m+1}dx, \quad m \geq 0. \tag{22}$$

It then follows that the superintegrable hierarchy (16) possesses the following super-Hamiltonian form:

$$u_{t_n} = K_n(u) = J\frac{\delta \mathscr{H}_n}{\delta u}, \quad n \geq 0, \tag{23}$$

where the super-Hamiltonian operator J is given by

$$J = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

We note that the recursion operator L is an integrodifferential operator, but the generalized superintegrable system (23) is pure differential equations according to [12].

4. Reductions

We now consider the possible reductions of our six-component super-AKNS hierarchy.

Assuming $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, hierarchy (23) reduces to the classical AKNS hierarchy [1]. Taking $\alpha_1 = -\alpha_2, \beta_2 = \beta_1$, we can have the super-AKNS hierarchy [6, 9, 10].

When $n = 1$ in (23), we obtain the first-order nonlinear superintegrable equations

$$\begin{aligned} q_{t_1} &= q_x - g_0\alpha_1\alpha_2, \\ r_{t_1} &= r_x + g_0\beta_1\beta_2, \\ \alpha_{1,t_1} &= \alpha_{1,x} - \frac{1}{2}g_0\alpha_{1,x} - \frac{1}{2}g_0 q\beta_1, \\ \alpha_{2,t_1} &= \alpha_{2,x} + \frac{1}{2}g_0\alpha_{2,x} - \frac{1}{2}g_0 q\beta_2, \\ \beta_{1,t_1} &= \beta_{1,x} + \frac{1}{2}g_0\beta_{1,x} + \frac{1}{2}g_0 r\alpha_1, \\ \beta_{2,t_1} &= \beta_{2,x} - \frac{1}{2}g_0\beta_{2,x} + \frac{1}{2}g_0 r\alpha_2. \end{aligned} \tag{25}$$

Taking $n = 2$ in (23), we can obtain the second-order nonlinear superintegrable equations

$$\begin{aligned} q_{t_2} &= -\frac{1}{2}q_{xx} + q^2 r + (g_0 - 1)(\alpha_{1,x}\alpha_2 - q\alpha_1\beta_2) \\ &\quad - (g_0 + 1)(\alpha_{2,x}\alpha_1 + q\alpha_2\beta_1), \\ r_{t_2} &= \frac{1}{2}r_{xx} - qr^2 + (g_0 + 1)(\beta_{1,x}\beta_2 + r\alpha_2\beta_1) \\ &\quad + (g_0 - 1)(r\alpha_1\beta_2 - \beta_{2,x}\beta_1), \\ \alpha_{1,t_2} &= \left(\frac{1}{2}g_0 - 1\right)\alpha_{1,xx} + \frac{1}{2}(g_0 - 1)\beta_1 q_x - q\beta_{1,x} \\ &\quad + \frac{1}{2}(1 - g_0)qr\alpha_1 - \alpha_1\alpha_2\beta_1, \\ \alpha_{2,t_2} &= \left(-\frac{1}{2}g_0 - 1\right)\alpha_{2,xx} + \frac{1}{2}(g_0 + 1)\beta_2 q_x + q\beta_{2,x} \\ &\quad + \frac{1}{2}(1 + g_0)qr\alpha_2 - \alpha_1\alpha_2\beta_2, \\ \beta_{1,t_2} &= \left(\frac{1}{2}g_0 + 1\right)\beta_{1,xx} + \frac{1}{2}(g_0 + 1)\alpha_1 r_x + r\alpha_{1,x} \\ &\quad - \frac{1}{2}(1 + g_0)qr\beta_1 + \alpha_1\beta_1\beta_2, \\ \beta_{2,t_2} &= \left(1 - \frac{1}{2}g_0\right)\beta_{2,xx} + \frac{1}{2}(g_0 - 1)\alpha_2 r_x - r\alpha_{2,x} \\ &\quad + \frac{1}{2}(g_0 - 1)qr\beta_2 + \alpha_2\beta_1\beta_2. \end{aligned} \tag{26}$$

In particular, letting $g_0 = 1$ in (26), we have

$$\begin{aligned} q_{t_2} &= -\frac{1}{2}q_{xx} + q^2 r - 2\alpha_{2,x}\alpha_1 - 2q\alpha_2\beta_1, \\ r_{t_2} &= \frac{1}{2}r_{xx} - qr^2 + 2\beta_{1,x}\beta_2 + 2r\alpha_2\beta_1, \\ \alpha_{1_{t_2}} &= -\frac{1}{2}\alpha_{1,xx} - q\beta_{1,x} - \alpha_1\alpha_2\beta_1, \\ \alpha_{2_{t_2}} &= -\frac{3}{2}\alpha_{2,xx} + \beta_2 q_x + q\beta_{2,x} + qr\alpha_2 - \alpha_1\alpha_2\beta_2, \\ \beta_{1_{t_2}} &= \frac{3}{2}\beta_{1,xx} + \alpha_1 r_x + r\alpha_{1,x} - qr\beta_1 + \alpha_1\beta_1\beta_2, \\ \beta_{2_{t_2}} &= \frac{1}{2}\beta_{2,xx} - r\alpha_{2,x} + \alpha_2\beta_1\beta_2, \end{aligned} \tag{27}$$

and taking $g_0 = -1$, (26) becomes

$$\begin{aligned} q_{t_2} &= -\frac{1}{2}q_{xx} + q^2 r - 2\alpha_{1,x}\alpha_2 + 2q\alpha_1\beta_2, \\ r_{t_2} &= \frac{1}{2}r_{xx} - qr^2 + 2\beta_{2,x}\beta_1 - 2r\alpha_1\beta_2, \\ \alpha_{1_{t_2}} &= -\frac{3}{2}\alpha_{1,xx} - \beta_1 q_x - q\beta_{1,x} + 2qr\alpha_1 - \alpha_1\alpha_2\beta_1, \\ \alpha_{2_{t_2}} &= -\frac{1}{2}\alpha_{2,xx} + q\beta_{2,x} - \alpha_1\alpha_2\beta_2, \\ \beta_{1_{t_2}} &= \frac{1}{2}\beta_{1,xx} + r\alpha_{1,x} + \alpha_1\beta_1\beta_2, \\ \beta_{2_{t_2}} &= \frac{3}{2}\beta_{2,xx} - r_x\alpha_2 - r\alpha_{2,x} - qr\beta_2 + \alpha_2\beta_1\beta_2. \end{aligned} \tag{28}$$

If we choose $\alpha_1 = -\alpha_2 = \alpha$, $\beta_2 = \beta_1 = \beta$, and $g_0 = 0$, (26) can be reduced to the second-order super-AKNS equations [10]

$$\begin{aligned} q_{t_2} &= -\frac{1}{2}q_{xx} + q^2 r + 2q\alpha\beta - 2\alpha\alpha_x, \\ r_{t_2} &= \frac{1}{2}r_{xx} - qr^2 - 2r\alpha\beta - 2\beta\beta_x, \\ \alpha_{t_2} &= -\alpha_{xx} - q\beta_x + \frac{1}{2}qr\alpha - \frac{1}{2}q_x\beta, \\ \beta_{t_2} &= \beta_{xx} + r\alpha_x + \frac{1}{2}r_x\alpha - \frac{1}{2}qr\beta, \end{aligned} \tag{29}$$

which is just the coupled nonlinear Schrödinger equations, also called Manakov equations

$$\begin{aligned} u_T &= -u_{XX} + u^2 v, \\ v_T &= v_{XX} - uv^2, \end{aligned} \tag{30}$$

as $\alpha = \beta = 0$, $q = u$, $r = v$, $t_2 = T$, and $x = 2X$.

5. Infinite Conservation Laws

In what follows, we will derive infinite conservation laws of (25) and (26). From the spectral problem (3), we can introduce the variables

$$M = \frac{\phi_2}{\phi_1}, \qquad N = \frac{\phi_3}{\phi_1}, \tag{31}$$

and then we obtain

$$\begin{aligned} M_x &= r + 2\lambda M + \beta_1 N - qM^2 - \alpha_1 MN, \\ N_x &= \beta_2 + \alpha_2 M + \lambda N - qMN. \end{aligned} \tag{32}$$

Next, we expand M and N as series of the spectral parameter λ,

$$M = \sum_{j=1}^{\infty} M_j \lambda^{-j}, \qquad N = \sum_{j=1}^{\infty} N_j \lambda^{-j}, \tag{33}$$

where M_j are even, $p(M_j) = 0$, and N_j are odd, $p(N_j) = 1$.

By substituting (33) into (32) and comparing the coefficients of λ, we raise the recursion formulas for M_j and N_j,

$$\begin{aligned} M_{j+1} &= \frac{1}{2}\Bigg(-r\delta_{n,0} + M_{j,x} - \beta_1 N_j \\ &\quad + q\sum_{l=1}^{j-1} M_j M_{j-l} + \alpha_1 \sum_{l=1}^{j-1} M_j N_{j-l}\Bigg), \\ &\quad (j \geq 2), \\ N_{j+1} &= -\beta_2\delta_{n,0} + N_{j,x} - \alpha_2 M_j \\ &\quad + q\sum_{l=1}^{j-1} M_j N_{j-l}, \quad (j \geq 2). \end{aligned} \tag{34}$$

We write below the first few terms of M_j and N_j:

$$\begin{aligned} M_1 &= -\frac{1}{2}r, \qquad N_1 = -\beta_2, \\ M_2 &= -\frac{1}{4}r_x + \beta_1\beta_2, \qquad N_2 = -\beta_{2,x} + \frac{1}{2}r\alpha_2, \\ M_3 &= -\frac{1}{8}r_{xx} + \frac{1}{2}\beta_{1,x}\beta_2 - \beta_{2,x}\beta_1 + \frac{1}{4}r\alpha_2\beta_1 \\ &\quad + \frac{1}{4}r\alpha_1\beta_2 + \frac{1}{8}qr^2, \\ N_3 &= -\beta_{2,xx} + \frac{3}{4}\alpha_2 r_x + \frac{1}{2}r\alpha_{2,x} - \alpha_2\beta_1\beta_2 + \frac{1}{2}qr\beta_2. \end{aligned} \tag{35}$$

On the other hand, it is easy to see that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\log\phi_1 = \frac{\partial}{\partial x}\frac{\partial}{\partial t}\log\phi_1 \quad \text{or} \quad \frac{\partial}{\partial t}\frac{\phi_{1,x}}{\phi_1} = \frac{\partial}{\partial x}\frac{\phi_{1,t}}{\phi_1}, \tag{36}$$

which implies

$$\frac{\partial}{\partial t}(-\lambda + qM + \alpha_1 N) = \frac{\partial}{\partial x}\left(\frac{1}{2}(A+G) + BM + \rho N\right), \tag{37}$$

and then the form of the conservation law is

$$\mu_t = \nu_x, \tag{38}$$

with the assumption that $\mu = -\lambda + qM + \alpha_1 N$, $\nu = (1/2)(A + G) + BM + \rho N$.

With regard to (25), we have

$$\begin{aligned} A = -2\lambda, \qquad B &= q, \qquad G = -g_0\lambda, \\ \rho &= \left(1 - \frac{1}{2}g_0\right)\alpha_1. \end{aligned} \tag{39}$$

Expanding μ and ν as

$$\mu = -\lambda + \sum_{j=1}^{\infty}\mu_j\lambda^{-j}, \qquad \nu = -\left(1 + \frac{1}{2}g_0\right)\lambda + \sum_{j=1}^{\infty}\nu_j\lambda^{-j}, \tag{40}$$

then we have

$$\mu_j = qM_j + \alpha_1 N_j, \qquad \nu_j = qM_j + \left(1 - \frac{1}{2}g_0\right)\alpha_1 N_j, \quad j \geq 1, \tag{41}$$

with the coefficients μ_j and ν_j, which are called conserved densities and currents, respectively. Then the first two conserved densities and currents of (25) read

$$\begin{aligned} \mu_1 &= -\frac{1}{2}qr - \alpha_1\beta_2, \\ \mu_2 &= -\frac{1}{4}qr_x + q\beta_1\beta_2 - \alpha_1\beta_{2,x} + \frac{1}{2}r\alpha_1\alpha_2, \\ \nu_1 &= -\frac{1}{2}qr - \left(1 - \frac{1}{2}g_0\right)\alpha_1\beta_2, \\ \nu_2 &= -\frac{1}{4}qr_x + q\beta_1\beta_2 - \left(1 - \frac{1}{2}g_0\right)\alpha_1\beta_{2,x} \\ &\quad + \frac{1}{2}\left(1 - \frac{1}{2}g_0\right)r\alpha_1\alpha_2. \end{aligned} \tag{42}$$

So the first two conservation laws of (25) are revealed as

$$\begin{aligned} &\left(-\frac{1}{2}qr - \alpha_1\beta_2\right)_t = \left[-\frac{1}{2}qr - \left(1 - \frac{1}{2}g_0\right)\alpha_1\beta_2\right]_x, \\ &\left(-\frac{1}{4}qr_x + q\beta_1\beta_2 - \alpha_1\beta_{2,x} + \frac{1}{2}r\alpha_1\alpha_2\right)_t \\ &\quad = \left[-\frac{1}{4}qr_x + q\beta_1\beta_2 - \left(1 - \frac{1}{2}g_0\right)\alpha_1\beta_{2,x} \right. \\ &\qquad \left. + \frac{1}{2}\left(1 - \frac{1}{2}g_0\right)r\alpha_1\alpha_2\right]_x. \end{aligned} \tag{43}$$

For (26), one infers

$$\begin{aligned} A &= -2\lambda^2 + qr + \alpha_1\beta_2 - \alpha_2\beta_1 - \frac{1}{2}g_0\alpha_1\beta_2 - \frac{1}{2}g_0\alpha_2\beta_1, \\ G &= -g_0\lambda^2 + \alpha_1\beta_2 + \alpha_2\beta_1 - \frac{1}{2}g_0\alpha_1\beta_2 + \frac{1}{2}g_0\alpha_2\beta_1, \\ \rho &= \left(1 - \frac{1}{2}g_0\right)\lambda\alpha_1 + \left(\frac{1}{2}g_0 - 1\right)\alpha_{1,x} + \frac{1}{2}g_0 q\beta_1, \\ B &= \lambda q - \frac{1}{2}q_x + \frac{1}{2}g_0\alpha_1\alpha_2. \end{aligned} \tag{44}$$

If we write μ and ν as

$$\mu = -\lambda + \sum_{j=1}^{\infty}\mu_j\lambda^{-j}, \qquad \nu = -\left(1 + \frac{1}{2}g_0\right)\lambda^2 + \sum_{j=1}^{\infty}\nu_j\lambda^{-j}, \tag{45}$$

the first two are

$$\begin{aligned} \mu_1 &= -\frac{1}{2}qr - \alpha_1\beta_2, \\ \mu_2 &= -\frac{1}{4}qr_x + q\beta_1\beta_2 - \alpha_1\beta_{2,x} + \frac{1}{2}r\alpha_1\alpha_2, \\ \nu_1 &= -\frac{1}{2}g_0\left(\beta_{2,x}\alpha_1 + r\alpha_1\alpha_2 + \alpha_{1,x}\beta_2 + q\beta_1\beta_2\right) \\ &\quad - \frac{1}{4}qr_x + q\beta_1\beta_2 + \frac{1}{4}q_x r + \alpha_{1,x}\beta_2 \\ &\quad + \beta_{2,x}\alpha_1 + \frac{1}{2}r\alpha_1\alpha_2, \\ \nu_2 &= -\frac{1}{2}g_0\left(\alpha_{1,x}\beta_{2,x} + \frac{1}{2}qr\alpha_2\beta_1 + \frac{1}{2}qr\alpha_1\beta_2 - q\beta_{2,x}\beta_1 \right. \\ &\quad - \frac{1}{2}r\alpha_{1,x}\alpha_2 + r_x\alpha_1\alpha_2 - \frac{1}{2}r\alpha_{2,x}\alpha_1 \\ &\quad \left. - 2\alpha_1\alpha_2\beta_1\beta_2 + \beta_{2,xx}\alpha_1\right) + \alpha_{1,x}\beta_{2,x} \\ &\quad + \frac{1}{4}qr\alpha_2\beta_1 - \frac{1}{2}r\alpha_{1,x}\alpha_2 - \frac{1}{2}r\alpha_{2,x}\alpha_1 - \alpha_1\alpha_2\beta_1\beta_2 \\ &\quad + \beta_{2,xx}\alpha_1 + \frac{3}{4}r_x\alpha_1\alpha_2 + \frac{1}{8}q^2r^2 - \frac{1}{8}qr_{xx} \\ &\quad + \frac{1}{2}q\beta_{1,x}\beta_2 - q\beta_{2,x}\beta_1 + \frac{1}{8}q_x r_x \\ &\quad - \frac{1}{2}q_x\beta_1\beta_2 + \frac{3}{4}qr\alpha_1\beta_2. \end{aligned} \tag{46}$$

The recursion relations for μ_j and ν_j $(j \geq 1)$ are as follows:

$$\begin{aligned} \mu_j &= qM_j + \alpha_1 N_j, \\ \nu_j &= qM_{j+1} + \frac{1}{2}\left(g_0\alpha_1\alpha_2 - q_x\right)M_j + \left(1 - \frac{1}{2}g_0\right)\alpha_1 N_{j+1} \\ &\quad - \left(\alpha_{1,x} - \frac{1}{2}g_0\alpha_{1,x} - \frac{1}{2}g_0 q\beta_2\right)N_j, \end{aligned} \tag{47}$$

where M_j and N_j can be recursively calculated from (34). Then we display the first two conservation laws of (26) as

$$\mu_{1,t} = \nu_{1,x}, \qquad \mu_{2,t} = \nu_{2,x}, \tag{48}$$

where μ_1, μ_2, ν_1, and ν_2 are defined in (46).

6. Concluding Remarks

In this paper, we proposed a six-component super-AKNS system from a 3×3 matrix superspectral problem. We obtained the super-Hamiltonian structure and different reductions for the superintegrable equations. Infinitely many conservation laws were also considered. The obtained results supplement the existing theories on the superintegrable systems. The bosonization approach for supersymmetric systems is a powerful tool to generate exact solutions. The superintegrable system (23) may admit bosonization. This and other related issues may be considered in further publication.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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