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Research Article

A Note on Optimality Conditions for DC Programs Involving Composite Functions

Xiang-Kai Sun[1,2] and Hong-Yong Fu[3]

[1] *College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China*
[2] *College of Automation, Chongqing University, Chongqing 400030, China*
[3] *School of Management, Southwest University of Political Science and Law, Chongqing 401120, China*

Correspondence should be addressed to Hong-Yong Fu; fuhongyong.cqu@gmail.com

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By using the formula of the ε-subdifferential for the sum of a convex function with a composition of convex functions, some necessary and sufficient optimality conditions for a DC programming problem involving a composite function are obtained. As applications, a composed convex optimization problem, a DC optimization problem, and a convex optimization problem with a linear operator are examined at the end of this paper.

1. Introduction

Let X and Y be two real locally convex Hausdorff topology vector spaces with their dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Let $K \subseteq Y$ be a nonempty closed convex cone which defined the partial order "$\leq_K$" of Y; namely,

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K, \quad \text{for any } y_1, y_2 \in Y. \tag{1}$$

We attach an element $\infty_Y \notin Y$ which is the greatest element with respect to "$\leq_K$" and let $Y^\bullet = Y \cup \{\infty_Y\}$. Then, for any $y \in Y^\bullet$, one has $y \leq_K \infty_Y$ and we define the following operations on $Y^\bullet$:

$$y + (\infty_Y) = (\infty_Y) + y = \infty_Y, \qquad t(\infty_Y) = \infty_Y, \quad \text{for any } y \in Y, t \geq 0. \tag{2}$$

Let $f_1, f_2 : X \to \overline{\mathbb{R}} := \overline{\mathbb{R}} \cup \{+\infty\}$ and $g : Y \to \overline{\mathbb{R}}$ be proper, convex, and lower semicontinuous functions, and let $h : X \to Y^\bullet$ be a proper, K-convex, and star K-lower semicontinuous function such that $h(\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \cap \operatorname{dom} g \neq \emptyset$. Moreover, we assume that g is a K-increasing function; that is,

$$\text{for any } x, y \in Y \text{ such that } x \leq_K y, \text{ we have } g(x) \leq g(y). \tag{3}$$

In this paper, we deal with a new class of DC programming involving a composite function given in the following form:

$$\inf_{x \in X} \{f_1(x) - f_2(x) + g \circ h(x)\}. \tag{P}$$

The problem (P) is very general in the sense that it includes, as particular cases, many different problems as, for example, a composed convex optimization problem, a DC optimization problem, and a convex optimization problem with a linear operator; see [1–12] and the references therein. The interest of such a general problem is that it unifies all these particular problems in a convenient way. Moreover, many results obtained for one of these problems can be extended with suitable modifications to the problem (P).

Recently, optimality conditions for global or local minimizers of some special kinds of the problem (P) have been studied by many researchers; see [13–25] and the references therein. Here, we specially mention the works on optimality defined via subdifferential calculus due to [18, 24, 25]. By using a formula for the ε-subdifferential of the sum of a convex function with a composition of convex functions, Boţ et al. [18] have obtained necessary and sufficient conditions for the ε-optimal solutions of composed convex optimization problems. By using some suitable conditions and the notions of strong subdifferential and epsilon-subdifferential, Guo and Li [24] obtained necessary and sufficient optimality conditions for an epsilon-weak Pareto minimal point and an epsilon-proper Pareto minimal point of a DC vector optimization problem. Fang and Zhao [25] introduced the local and global KKT type conditions for a DC optimization problem. Then, by using properties of the subdifferentials of the involved functions, they obtained some sufficient and/or necessary conditions for these two types of optimality conditions. The purpose of this paper is to establish optimality conditions for this optimization problem (P). To do that, by using the properties of the epigraph of the conjugate functions, we first introduce some closedness conditions and investigate some characterizations of these closedness conditions via the formula of the ε-subdifferential. Then, we obtain some necessary and sufficient optimality conditions. Moreover, at the end of this paper, we examine a composed convex optimization problem, a DC optimization problem, and a convex optimization problem with a linear operator.

The paper is organized as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we obtain some optimality conditions for the problem (P) in terms of the subdifferentials and the ε-subdifferentials of the functions. In Section 4, we give some special cases of our general results, which have been treated in the previous papers.

2. Mathematical Preliminaries

Throughout this paper, let X and Y be two real locally convex Hausdorff topology vector spaces. Let D be a set in X; the interior (resp., closure, convex hull, and convex cone hull) of D is denoted by $\operatorname{int} D$ (resp., $\operatorname{cl} D$, $\operatorname{co} D$, and $\operatorname{cone} D$). Thus, if $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak* closure of W. We shall adopt the convention that $\operatorname{cone} D = \{0\}$ when D is an empty set. Let $D^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \forall x \in X\}$ be the dual cone of D. The indicator function $\delta_D : X \to \overline{\mathbb{R}}$ of X is defined by

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{if } x \notin D. \end{cases} \tag{4}$$

The support function $\sigma_D : X^* \to \overline{\mathbb{R}}$ of D is defined by

$$\sigma_D(x^*) = \sup_{x\in D} \langle x^*, x\rangle . \tag{5}$$

Let $f : X \to \overline{\mathbb{R}}$ be an extended real valued function. The effective domain and the epigraph are defined by

$$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}, \tag{6}$$

$$\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}, \tag{7}$$

respectively. f is said to be proper if and only if its effective domain is nonempty and $f(x) > -\infty$. The conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ of f is defined by

$$f^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle - f(x)\}. \tag{8}$$

Let $\overline{x} \in \operatorname{dom} f$. For any $\varepsilon \geq 0$, the ε-subdifferential of f at $\overline{x}$ is the convex set defined by

$$\partial_\varepsilon f(\overline{x}) = \{x^* \in X^* : f(x) \geq f(\overline{x}) + \langle x^*, x - \overline{x}\rangle - \varepsilon, \forall x \in X\}. \tag{9}$$

When $\overline{x} \notin \operatorname{dom} f$, we define that $\partial_\varepsilon f(\overline{x}) = \emptyset$. If $\varepsilon = 0$, the set $\partial f(\overline{x}) := \partial_0 f(\overline{x})$ is the classical subdifferential of convex analysis. It is easy to prove that, for any $\overline{x} \in \operatorname{dom} f$ and $x^* \in X^*$,

$$f(\overline{x}) + f^*(x^*) \leq \langle x^*, \overline{x}\rangle + \varepsilon \Longleftrightarrow x^* \in \partial_\varepsilon f(\overline{x}). \tag{10}$$

Let E be a convex set of X. The ε-normal cone to E at a point $\overline{x} \in E$ is defined by $N_\varepsilon(E, \overline{x}) = \{x^* \in X^* : \langle x^*, x - \overline{x}\rangle \leq \varepsilon$, for any $x \in E\}$. If $\varepsilon = 0$, $N_0(E, \overline{x})$ is the normal cone $N(E, \overline{x})$ of convex analysis. Moreover, it is easy to see that $N_\varepsilon(E, \overline{x}) = \partial_\varepsilon \delta_E(\overline{x})$.

Let $A : X \to Y$ be a linear continuous mapping. The adjoint mapping $A^* : Y^* \to X^*$ of A is defined by

$$\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle, \quad \text{for any } (x, y^*) \in X \times Y^*. \tag{11}$$

The infimal function $Af : Y \to \overline{\mathbb{R}}$ of f through A is defined by

$$Af(y) = \inf\{f(x) : x \in X, Ax = y\}, \quad \text{for any } y \in Y. \tag{12}$$

Let $h : X \to Y^\bullet$ be an extended vector valued function. The domain and the K-epigraph of h are defined by

$$\operatorname{dom} h = \{x \in X : h(x) \in Y\}, \tag{13}$$

$$\operatorname{epi}_K h = \{(x, y) \in X \times Y : y \in h(x) + K\}, \tag{14}$$

respectively. h is said to be proper if and only if $\operatorname{dom} h \neq \emptyset$. h is said to be a K-convex function if and only if, for any $x, y \in X$ and $t \in [0, 1]$, we have

$$h(tx + (1 - t) y) \leq_K t h(x) + (1 - t) h(y). \tag{15}$$

For any subset $W \subseteq Y$, we denote

$$h^{-1}(W) = \{x \in X : \text{there exists } y \in W \text{ such that } h(x) = y\}. \tag{16}$$

Moreover, let $\lambda \in K^*$. The function $(\lambda h) : X \to \overline{\mathbb{R}}$ is defined by

$$(\lambda h)(x) = \begin{cases} \langle \lambda, h(x)\rangle, & \text{if } x \in \operatorname{dom} h, \\ +\infty, & \text{otherwise}. \end{cases} \tag{17}$$

We say that h is star K-lower semicontinuous if and only if (λh) is lower semicontinuous, for any $\lambda \in K^*$.

Now, let us recall the following result which will be used in the following section.

Lemma 1 (see [26]). *Let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be proper, convex, and lower semicontinuous functions. Then*

(i) *$\overline{x}$ is a global optimal solution of $\inf_{x\in X}\{f_1(x) - f_2(x)\}$ if and only if, for any $\varepsilon \geq 0$, $\partial_\varepsilon f_2(\overline{x}) \subset \partial_\varepsilon f_1(\overline{x})$.*

(ii) *If $\overline{x}$ is a local optimal solution of $\inf_{x\in X}\{f_1(x)-f_2(x)\}$, then $\partial f_2(\overline{x}) \subset \partial f_1(\overline{x})$.*

3. Optimality Conditions for (P)

In this section, we will employ the closedness qualification condition to derive necessary optimality conditions as well as necessary and sufficient optimality conditions for local and global minimizers in DC programs of type (P). Now, we first recall the closedness qualification condition (CQC).

Definition 2 (see [3]). The problem (P) is said to satisfy the closedness qualification condition (CQC) if the set

$$\operatorname{epi} f_1^* + \bigcup_{\lambda\in\operatorname{dom} g^*} \left(\operatorname{epi}(\lambda h)^* + (0, g^*(\lambda))\right) \quad \text{(CQC)}$$

is weak* closed in the space $X^* \times \mathbb{R}$.

The next lemma provides several characterizations of the closedness qualification condition (CQC). Moreover, the condition will be crucial in the sequel and it also deserves some attention for its independent interest.

Lemma 3 (see [3]). *The closedness qualification condition (CQC) holds if and only if, for any $x \in \operatorname{dom} f_1 \cap h^{-1}(\operatorname{dom} g)$ and any $\varepsilon \geq 0$,*

$$\partial_\varepsilon (f_1 + g\circ h)(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon}} \left\{\partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))\right\}. \quad (18)$$

Taking $\varepsilon = 0$ in Lemma 3, we can easily obtain the following subdifferential sum rule.

Corollary 4. *If the closedness qualification condition (CQC) holds, then, for any $x \in \operatorname{dom} f_1 \cap h^{-1}(\operatorname{dom} g)$,*

$$\partial (f_1 + g\circ h)(x) = \partial f_1(\overline{x}) + \bigcup_{\lambda\in K^*\cap\partial g(h(x))} \partial(\lambda h)(x). \quad (19)$$

Now, by using the closedness qualification condition and the ε-subdifferential sum rule, we establish necessary and sufficient optimality conditions for global optimal solution of (P).

Theorem 5. *Let $\overline{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap h^{-1}(\operatorname{dom} g)$. Suppose that the closedness qualification condition (CQC) holds. Then, $\overline{x}$ is a global optimal solution of (P) if and only if, for any $\varepsilon \geq 0$, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ and $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))$ such that*

$$\partial_\varepsilon f_2(\overline{x}) \subseteq \partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x}), \quad \varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon. \quad (20)$$

Proof. It is clear that (P) can be rewritten as

$$\inf_{x\in X}\{(f_1 + g\circ h)(x) - f_2(x)\}. \quad (21)$$

Then, by Lemma 1, $\overline{x}$ is a global optimal solution of (P) if and only if, for any $\varepsilon \geq 0$,

$$\partial_\varepsilon f_2(\overline{x}) \subset \partial_\varepsilon (f_1 + g\circ h)(\overline{x}). \quad (22)$$

Moreover, by Lemma 3, this is further equivalent to

$$\partial_\varepsilon f_2(\overline{x}) \subseteq \bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon}} \left\{\partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))\right\}. \quad (23)$$

This means that, for any $\varepsilon \geq 0$, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ and $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))$ such that $\varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon$ and $\partial_\varepsilon f_2(\overline{x}) \subset \partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x})$. This completes the proof. □

The following result establishes necessary optimality conditions for local optimal solution of (P).

Corollary 6. *Let $\overline{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap h^{-1}(\operatorname{dom} g)$. Suppose that the closedness qualification condition (CQC) holds. If $\overline{x}$ is a local optimal solution of (P), then there exists $\lambda \in K^* \cap \partial g(h(x))$ such that*

$$\partial f_2(\overline{x}) \subseteq \partial f_1(\overline{x}) + \partial(\lambda h)(\overline{x}). \quad (24)$$

Proof. If $\overline{x}$ is a local optimal solution of (P), then, by Lemma 1,

$$\partial f_2(\overline{x}) \subset \partial (f_1 + g\circ h)(\overline{x}). \quad (25)$$

By Corollary 4,

$$\partial f_2(\overline{x}) \subseteq \partial f_1(\overline{x}) + \bigcup_{\lambda\in K^*\cap\partial g(h(x))} \partial(\lambda h)(x), \quad (26)$$

which means that there exists $\lambda \in K^* \cap \partial g(h(x))$ such that

$$\partial f_2(\overline{x}) \subseteq \partial f_1(\overline{x}) + \partial(\lambda h)(\overline{x}). \quad (27)$$

This completes the proof. □

4. The Special Cases

In this section, we will give some special cases of our general results, which have been treated in the previous papers.

4.1. A Composed Convex Optimization Problem. When $f_2(x) = 0$, (P) becomes the following composed convex optimization problem:

$$\inf_{x\in X}\{f_1(x) + g\circ h(x)\}. \quad (P_1)$$

As some consequences of the results which have been treated in Section 3, we obtain the following results for (P_1) which was established in [18].

Theorem 7. *Let $\overline{x} \in \operatorname{dom} f_1 \cap h^{-1}(\operatorname{dom} g)$. Suppose that the closedness qualification condition (CQC) holds. Then, $\overline{x}$ is an ε-optimal solution of (P_1) if and only if, for any $\varepsilon \geq 0$, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ and $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(\overline{x}))$ such that*

$$0 \in \partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x}), \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon. \tag{28}$$

Corollary 8. *Let $\overline{x} \in \operatorname{dom} f_1 \cap h^{-1}(\operatorname{dom} g)$. Suppose that the closedness qualification condition (CQC) holds. Then, $\overline{x}$ is a global optimal solution of (P) if and only if there exists $\lambda \in K^* \cap \partial g(h(\overline{x}))$ such that*

$$0 \in \partial f_1(\overline{x}) + \partial(\lambda h)(\overline{x}). \tag{29}$$

4.2. A Constrained DC Optimization Problem. In this subsection, we consider the following DC optimization problem:

$$\inf_{\substack{x \in X,\\ h(x) \in -K}} \{f_1(x) - f_2(x)\}. \tag{P_2}$$

Let $g = \delta_{\{-K\}}$. Obviously, g is a proper, convex, lower semicontinuous, and K-increasing function. Then (P_2) can be seen as a particular case of the problem (P), since it can be rewritten as

$$\inf_{x \in X} \{f_1(x) - f_2(x) + \delta_{-K} \circ h\}. \tag{30}$$

Since $g^* = \delta_{K^*}$, we obtain that $\operatorname{dom} g^* = K^*$. Then, condition (CQC) becomes

$$\operatorname{epi} f_1^* + \bigcup_{\lambda \in K^*} \operatorname{epi}(\lambda h)^* \text{ is weak}^* \text{ closed in the space } X^* \times \mathbb{R}. \tag{CQC_1}$$

Moreover, by (10), we obtain that, for any $y \in -K$, $\lambda \in \partial_\varepsilon \delta_{\{-K\}}(y)$ if and only if $\lambda \in K^*$ and $0 \leq \langle \lambda, y \rangle + \varepsilon$. Then, by Lemma 3, we get the following result.

Lemma 9. *The closedness qualification condition $(CQC)_1$ holds if and only if, for any $x \in \operatorname{dom} f_1 \cap h^{-1}(-K)$ and any $\varepsilon \geq 0$,*

$$\begin{aligned}\partial_\varepsilon (f_1 + \delta_{-K} \circ h)(x) &= \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon}} \{\partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^*, \\ &\qquad 0 \leq \langle \lambda, h(x) \rangle + \varepsilon_3\}.\end{aligned} \tag{31}$$

Taking $\varepsilon = 0$ in Lemma 9, we can easily obtain the following subdifferential sum rule.

Corollary 10. *If the closedness qualification condition $(CQC)_1$ holds, then, for any $x \in \operatorname{dom} f_1 \cap h^{-1}(-K)$,*

$$\partial(f_1 + \delta_{-K} \circ h)(x) = \partial f_1(\overline{x}) + \bigcup_{\lambda \in K^*} \partial(\lambda h)(x). \tag{32}$$

Similarly, by using Lemma 9 and Corollary 10, we obtain the following results.

Theorem 11. *Let $\overline{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap h^{-1}(-K)$. Suppose that the closedness qualification condition $(CQC)_1$ holds. Then, $\overline{x}$ is a global optimal solution of (P_2) if and only if, for any $\varepsilon \geq 0$, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\lambda \in K^*$ and $0 \leq \langle \lambda, h(x) \rangle + \varepsilon_3$ such that*

$$\partial_\varepsilon f_2(\overline{x}) \subseteq \partial_{\varepsilon_1} f_1(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x}), \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon. \tag{33}$$

Corollary 12. *Let $\overline{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap h^{-1}(-K)$. Suppose that the closedness qualification condition $(CQC)_1$ holds. If $\overline{x}$ is a local optimal solution of (P_2) then there exists $\lambda \in K^*$ such that*

$$\partial f_2(\overline{x}) \subseteq \partial f_1(\overline{x}) + \partial(\lambda h)(\overline{x}). \tag{34}$$

4.3. A Convex Optimization Problem with a Linear Operator. Let $f_2 \equiv 0$ and $h(x) = Ax$, for any $x \in X$, where $A : X \to Y$ is a linear continuous mapping. Taking $K = \{0\}$, one has that h is a K-convex function and $K^* = Y^*$. So, the problem (P) becomes

$$\inf_{x \in X} \{f(x) + g(A(x))\}. \tag{P_3}$$

Since

$$(\lambda h)^*(p) = \begin{cases} 0, & \text{if } A^*\lambda = p, \\ +\infty, & \text{otherwise,} \end{cases} \tag{35}$$

we get

$$\operatorname{epi}(\lambda h)^* = \{(p, r) \in X^* \times \mathbb{R} : p = A^*\lambda, r \geq 0\}. \tag{36}$$

Then

$$\begin{aligned}\operatorname{epi} f^* + \bigcup_{\lambda \in \operatorname{dom} g^*} (\operatorname{epi}(\lambda h)^* + (0, g^*(\lambda))) \\ = \operatorname{epi} f^* + (A^* \times \mathrm{id}_{\mathbb{R}}) \operatorname{epi} g^*.\end{aligned} \tag{37}$$

Thus, the condition (CQC) becomes in this special case

$$\operatorname{epi} f^* + (A^* \times \mathrm{id}_{\mathbb{R}}) \operatorname{epi} g^* \text{ is weak}^* \text{ closed in the space } X^* \times \mathbb{R}. \tag{RC_A}$$

Moreover, for any $\varepsilon \geq 0$, it is easy to see that $\partial_\varepsilon(\lambda h)(x) = A^*\lambda$. Then, by Lemma 3, we get the following result.

Lemma 13. *The closedness qualification condition (RC_A) holds if and only if, for any $x \in \operatorname{dom} f_1 \cap A^{-1}(\operatorname{dom}(g))$ and any $\varepsilon \geq 0$,*

$$\partial_\varepsilon (f_1 + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_3 \geq 0,\\ \varepsilon_1+\varepsilon_3=\varepsilon}} \{\partial_{\varepsilon_1} f_1(\overline{x}) + A^* \partial_{\varepsilon_3} g(Ax)\}. \tag{38}$$

Taking $\varepsilon = 0$ in Lemma 13, we can easily obtain the following subdifferential sum rule.

Corollary 14. *If the closedness qualification condition* (RC_A) *holds, then, for any* $x \in \operatorname{dom} f_1 \cap A^{-1}(\operatorname{dom}(g))$,

$$\partial (f_1 + g \circ A)(x) = \partial f_1(\overline{x}) + A^* \partial g(Ax). \tag{39}$$

Similarly, by using Lemma 13 and Corollary 14, we obtain the following results.

Theorem 15. *Let* $\overline{x} \in \operatorname{dom} f_1 \cap A^{-1}(\operatorname{dom}(g))$. *Suppose that the closedness qualification condition* (RC_A) *holds. Then,* $\overline{x}$ *is a* ε*-optimal solution of* (P_3) *if and only if, for any* $\varepsilon \geq 0$, *there exist* $\varepsilon_1, \varepsilon_3 \geq 0$, *such that*

$$\begin{gathered} 0 \in \partial_{\varepsilon_1} f_1(\overline{x}) + A^* \partial_{\varepsilon_3} g(Ax), \\ \varepsilon_1 + \varepsilon_3 = \varepsilon. \end{gathered} \tag{40}$$

Corollary 16. *Let* $\overline{x} \in \operatorname{dom} f_1 \cap A^{-1}(\operatorname{dom}(g))$. *Suppose that the closedness qualification condition* (RC_A) *holds. Then,* $\overline{x}$ *is a global optimal solution of* (P_3) *if and only if there exists* $\lambda \in K^*$ *such that*

$$0 \in \partial f_1(\overline{x}) + A^* \partial g(Ax). \tag{41}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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