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Research Article

Existence of Traveling Wave Solutions for Cholera Model

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To investigate the spreading speed of cholera, Codeço's cholera model (2001) is developed by a reaction-diffusion model that incorporates both indirect environment-to-human and direct human-to-human transmissions and the pathogen diffusion. The two transmission incidences are supposed to be saturated with infective density and pathogen density. The basic reproduction number R_0 is defined and the formula for minimal wave speed c^* is given. It is proved by shooting method that there exists a traveling wave solution with speed c for cholera model if and only if $c \geq c^*$.

1. Introduction

Cholera has been a serious threat to human health in the past and at present, which is an acute, diarrheal illness caused by infection of the intestine with the bacterium *Vibrio cholera*. An estimated 3–5 million cases and over 100,000 deaths occur each year around the world [1]. The cholera bacterium is usually found in water or food sources that have been contaminated by feces from a person infected with cholera. Cholera is most likely to be found and to spread in places with inadequate water treatment, poor sanitation, and inadequate hygiene. Therefore, cholera outbreaks have occurred in developing countries, for example, Iraq (2007-2008), Guinea Bissau (2008), Zimbabwe (2008-2009), Haiti (2010), Democratic Republic of Congo (2011-2012), and Sierra Leone (2012) [2].

To understand the propagation mechanism of cholera, many mathematical models were proposed, whose earlier one was established by Capasso and Paveri-Fontana [3] to study the 1973 cholera epidemic in the Mediterranean region as follows:

$$\frac{dI}{dt} = g(B) - a_{22}I, \qquad \frac{dB}{dt} = -a_{11}B + a_{12}I, \tag{1}$$

where $B(t)$ and $I(t)$ denote the concentrations of the pathogen and the infective populations, respectively. In addition, Codeço [4] investigated the role of the aquatic pathogen in dynamics of cholera through the following susceptible-infective-pathogen model:

$$\frac{dS}{dt} = n(H-S) - a\frac{SB}{K+B}, \qquad \frac{dI}{dt} = a\frac{SB}{K+B} - rI,$$
$$\frac{dB}{dt} = eI - (mb - nb)B, \tag{2}$$

where $S(t)$ is the susceptible individuals. In this model, human is divided into two groups: the susceptible group and the infective group. As pointed out in [4–8], bacterium *Vibrio cholera* can spread by direct human-to-human and indirect environment-to-human modes. To understand the complex dynamics of cholera, model (2) is extended by [8–15] and so forth.

In all previous models the influences of space distribution of human on the transmission of cholera are omitted. Cholera usually spreads in spatial wave [16]. Cholera bacteria live in rivers and interact with the plankton on the surface of the water [17]. When individuals drink contaminated water and are infected, they will release cholera bacteria through excretion [18]. Capasso et al. [19–23] developed model (1) by incorporating the bacterium diffusion in a bounded area and studied the existence and stability of solutions. To deeply investigate the interaction of transmission modes and bacterium diffusion, Bertuzzo et al. [24, 25] incorporated patchy structure into model (2) and supposed that pathogen in water could diffuse among these patches. Furthermore, Mari et al. [26] studied the influence of diffusion of both human and pathogen on cholera dynamics through a patchy model.

Infectious case is usually found firstly at some location and then spreads to other areas. Consequently, the most important question for cholera is what the spreading speed of cholera is. However, the above spatial models mainly focus on the stability of solutions not the spreading speed. Traveling wave solution is an important tool used to study the spreading speed of infectious diseases [27–29]. Based on Capasso's model (1), Zhao and Wang [30], Xu and Zhao [31], Jin and Zhao [32], and Hsu and Yang [33] studied the influences of pathogen diffusion on the spread speed of cholera.

The studies of traveling wave solutions of Capasso's model (1) incorporating pathogen diffusion provide insight into the spreading speed of cholera. However, some pieces of information are omitted, such as the interaction of direct human-to-human and indirect environment-to-human transmissions. In this paper, a reaction-diffusion model with pathogen diffusion and both transmission paths is proposed by developing Codeço's model (2). Based on model (2) and ignoring the disease-related death, a general diffusive cholera model can be formulated as the following reaction-diffusion system:

$$\begin{aligned}\frac{\partial S}{\partial t} &= b(N-S) - f(I)S - g(B)S,\\ \frac{\partial I}{\partial t} &= f(I)S + g(B)S - bI,\\ \frac{\partial B}{\partial t} &= d\frac{\partial^2 B}{\partial x^2} + eI - mB,\end{aligned} \tag{3}$$

where $S = S(x,t)$ and $I = I(x,t)$ denote the concentrations of susceptible and infected individuals, respectively, and $B = B(x,t)$ is the concentration of the infectious agents. N is the total human population, b stands for the natural birth and death rate, e denotes the contribution of each infected person to the concentration of cholera, and m is the net death rate of *vibrio cholera*. $f(I)$ and $g(B)$ are the human-to-human and environment-to-human transmission incidences, respectively. Similar to [10], we assume that $f(I)$ and $g(B)$ satisfy

(A1) $f(0) = 0$, $f'(I) \geq 0$, $f''(I) \leq 0$;

(A2) $g(0) = 0$, $g'(0) > 0$, $g'(B) \geq 0$, $g''(B) \leq 0$, and $g(B)$ is strictly monotonously increasing in $[0, +\infty)$.

It is easy to conclude that $f(I) \leq f'(0)I$, $g(B) \leq g'(0)B$, and $f(I)/I$ and $g(B)/B$ are nonincreasing. Obviously, hypotheses (A1) and (A2) imply that the two transmission paths are saturated. In Tian and Wang [10], $f(I)$ and $g(B)$ have the following expressions:

$$f(I) = \beta_1 I, \qquad g(B) = \frac{\beta_2 B}{K + B}. \tag{4}$$

Obviously, as a special case, such selections satisfy (A1) and (A2).

Shooting method is very important in proving the existence of traveling wave solutions, which was proposed by Dunbar [34, 35] and was applied to many models (e.g., [36–40]). In this paper, the existence of traveling wave solutions of system (3) will be proved by shooting method and the formula for minimal wave speed will be given.

This paper is organized as follows. In next section, the main theorem and the formula for minimal wave speed will be given. In Section 3, the nonexistence of the traveling wave solutions for $c < c^*$ is proved by geometric method. Section 4 is devoted to shooting arguments and the construction of Wazewski set. In Section 5, we prove the existence of traveling wave solutions for $c > c^*$ and then give the existence of traveling wave solution for $c = c^*$ by limit arguments. The final section is devoted to the simulations.

2. Main Results

For convenience, we introduce dimensionless variables and parameters. By setting

$$u_1 = \frac{S}{bN}, \qquad u_2 = \frac{I}{bN}, \qquad u_3 = \frac{m}{ebN}B, \qquad y = \frac{x}{\sqrt{d}}, \tag{5}$$

model (3) has the form

$$\begin{aligned}u_{1,t} &= 1 - bu_1 - f_1(u_2)u_1 - g_1(u_3)u_1,\\ u_{2,t} &= f_1(u_2)u_1 + g_1(u_3)u_1 - bu_2,\\ u_{3,t} &= u_{3,yy} + m(u_2 - u_3),\end{aligned} \tag{6}$$

where $f_1(u_2) = f(bNu_2)$ and $g_1(u_3) = g(ebNu_3/m)$.

Denote $R_0 = [f_1'(0) + g_1'(0)]/b^2$, which is the basic reproduction number of (6). Then hypotheses (A1) and (A2) imply that system (6) has two nonnegative constant solutions $P_1(1/b, 0, 0)$ and $P_2(1/b - u^*, u^*, u^*)$ if and only if $R_0 > 1$, where u^* is the only one positive root of equation

$$[f_1(u^*) + g_1(u^*)]\left(\frac{1}{b} - u^*\right) = bu^* \tag{7}$$

and $0 < u^* < 1/b$. Biologically, P_1 corresponds to disease-free equilibrium and P_2 corresponds to endemic equilibrium. To study the spreading wave of cholera, it is assumed that $R_0 > 1$ holds in this paper; that is

$$f_1'(0) + g_1'(0) > b^2. \tag{8}$$

A traveling wave solution of system (6) with speed c is a nonnegative solution of the form

$$\begin{aligned}u_1(y,t) &= u_1(s), \qquad u_2(y,t) = u_2(s),\\ u_3(y,t) &= u_3(s), \qquad s = y + ct.\end{aligned} \tag{9}$$

Substituting traveling profile $(u_1(s), u_2(s), u_3(s))$ into system (6) yields the following equations:

$$\begin{aligned}cu_1' &= 1 - bu_1 - f_1(u_2)u_1 - g_1(u_3)u_1,\\ cu_2' &= f_1(u_2)u_1 + g_1(u_3)u_1 - bu_2,\\ cu_3' &= u_3'' + m(u_2 - u_3),\end{aligned} \tag{10}$$

where $'$ denotes d/ds. To investigate invasion question by cholera, we will study the positive solutions of (10) such that

$$\begin{aligned}(u_1(+\infty), u_2(+\infty), u_3(+\infty)) &= \left(\frac{1}{b} - u^*, u^*, u^*\right), \\ (u_1(-\infty), u_2(-\infty), u_3(-\infty)) &= \left(\frac{1}{b}, 0, 0\right).\end{aligned} \tag{11}$$

Before giving the main theorem, we introduce the equation for minimal wave speed

$$\Delta(c) := b_3 c^6 + b_2 c^4 + b_1 c^2 + b_0 = 0, \tag{12}$$

where

$$\begin{aligned}\epsilon &= f_1'(0) - b^2, \\ b_3 &= b^2\epsilon^2 + 2b^3 m\left(f_1'(0) + g_1'(0) - b^2\right) \\ &\quad + 2b^3 m g_1' + b^4 m^2, \\ b_2 &= -2b\epsilon^3 + 2b^2 m\epsilon^2 + \left(8b^3m^2 - 6b^2 m g_1'\right)\epsilon \\ &\quad + 4m^3b^4 + 18b^3m^2 g_1'(0), \\ b_1 &= \epsilon^4 - 8mb\epsilon^3 - \left(8b^2m^2 + 6bmg_1'\right)\epsilon^2 \\ &\quad - 36b^2m^2 g_1'(0)\epsilon - 27m^2b^2 g_1'(0)^2, \\ b_0 &= 4m\left(b^2 - f_1'(0)\right)^3\left(b^2 - f_1'(0) - g_1'(0)\right).\end{aligned} \tag{13}$$

Theorem 1. *There exists a constant $c^* > 0$ which is the greatest positive root of* (12). *When $c \geq c^*$, system* (6) *has a traveling wave solution satisfying boundary condition* (11). *When $0 < c < c^*$, system* (6) *has no traveling wave solutions satisfying boundary condition* (11).

3. Nonexistence of Traveling Wave Solutions for $c < c^*$

From (10), we have

$$[u_1(s) + u_2(s)]' = \frac{[1 - b(u_1(s) + u_2(s))]}{c}. \tag{14}$$

Consequently, if $u_1(0) + u_2(0) \neq 1/b$, then

$$|u_1(s) + u_2(s)| \longrightarrow \infty \quad \text{when } s \longrightarrow -\infty. \tag{15}$$

Hence, the traveling profile $(u_1(s), u_2(s), u_3(s))$ with boundary condition (11) must satisfy

$$u_1(s) + u_2(s) = \frac{1}{b} \quad \text{for any } s \in R. \tag{16}$$

Therefore, to study traveling wave solutions we assume (16) satisfies. Setting $u_3' = z$ in system (10) and noticing (16), it follows

$$\begin{aligned}u_2' &= \frac{[(f_1(u_2) + g_1(u_3))((1/b) - u_2) - bu_2]}{c}, \\ u_3' &= z, \\ z' &= cz + m(u_3 - u_2).\end{aligned} \tag{17}$$

If $u_1(s) = 0$, then $u_1'(s) = 1/c > 0$ by system (10). Therefore, we suppose $u_1(s) = 1/b - u_2(s) > 0$ for any s; that is, $u_2(s) < 1/b$.

Obviously, system (17) has two equilibria $E_1(0,0,0)$ and $E_2(u^*, u^*, 0)$. A profile solution of (10) which satisfies boundary condition (11) corresponds to the positive solution $(u_2(s), u_3(s), z(s))$ of system (17) which satisfies

$$u(+\infty) = (u_2^*, u_2^*, 0), \qquad u(-\infty) = (0,0,0), \tag{18}$$

where $u(s) = (u_2(s), u_3(s), z(s))$. Therefore, to study the solutions of (10), it is sufficient to study those of system (17) satisfying boundary condition (18).

Firstly, we investigate the dynamics near E_1. Simple calculations show that the characteristic equation of the linearization of system (17) at E_1 is

$$H(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{19}$$

where

$$a_0 = \frac{m\left(f_1'(0) + g_1'(0) - b^2\right)}{bc}, \qquad a_1 = \frac{f_1'(0) - b^2 - mb}{b}, \qquad a_2 = \frac{b^2 - f_1'(0) - bc^2}{bc}. \tag{20}$$

Because $a_0 > 0$ (19) has a negative real root, which is denoted by λ_3. Let λ_1 and λ_2 be the other two eigenvalues of (19) and suppose that $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2$. To investigate the distribution of roots of (19), denote

$$p = a_1 - \frac{a_2^2}{3}, \qquad q = \frac{2a_2^3}{27} - \frac{a_1a_2}{3} + a_0, \qquad \Delta_0 = \frac{q^2}{4} + \frac{p^3}{27} \tag{21}$$

and introduce the following lemma [41].

Lemma 2. (a) *If $\Delta_0 > 0$,* (19) *has one real root and two nonreal complex conjugate roots.*

(b) *If $\Delta_0 = 0$,* (19) *has a multiple root and all its roots are real.*

(c) *If $\Delta_0 < 0$,* (19) *has three distinct real roots.*

Direct calculations show that $\Delta_0 = -\Delta/(108b^4c^4)$, where Δ is defined by (12).

Lemma 3. (a) *The real parts of λ_1 and λ_2 are positive.*

(b) *Assume $f_1'(0) \leq b^2$. Then, there exists $c^* > 0$ which is the only positive root of $\Delta(c) = 0$. When $c \geq c^*$, λ_1, and λ_2 are real. When $0 < c < c^*$, λ_1, and λ_2 are complex and nonreal.*

(c) *Assume that $f_1'(0) > b^2$. Then, there exist two positive constants $c_1^* < c^*$ which are all positive roots of $\Delta(c) = 0$. λ_1 and λ_2 are complex and nonreal if and only if $c_1^* < c < c^*$. If $c > c^*$, then $\lambda^* < \lambda_2 < \lambda_1$; if $0 < c \leq c_1^*$, then $\lambda_2 \leq \lambda_1 < \lambda^*$, where $\lambda^* = (f_1'(0) - b^2)/(bc)$.*

(d) *$\lambda_1 = \lambda_2$ if and only if $c = c^*$ or c_1^*.*

Proof. Suppose $\lambda = \beta i \neq 0$ is the root of (19). Substituting $\lambda = \beta i$ into (19) and comparing real and imaginary parts show that $a_1 = \beta^2 > 0$ and $a_0 = a_1 a_2$. Since $a_0 > 0$, then $a_2 > 0$. However, it is impossible that $a_1 > 0$ and $a_2 > 0$ by the expressions of a_1 and a_2. Therefore, the real parts of λ_1 and λ_2 are not zero. Furthermore, since it is impossible that $a_1 > 0$ and $a_2 > 0$, Routh-Hurwitz theorem implies that it is impossible that the real parts of both λ_1 and λ_2 are negative. Consequently, there are two cases: (i) λ_1 and λ_2 are complex conjugate roots with positive real parts; (ii) λ_1 and λ_2 are real and at least one is positive. However, Descartes' rule of signs shows that the number of positive roots of (19) is zero or two. Thus, if case (ii) is true, both of λ_1 and λ_2 are real and positive. Therefore, (a) is proved.

In this paragraph, we consider the case $f_1'(0) \le b^2$. Firstly, suppose that $f_1'(0) < b^2$. Obviously, $b_0 < 0$ and $b_3 > 0$. By the expression of b_2, we have

$$\begin{aligned} b_2 &= -2b\epsilon^3 + 2b^2 m\epsilon^2 - 6b^2 m g_1' \epsilon \\ &\quad + 8b^3 m^2 \left(f_1'(0) + g_1'(0) - b^2\right) + 4m^3 b^4 + 10b^3 m^2 g_1'(0) \\ &> 0 \end{aligned} \tag{22}$$

since $\epsilon = f_1'(0) - b^2 < 0$. Now, assume $f_1'(0) = b^2$; that is, $\epsilon = 0$. Then. $b_3 > 0$, $b_2 > 0$, $b_1 < 0$, and $b_0 = 0$. Then, if $f_1'(0) \le b^2$, Descartes' rule of signs shows that there exists $c^* > 0$ which is the only positive root of $\Delta(c) = 0$, where $\Delta(c) < 0$ for $0 < c < c^*$ and $\Delta(c) > 0$ for $c > c^*$. Using Lemma 2 completes the proof of (b).

Suppose that $f_1'(0) > b^2$ in this paragraph and, thus, $\epsilon > 0$. Calculations show that

$$\begin{aligned} H(\lambda^*) &= \frac{m g_1'}{bc} > 0, \\ H'(\lambda^*) &= \frac{\epsilon^2 - bc^2\epsilon - mb^2c^2}{b^2c^2} \end{aligned} \tag{23}$$

and that $H'(\lambda) = 0$ has two roots λ_1^* and λ_2^*, where

$$\begin{aligned} \lambda_1^* &= \frac{bc^2 + \epsilon + \sqrt{b^2c^4 + (3mb^2 - b\epsilon)c^2 + \epsilon^2}}{3bc}, \\ \lambda_2^* &= \frac{bc^2 + \epsilon - \sqrt{b^2c^4 + (3mb^2 - b\epsilon)c^2 + \epsilon^2}}{3bc}, \end{aligned} \tag{24}$$

and $\lambda_1^* > \lambda_2^*$. By letting $c_0 \triangleq \epsilon/\sqrt{b\epsilon + mb^2}$ and using trivial calculations, we get (see Figure 1)

$$\begin{aligned} \lambda^* = \lambda_1^* &\Longleftrightarrow c = c_0 \Longleftrightarrow H'(\lambda^*) = 0, \\ \lambda^* > \lambda_1^* &\Longleftrightarrow c < c_0 \Longleftrightarrow H'(\lambda^*) > 0, \\ \lambda^* < \lambda_1^* &\Longleftrightarrow c > c_0 \Longleftrightarrow H'(\lambda^*) < 0. \end{aligned} \tag{25}$$

Therefore, if $c = c_0$, then $H(\lambda_1^*) = H(\lambda^*) > 0$. Since λ_1^* is the only minimum-value point of $H(\lambda)$, and then $H(\lambda) > 0$ for any $\lambda > 0$ and both of λ_1 and λ_2 are not real. Lemma 2 shows that $\Delta(c_0) < 0$. Thus, since $b_0 > 0$ and $b_3 > 0$, there exist two positive roots $c_1^* < c^*$ for equation $\Delta(c) = 0$ such that $c_1^* < c_0 < c^*$. Then, using (25) and Lemma 2 completes the proof of (c) and (d). □

Direct calculations show that corresponding eigenvectors of eigenvalue λ_i are

$$e_i = \left(1 - \frac{\lambda_i(\lambda_i - c)}{m}, 1, \lambda_i\right), \tag{26}$$

where $i = 1, 2, 3$. Since

$$\begin{aligned} H(\lambda_i) &= -m\left[1 - \frac{\lambda_i(\lambda_i - c)}{m}\right]\left[\lambda_i - \frac{f_1'(0) - b^2}{bc}\right] + \frac{m g_1'}{bc} \\ &= 0, \end{aligned} \tag{27}$$

and thus

$$1 - \frac{\lambda_i(\lambda_i - c)}{m} = \frac{g_1'(0)}{bc\lambda_i + b^2 - f_1'(0)} = \frac{g_1'(0)}{bc(\lambda_i - \lambda^*)}. \tag{28}$$

Then, we have the following lemma.

Lemma 4. *If $0 < c < c^*$, there exist no traveling wave solutions which satisfy boundary condition* (11).

Proof. Assume that $f_1'(0) \le b^2$ and $0 < c < c^*$. Then, (b) of Lemma 3 implies that λ_1 and λ_2 are complex conjugate eigenvalues and there exits locally unstable manifold $\mathscr{W}^u$ and locally stable manifold $\mathscr{W}^s$. If a solution of (17) tends to E_1 when $s \to -\infty$, then it will be spiral on $\mathscr{W}^u$. By the structures of e_1 and e_2, $u_2(s) < 0$ at some time $s < 0$, which shows that there exist no traveling wave solutions departing from E_1.

Suppose that $f_1'(0) > b^2$. If $c_1^* < c < c^*$, (c) of Lemma 3 shows that λ_1 and λ_2 are complex conjugate eigenvalues and similar arguments to that of previous paragraph finish the proof. If $0 < c \le c_1^*$, (c) of Lemma 3 shows that λ_1 and λ_2 are real; however, $\lambda_2 \le \lambda_1 < \lambda^*$. If a solution of (17) tends to E_1 when $s \to -\infty$, structures of e_1 and e_2 indicate that there is an $s < 0$ such that $u_2(s) < 0$. The proof is completed. □

From Section 4 to Section 5.2, we suppose that $c > c^*$, which implies $\lambda^* < \lambda_2 < \lambda_1$.

4. Shooting Method and Wazewski Set

To prove the existence of traveling wave, shooting method developed by Dunbar [34] is used. Firstly, we give the shooting arguments.

Consider the differential equation

$$\frac{dy}{ds} = f(y), \tag{29}$$

where $f(y)$ from R^n to R^n satisfies Lipschitz condition about y. Let $y(s; y_0)$ denote the unique solution of (29) with initial

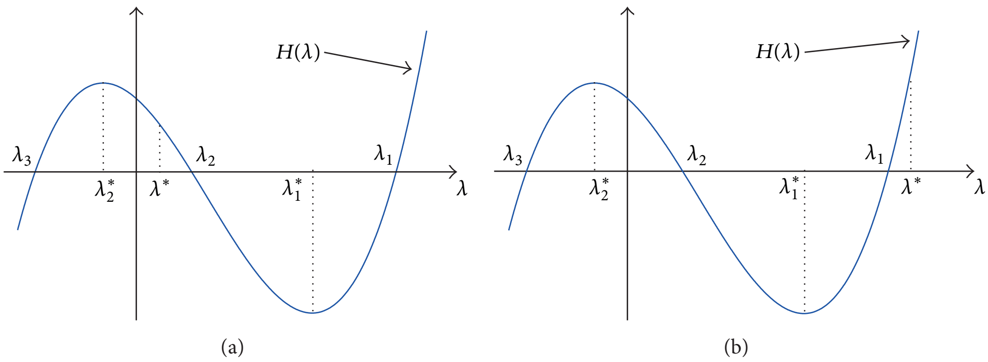

Figure 1: Distribution of eigenvalues of (19) when $f_1'(0) > b^2$, (a) for $c > c^*$ and (b) for $c < c_1^*$.

value $y(0) = y_0$. It is convenient to give the notations $y_0 \cdot s \triangleq y(s; y_0)$ and $y_0 \cdot S \triangleq \{y_0 \cdot s \mid s \in S \subset R\}$. To describe the shooting method (or Wazewski theorem), some definitions are necessary.

Definition 5. (a) For $W \subseteq R^n$, define immediate exit set W^- of W as

$$W^- \triangleq \{y_0 \in W \mid \forall s > 0, y_0 \cdot [0, s) \nsubseteq W\}. \tag{30}$$

(b) For $\Sigma \subseteq W$, let $\Sigma^0 \triangleq \{y_0 \in \Sigma \mid \exists s_0 > 0$ such that $y_0 \cdot s_0 \notin W\}$.

(c) Given $y_0 \in \Sigma^0$, define exit time $T(y_0)$ of y_0 by

$$T(y_0) \triangleq \sup \{s \mid y_0 \cdot [0, s) \subseteq W\}. \tag{31}$$

Then, Wazewski theorem is formulated as follows.

Lemma 6 (see [34]). *Suppose that*

(1) *if $y_0 \in \Sigma$ and $y_0 \cdot [0, s] \subseteq \mathrm{cl}(W)$, then $y_0 \cdot [0, s] \subseteq W$.*

(2) *If $y_0 \in \Sigma$, $y_0 \cdot s \in W$ and $y_0 \cdot s \notin W^-$, then there exists an open set V_s about $y_0 \cdot s$ disjoint from W^-.*

(3) *If $\Sigma = \Sigma^0$, Σ is compact and Σ intersects a trajectory of* (29) *only once.*

Then, the mapping $H(y_0) = y_0 \cdot T(y_0)$ is a homeomorphism from Σ to its image on W^-.

A set $W \subseteq R^n$ satisfying conditions (1) and (2) of Lemma 6 is called a Wazewski set. In the following, we first construct the Wazewski set W. Fundamental idea to construct a Wazewski set is that the characteristic vectors corresponding eigenvalues with positive real parts should be removed from W and that those characteristic vectors corresponding eigenvalues with negative real parts should be included. Therefore, we set

$$W = \mathbb{R}^3 \setminus (P \cup Q), \tag{32}$$

where

$$\begin{gathered} P = \{(u_1, u_2, u_3, z) : u_3 > u_2 > u^*, z > 0\}, \\ Q = \{(u_1, u_2, u_3, z) : 0 < u_3 < u_2 < u^*, z < 0\}. \end{gathered} \tag{33}$$

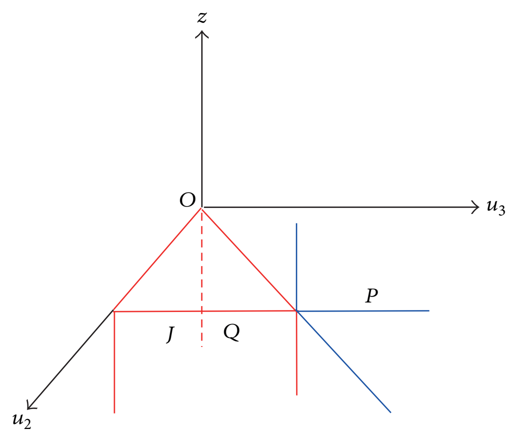

Figure 2: The construction of W and W^-.

It is obvious that $\partial W = \partial P \cup \partial Q$. Firstly, we give the construction of W^-, which is described in Figure 2.

Lemma 7. *The construction of W^- is as follows:*

$$W^- = \partial W \setminus (J \cup E_2), \tag{34}$$

where $J = \{(u_2, u_3, z) : 0 \le u_2 \le u^*, u_3 = 0, z \le 0\}$.

Proof. It is enough to analyze the behavior of solution on $\partial P \cup \partial Q$. We only study ∂Q and omit the proof of ∂P since the analysis of ∂P is similar to that of ∂Q and is simpler. In the process of this proof, we use some notations to simplify the proof. Set

$$\begin{gathered} u_i' = \left.\frac{du_i}{ds}\right|_{(u_2,u_3,z)\in\partial Q}, \qquad z' = \left.\frac{dz}{ds}\right|_{(u_2,u_3,z)\in\partial Q}, \qquad i = 2, 3, \\ h(u_2) = \left[\frac{f_1(u_2)}{u_2} + \frac{g_1(u_2)}{u_2}\right]\left(\frac{1}{b} - u_2\right) - b. \end{gathered} \tag{35}$$

From hypotheses (A1) and (A2), we find that $f_1(u_2)/u_2$ and $g_1(u_2)/u_2$ are monotonously decreasing, $h(u_2)$ is strictly monotonously decreasing for $u_2 \in (0, 1/b)$, and u^* is the only positive root of $h(u_2) = 0$. The set ∂Q is classified into two cases according to variable z.

(a) Case $z < 0$. This case is classified as follows.

(1) Case $0 = u_3 < u_2 < u^*$. Then $u_3' = z < 0$ and the solution of (17) will enter $\mathrm{int}(W)$.

(2) Case $0 < u_3 = u_2 < u^*$. Then

$$(u_3 - u_2)' = \frac{z - h(u_2)u_2}{c} < 0. \tag{36}$$

The solution of (17) will enter Q.

(3) Case $0 < u_3 < u_2 = u^*$. Then

$$\begin{aligned} u_2' &= \frac{[(f_1(u^*)/u^* + g_1(u_3)/u^*)(1/b - u^*) - b]u^*}{c} \\ &< \frac{h(u^*)u^*}{c} = 0. \end{aligned} \tag{37}$$

The solution of (17) will enter Q.

(4) Case $0 = u_3 = u_2 < u^*$. Then $u_3' = z < 0$ and the solution of (17) will enter $\mathrm{int}(W)$.

(5) Case $0 = u_3$ and $u_2 = u^*$. The solution of (17) will enter $\mathrm{int}(W)$.

(6) Case $u_3 = u_2 = u^*$. Then $u_2' = 0$,

$$\begin{aligned} u_2'' &= \left[\left(f_1'(0)(u^*)u_2' + g_1'(0)(u^*)z\right)\left(\frac{1}{b} - u^*\right)\right. \\ &\quad \left. - (f_1(u^*) + g_1(u^*))u_2' - b\right] \times (c)^{-1} \\ &= \frac{[g_1'(0)(u^*)z((1/b) - u^*) - b]}{c} < 0, \end{aligned} \tag{38}$$

and $(u_3 - u_2)' = z < 0$. Therefore, the solution of (17) will enter Q.

(b) Case $z = 0$. This case is classified as follows.

(1) Case $0 < u_3 < u_2 < u^*$. Then $z' = m(u_3 - u_2) < 0$ and the solution of (17) will enter Q.

(2) Case $0 = u_3 < u_2 < u^*$. Then $u_3' = z = 0$, $u_3'' = z' = -mu_2 < 0$. The solution of (17) will enter $\mathrm{int}(W)$.

(3) Case $0 < u_3 = u_2 < u^*$. Then $(u_3 - u_2)' = -h(u_2)u_2/c < 0$, $z' = 0$, and $z'' = cz' + m(u_3 - u_2)' < 0$. The solution of (17) will enter Q.

(4) Case $0 < u_3 < u_2 = u^*$. Then $u_2' < 0$ and $z' = m(u_3 - u_2) < 0$. The solution of (17) will enter Q.

(5) Case $0 = u_3 = u_2 < u^*$. In this case, $(0,0,0)$ is equilibrium and is constant.

(6) Case $0 = u_3$ and $u_2 = u^*$. Then $u_3' = z = 0$ and $u_3'' = z' = -mu_2 < 0$. The solution of (17) will enter $\mathrm{int}(W)$.

(7) Case $u_3 = u_2 = u^*$. Then $(u^*, u^*, 0)$ is equilibrium and is constant.

The proof is completed. □

5. Existence of Traveling Wave Solution for $c \geq c^*$

In this section, we prove the existence of traveling wave solution for $c \geq c^*$. Firstly, we study the behaviors of solutions near E_1.

5.1. Behaviors of Solutions Near E_1

Lemma 8. *Suppose $(u_2(s), u_3(s), z(s))$ is a solution of (17) satisfying initial conditions*

$$z(0) > ku_3(0), \qquad u_3(0) > \frac{bck + b^2 - f_1'(0)}{g_1'(0)} u_2(0) > 0, \tag{39}$$

where $k = (\lambda_1 + \lambda_2)/2$. Then, for every $s > 0$, we have

$$z(s) > ku_3(s), \qquad u_3(s) > \frac{bck + b^2 - f_1'(0)}{g_1'(0)} u_2(s) > 0. \tag{40}$$

Proof. From Lemma 3, we have $(bck + b^2 - f_1'(0))/g_1'(0) > 0$. To finish the proof, it is sufficient to prove that the set

$$\Psi = \left\{(u_2, u_3, z) : z > ku_3, u_3 > \frac{bck + b^2 - f_1'(0)}{g_1'(0)} u_2 > 0\right\} \tag{41}$$

is positively invariant. It is obvious that

$$\partial\Psi = \partial\Psi_1 \cup \partial\Psi_2 \cup \partial\Psi_3 \cup E_1, \tag{42}$$

where

$$\begin{aligned} \partial\Psi_1 &= \left\{(u_2, u_3, z) : z = ku_3, u_3 \geq \frac{bck + b^2 - f_1'(0)}{g_1'(0)} u_2 > 0\right\}, \\ \partial\Psi_2 &= \left\{(u_2, u_3, z) : z > ku_3, u_3 = \frac{bck + b^2 - f_1'(0)}{g_1'(0)} u_2 \geq 0\right\}, \\ \partial\Psi_3 &= \{(u_2, u_3, z) : z \geq ku_3, u_3 > u_2 = 0\}. \end{aligned} \tag{43}$$

Suppose that $(u_2(s_0), u_3(s_0), z(s_0)) \in \partial\Psi_1$. Then, $z(s_0) = ku_3(s_0)$ and

$$\begin{aligned} &\frac{d}{ds}[z(s) - ku_3(s)]_{s=s_0} \\ &\quad = cz(s_0) + m[u_3(s_0) - u_2(s_0)] - kz(s_0) \\ &\quad = (c-k)z(s_0) + m[u_3(s_0) - u_2(s_0)] \\ &\quad = (c-k)ku_3(s_0) + m[u_3(s_0) - u_2(s_0)] \\ &\quad = [(c-k)k + m]u_3(s_0) - mu_2(s_0) \\ &\quad \geq \left\{[(c-k)k + m]\frac{bck + b^2 - f_1'(0)}{g_1'(0)} - m\right\} u_2(s_0) \\ &\quad = -\frac{bc}{g_1'(0)} H(k) u_2(s_0) > 0. \end{aligned} \tag{44}$$

The last inequality is given since $\lambda_2 < k < \lambda_1$. Suppose that $(u_2(s_0), u_3(s_0), z(s_0)) \in \partial\Psi_2$. If $u_2(s_0) > 0$, then

$$\begin{aligned}
&\frac{d}{ds}\left[u_3(s) - \frac{bck + b^2 - f_1'(0)}{g_1'(0)}u_2(s)\right]_{s=s_0} \\
&= \left\{z - \frac{bck + b^2 - f_1'(0)}{g_1'(0)} \cdot \frac{1}{c} \right.\\
&\quad \left. \times \left[(f_1(u_2) + g_1(u_3))\left(\frac{1}{b} - u_2\right) - bu_2\right]\right\}_{s=s_0} \\
&> \left\{ku_3 - \frac{bck + b^2 - f_1'(0)}{cg_1'} \right.\\
&\quad \left. \times \left[\frac{f_1'(0)u_2 + g_1'(0)u_3}{b} - bu_2\right]\right\}_{s=s_0} \\
&= \left\{\frac{b^2 - f_1'(0)}{bcg_1'}\left[\left(bck + b^2 - f_1'(0)\right)u_2 - g_1'(0)u_3\right]\right\}_{s=s_0} \\
&= 0.
\end{aligned} \tag{45}$$

If $u_2(s_0) = 0$, we have

$$\begin{aligned}
&\frac{d}{ds}\left[u_3(s) - \frac{bck + b^2 - f_1'(0)}{g_1'(0)}u_2(s)\right]_{s=s_0} \\
&\quad = z(s_0) > 0.
\end{aligned} \tag{46}$$

Consequently, the solution of system (17) departing from Ψ cannot intersect $\partial\Psi_1 \cup \partial\Psi_2$. If $(u_2(s_0), u_3(s_0), z(s_0)) \in \partial\Psi_3$, then $u_2'(s_0) = g_1(u_3(s_0))/(bc) > 0$. Since E_1 is equilibrium, in summary, Ψ is positive invariant. □

Since $\lambda_1 > \lambda_2 > 0$, stable manifold theorem implies that there exists a one-dimensional strong unstable manifold $\mathscr{W}_1$ tangent to e_1 at E_1 such that the point on $\mathscr{W}_1$ near E_1 can be expressed by

$$G_1(\varepsilon) = \varepsilon e_1 + o(\varepsilon). \tag{47}$$

Furthermore, there is a two-dimensional unstable manifold $\mathscr{W}_2$ tangent to span $\{e_1, e_2\}$ at E_1 such that $\mathscr{W}_2$ near E_1 can be expressed by

$$G_2(\varepsilon_1, \varepsilon_2) = \varepsilon_1 e_1 + \varepsilon_2 e_2 + o\left(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right). \tag{48}$$

Lemma 9. *Suppose that $u(s) \triangleq (u_2(s), u_3(s), z(s))$ is a solution of (17) such that $u(0) \in \mathscr{W}_1$ for small $\varepsilon > 0$. Then, $u(s)$ will leave W and enter P.*

Proof. Obviously, $u(s)$ satisfies initial condition (39) by the structure of e_1, and Lemma 8 implies $u(s) > 0$ ($u(s) > 0$ means that $u_i(s) > 0$ and $z(s) > 0$, $i = 2, 3$) for every $s > 0$.

Furthermore, Lemma 8 shows that $u_3'(s) = z(s) > ku_3(s)$, implying $\lim_{s\to+\infty} u_3(s) = +\infty$. Since $u_2(s) < 1/b$, it follows $\lim_{s\to+\infty} z(s) = +\infty$. Suppose that $u_2(s) < u^*$ for every $s > 0$. Then

$$\begin{aligned}
u_2' &> \frac{[((f_1(u^*)/u^*) + (g_1(u_3)/u^*))((1/b) - u^*) - b]u_2}{c} \\
&> \frac{[((f_1(u^*)/u^*) + (g_1(2u^*)/u^*))((1/b) - u^*) - b]u_2}{c} \\
&= \frac{Mu_2}{c} > 0
\end{aligned} \tag{49}$$

for large s since $u_3(s)$ and $g_1(u_3)$ are strictly monotonous increasing with respect to s and u_3, respectively. Thus, we have that $\lim_{s\to+\infty} u_2(s) = +\infty$, contradicting $u_2(s) < 1/b$ for any $s \in R$. Therefore, there exists $s_1 > 0$ such that $u_2(s_1) = u^*$. Without losing generality, let $s_1 = \inf\{s > 0 : u_2(s) = u^*\}$. Obviously, we have $u_2'(s_1) \geq 0$. If $u_3(s_1) < u^*$, then

$$\begin{aligned}
&u_2'(s_1) \\
&\quad = \frac{[((f_1(u^*)/u^*) + (g_1(u_3(s_1))/u^*))((1/b) - u^*) - b]u^*}{c} \\
&\quad < 0,
\end{aligned} \tag{50}$$

which is a contradiction. Therefore, $u_3(s_1) \geq u^*$ and $u(s_1) \in \partial P$. Then, the construction of W^- shows that $u(s)$ will leave W and enter P. □

Let C be a small circle on $\mathscr{W}_2$ centered at E_1. Then, points on C can be expressed in terms of local coordinate by

$$F(\theta) \triangleq G_2(\varepsilon\cos\theta, \varepsilon\sin\theta) = \varepsilon\left[e_1\cos\theta + e_2\sin\theta + O(\varepsilon)\right], \tag{51}$$

where $\theta \in [\theta_1, 2\pi + \theta_1)$, $\varepsilon > 0$, and θ_1 is chosen such that $F(\theta_1)$ lies on $\mathscr{W}_1$ with $z > 0$. Then, stable manifold theorem shows that $\theta_1 \to 0$ when $\varepsilon \to 0$. Denote $F(\theta) \triangleq (\overline{u}_2(\theta), \overline{u}_3(\theta), \overline{z}(\theta))$.

Lemma 10. *There exists a $\theta_2 \in (\pi/2, 3\pi/4)$ such that*

$$\overline{z}(\theta_2) = 0, \qquad 0 < \overline{u}_3(\theta_2) < \overline{u}_2(\theta_2) < u^*, \tag{52}$$

and that

$$\overline{z}(\theta) > 0, \qquad 0 < \overline{u}_2(\theta_2) < u^*, \qquad 0 < \overline{u}_3(\theta) < u^* \tag{53}$$

for $\theta \in [\theta_1, \theta_2)$.

Proof. From (51), we have

$$\begin{aligned}
\overline{z}(\theta) &= \varepsilon\left[\lambda_1\cos\theta + \lambda_2\sin\theta + O(\varepsilon)\right] \\
&= \varepsilon\sqrt{\lambda_1^2 + \lambda_2^2} \\
&\quad \times \left[\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\cos\theta + \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}\sin\theta + O(\varepsilon)\right] \\
&= \varepsilon\sqrt{\lambda_1^2 + \lambda_2^2}\left[\sin(\varphi_0 + \theta) + O(\varepsilon)\right],
\end{aligned} \tag{54}$$

where $\sin(\varphi_0) = \lambda_1/\sqrt{\lambda_1^2+\lambda_2^2}$, $\cos(\varphi_0) = \lambda_2/\sqrt{\lambda_1^2+\lambda_2^2}$, and $\varphi_0 \in (\pi/4, \pi/2)$ since $\lambda_1 > \lambda_2$. Therefore, $\bar{z}(\theta_2) = 0$ and $\theta_2 \in [0,\pi]$ imply that $\theta_2 = \pi - \varphi_0 + O(\varepsilon) \in (\pi/2, 3\pi/4)$. Obviously, $\bar{z}(\theta) > 0$ for any $\theta \in [\theta_1, \theta_2)$. However,

$$\begin{aligned}\bar{u}_2(\theta) &= \frac{\varepsilon g_1'(0)}{bc}\left[\frac{\cos\theta}{\lambda_1-\lambda^*} + \frac{\sin\theta}{\lambda_2-\lambda^*} + O(\varepsilon)\right] \\ &= \frac{\varepsilon}{m}\left[\left(-\lambda_1^2+\lambda_1 c+m\right)\cos\theta \right.\\ &\qquad \left. +\left(-\lambda_2^2+\lambda_2 c+m\right)\sin\theta + O(\varepsilon)\right], \\ \bar{u}_3(\theta) &= \varepsilon\left[\cos\theta+\sin\theta+O(\varepsilon)\right], \\ \bar{u}_2(\theta)-\bar{u}_3(\theta) &= \frac{\varepsilon}{m}\left[\lambda_1(c-\lambda_1)\cos\theta+\lambda_2(c-\lambda_2)\sin\theta+O(\varepsilon)\right].\end{aligned} \tag{55}$$

Then, equality $\bar{z}(\theta_2) = \varepsilon[\lambda_1\cos\theta_2 + \lambda_2\sin\theta_2 + O(\varepsilon)] = 0$, together with the last of (55), reveals $\bar{u}_2(\theta_2)-\bar{u}_3(\theta_2) = \varepsilon[(\lambda_1-\lambda_2)\lambda_2\sin\theta_2 + O(\varepsilon)]/m > 0$; that is, $\bar{u}_2(\theta_2) > \bar{u}_3(\theta_2)$. For $\theta \in [\theta_1,\theta_2]$, the first and second equalities of (55) imply that $0 < \bar{u}_i(\theta) < u^*$ where $i = 2, 3$ since $\lambda_1 > \lambda_2$ and $0 < \varepsilon \ll 1$. □

Let

$$\Sigma = \{F(\theta) \mid \theta \in [\theta_1,\theta_2], \varepsilon \text{ is small enough}\}. \tag{56}$$

By Lemma 10, Σ is an arc of circle, $\Sigma \subseteq W$, and the solution of (17) with initial value being the endpoint $F(\theta_2)$ will enter Q since $F(\theta_2) \in W^- \cap \partial Q$. From Lemma 9, the solution of (17) with initial value being the endpoint $F(\theta_1)$ will enter P.

5.2. Traveling Wave Solution for $c > c^*$

Lemma 11. *Let $u(s) = (u_2(s), u_3(s), z(s))$ be a solution of (17) such that $u(0) \in \Lambda$. If $u(s) \in W$ for any $s \ge 0$, then $u(s) \in \Lambda$ for any $s > 0$, where*

$$\Lambda = \{(u_2,u_3,z) : 0 < u_2 < u^*, u_3 > 0, 0 < z < ku_3\} \tag{57}$$

and $k = c + \sqrt{c^2+4m}$.

Proof. Set $s_0 = \inf\{s : u(s) \notin \Lambda, s \ge 0\}$. Suppose the conclusion is false; that is, $s_0 < +\infty$. Obviously, $s_0 > 0$ and $u(s_0) \in \partial\Lambda$ where

$$\begin{aligned}\partial\Lambda &= \left(\cup_{i=1}^{7}\partial\Lambda_i\right) \cup E_2, \\ \partial\Lambda_1 &= \{(u_2,u_3,z) : u_2 = u^*, u_3 \ge u^*, 0 \le z \le ku_3\} \setminus E_2, \\ \partial\Lambda_2 &= \{(u_2,u_3,z) : 0 < u_3 \le u_2 \le u^*, z = 0\} \setminus E_2, \\ \partial\Lambda_3 &= \{(u_2,u_3,z) : u_2 = 0, u_3 > 0, 0 \le z \le ku_3\}, \\ \partial\Lambda_4 &= \{(u_2,u_3,z) : u_2 = u^*, 0 < u_3 < u^*, 0 < z \le ku_3\}, \\ \partial\Lambda_5 &= \{(u_2,u_3,z) : 0 \le u_2 \le u^*, u_3 > 0, z = ku_3\}, \\ \partial\Lambda_6 &= \{(u_2,u_3,z) : u_3 > u_2, 0 \le u_2 < u^*, u_3 > 0, z = 0\}, \\ \partial\Lambda_7 &= \{(u_2,u_3,z) : 0 \le u_2 \le u^*, u_3 = z = 0\}.\end{aligned} \tag{58}$$

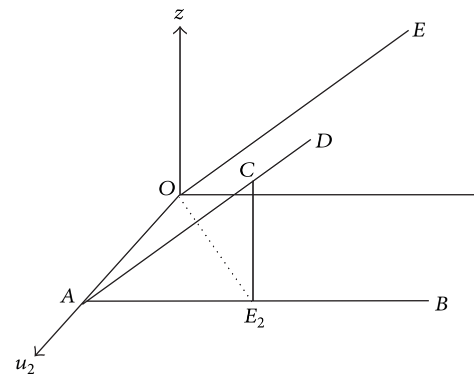

Figure 3: The construction of $\partial\Lambda$.

In Figure 3, we find $\partial\Lambda_1$ = {unbounded area BE_2CD}, $\partial\Lambda_2$ = {triangle OAE_2O}, $\partial\Lambda_3$ = {unbounded cone u_3OE}, $\partial\Lambda_4$ = {triangle AE_2CA}, $\partial\Lambda_5$ = {unbounded area $DCAOE$}, $\partial\Lambda_6$ = {unbounded area BE_2Ou_3}, and $\partial\Lambda_7$ = {segment OA}.

Since $\partial\Lambda_1 \cup \partial\Lambda_2 \subset W^-$, thus $u(s_0) \notin \partial\Lambda_1 \cup \partial\Lambda_2$. If $u(s_0) \in \partial\Lambda_3$, we have $u_2'(s_0) \le 0$ because $u_2(s) > 0$ for $0 < s < s_0$ and $u_2(s_0) = 0$. However, $u_2'(s_0) = g_1(u_3(s_0))/(bc) > 0$ which is a contradiction. Therefore, $u(s_0) \notin \partial\Lambda_3$. If $u(s_0) \in \partial\Lambda_4$, then

$$\begin{aligned}&u_2'(s_0) \\ &= \frac{\left[\left(\left(f_1(u^*)/u^*\right) + \left(g_1(u_3(s_0))/u^*\right)\right)\left((1/b) - u^*\right) - b\right]u^*}{c} \\ &< 0,\end{aligned} \tag{59}$$

contradicting $u_2'(s_0) \ge 0$. If $u(s_0) \in \partial\Lambda_5$, then

$$\begin{aligned}&[z(s) - ku_3(s)]'_{s=s_0} \\ &\quad = (c-k)z(s_0) + m[u_3(s_0) - u_2(s_0)] \\ &\quad = [(c-k)k + m]u_3(s_0) - mu_2(s_0) < 0\end{aligned} \tag{60}$$

since $(c-k)k + m < 0$, contradicting $[z(s) - ku_3(s)]'_{s=s_0} \ge 0$. If $u(s_0) \in \partial\Lambda_6$, then $z'(s_0) = m[u_3(s_0) - u_2(s_0)] > 0$ which is a contradiction. In conclusion, $u(s_0) \notin \partial\Lambda_4 \cup \partial\Lambda_5 \cup \partial\Lambda_6$. If $u(s_0) \in \partial\Lambda_7$, then $u_3(s) > 0$ and $z(s) > 0$ for any $0 < s < s_0$. Hence, $u_3'(s) = z(s) > 0$ for any $0 < s < s_0$, which implies that $u_3(s_0) > u_3(0) > 0$. From this contradiction we find $u(s_0) \notin \partial\Lambda_7$. Because E_2 is a constant solution, we get $u(s_0) \ne E_2$. In summary, $u(s_0) \notin \partial\Lambda$ and $s_0 = +\infty$. The proof is completed. □

Lemma 12. *There exists a point $u_0 = (u_{20}, u_{30}, z_0) \in \Sigma$ such that the solution $u(s; u_0) = (u_2(s), u_3(s), z(s))$ of (17) with initial value being u_0 will stay in W for any $s > 0$.*

Proof. It is sufficient to prove $\Sigma \ne \Sigma_0$. Suppose that $\Sigma = \Sigma_0$. Firstly, we verify Conditions (1) and (2) of Lemma 6. Condition (1) of Lemma 6 is valid since W is closed.

Suppose $u_0 = (u_{20}, u_{30}, z_0) \in \Sigma$, $s < T(u_0)$ and $u(s; u_0) \in W \setminus W^-$. Then, $u(s; u_0) \in \operatorname{int} W \cup J$ and $u_0 \neq F(\theta_2)$ since $F(\theta_2) \in W^-$. The structure of Σ implies that $u_{20} > 0$, $u_{30} > 0$, and $z_0 > 0$. By the proof of Lemma 11, we have that $u(s; u_0) > 0$ for $s < T(u_0)$. Therefore, $u(s; u_0) \notin J$ and $u(s; u_0) \in \operatorname{int} W$. Condition (2) of Lemma 6 holds.

Lemma 6 shows that Σ is homeomorphic to $H(\Sigma)$. Since

$$H(F(\theta_1)) \in \partial P \cap W^-, \qquad H(F(\theta_2)) \in \partial Q \cap W^-, \quad (61)$$

and W^- is disconnected, we have that $H(\Sigma)$ is disconnected, contradicting the connection of Σ. Thus, $\Sigma \neq \Sigma_0$ and the proof is completed. □

Lemma 13. *Let $c > c^*$. Then, there exists a positive solution $u(s) = (u_2(s), u_3(s), z(s))$ of (17) such that*

$$u(+\infty) = E_2, \qquad u(-\infty) = E_1. \quad (62)$$

Proof. By Lemma 12 there exists a point $u_0 = (u_{20}, u_{30}, z_0) \in \Sigma$ such that the solution $u(s; u_0) = (u_2(s), u_3(s), z(s))$ of (17) with initial value being u_0 will stay in W for any $s > 0$. Furthermore, Lemma 11 shows $u(s; u_0) > 0$ for any $s \geq 0$. Stable manifold theorem implies that $u(s; u_0) > 0$ for any $s \leq 0$ and $\lim_{s\to-\infty} u(s; u_0) = E_1$. Therefore, $u(s; u_0)$ is a positive solution.

To complete the proof, it is sufficient to show that $\lim_{s\to+\infty} u(s; u_0) = E_2$. By Lemma 11, we know that $u_2(s) < u^*$ for any $s > 0$ since $u(s; u_0)$ remains in W for all s. Because $u_3'(s) = z(s) > 0$, then the limit of $u_3(s)$ exists; that is, $\lim_{s\to+\infty} u_3(s) = u_3^*$ and $0 < u_3^* \leq +\infty$. Suppose that $u^* < u_3^* \leq +\infty$. The first equation of (17) shows that

$$\begin{aligned} u_2' &> \frac{[((f_1(u^*)/u^*)+(g_1(u_3)/u^*))((1/b) - u^*) - b]\,u_2}{c} \\ &> \left(\left[\left(\frac{f_1(u^*)}{u^*} + \frac{g_1((u_3^* + u^*)/2)}{u^*}\right)\left(\frac{1}{b} - u^*\right) - b\right] u_2\right) \times (c)^{-1} \\ &= \frac{M u_2}{c} > 0 \end{aligned} \quad (63)$$

for large s, which implies that there is an $s^* > 0$ such that $u_2(s^*) > u^*$. This is a contradiction, and thus $0 < u_3^* \leq u^*$. From the first equation of (17), we have $\lim_{s\to+\infty} u_2(s) = u_2^*$ where u_2^* is the only positive root of algebra equation

$$[f_1(u_2) + g_1(u_3^*)]\left(\frac{1}{b} - u_2\right) - b u_2 = 0. \quad (64)$$

At the same time, the third equation of (17) implies $\lim_{s\to+\infty} z(s) = z^*$ and $z^* = m(u_2^* - u_3^*)/c$ or $\pm\infty$. It is impossible that $z^* = \pm\infty$ due to the boundedness of $u_3(s)$. In conclusion, the limit $\lim_{s\to+\infty} u(s) = (u_2^*, u_3^*, z^*)$ exists and is finite. By [42], (u_2^*, u_3^*, z^*) must be equilibrium. Since $u_3^* > 0$, then $(u_2^*, u_3^*, z^*) = E_2$. □

Noticing the relation of systems (17) and (10) completes the proof of Theorem 1 for case $c > c^*$.

5.3. Traveling Wave Solution for $c = c^*$. Firstly, suppose $c > c^*$ and let $u(s; u_0) = (u_2(s), u_3(s), z(s))$ be the traveling wave solution of (17). Then, Lemma 11 implies that $u(s; u_0) \in \Lambda$ for all s. From the proof of Lemma 13, we find $u_3(s) \leq u^*$. Therefore, for all s, we have $u(s; u_0) \in \Pi$ where

$$\Pi = \{(u_2, u_3, z) : 0 < u_2 < u^*, 0 < u_3 \leq u^*, 0 < z < k u_3\}. \quad (65)$$

Let $\{c_n\}$ be a sequence such that $c^* < c_n < c_{n+1}$ for any n and $\lim_{n\to\infty} c_n = c^*$. Set $k_n = c_n + \sqrt{c_n^2 + 4m}$ and

$$\Pi_n = \{(u_2, u_3, z) : 0 < u_2 \leq u^*, 0 < u_3 \leq u^*, 0 < z \leq k_n u_3\}. \quad (66)$$

Then, $\Pi_n \subseteq \Pi_1$ for any n.

Lemma 13 shows that there is a positive solution $w_n(s) = (u_{2,n}(s), u_{3,n}(s), z_n(s))$ for system

$$\begin{aligned} u_2' &= \frac{[(f_1(u_2) + g_1(u_3))((1/b) - u_2) - b u_2]}{c_n}, \\ u_3' &= z, \\ z' &= c_n z + m(u_3 - u_2), \end{aligned} \quad (67)$$

satisfying boundary condition (62) such that $w_n(s) \in \Pi_n \subseteq \Pi_1$ for any s.

Lemma 14. *Let $c = c^*$. Then, there exists a traveling wave solution for system (6) satisfying boundary condition (11).*

Proof. It is sufficient to prove that there exists a positive solution $u(s) = (u_2(s), u_3(s), z(s))$ of (17) satisfying boundary condition (62).

Firstly, we show that sequences $\{u_{2,n}\}$, $\{u_{3,n}\}$, $\{z_n\}$, $\{u_{2,n}'\}$, $\{u_{3,n}'\}$, and $\{z_n'\}$ are uniformly bounded and equicontinuous. The idea of Lemma 11 in [34] is used. Obviously, $\{u_{2,n}\}$, $\{u_{3,n}\}$, and $\{z_n\}$ are uniformly bounded since $w_n(s) \subseteq \Pi_1$ for any s. Because $w_n(s) = (u_{2,n}(s), u_{3,n}(s), z_n(s))$ is the solution of (67), $\{u_{2,n}'\}$, $\{u_{3,n}'\}$, and $\{z_n'\}$ are also uniformly bounded. Since $|z_n(s_1) - z_n(s_2)| = z_n'(s_3)|s_1 - s_2|$ where $s_1 < s_3 < s_2$, then $\{z_n\}$ is equicontinuous. Similarly, $\{u_{2,n}\}$ and $\{u_{3,n}\}$ are also equicontinuous. By differentiating the equations of (67) and using the previous bounds, we can get that $\{u_{2,n}''\}$, $\{u_{3,n}''\}$, and $\{z_n''\}$ are uniformly bounded, and hence $\{u_{2,n}'\}$, $\{u_{3,n}'\}$, and $\{z_n'\}$ are equicontinuous.

The previous paragraph and Arzelà-Ascoli theorem imply that there exist subsequences, again denoted by $\{u_{2,n}\}$, $\{u_{3,n}\}$, and $\{z_n\}$ and functions u_2, u_3, and z such that

$$u_{2,n} \longrightarrow u_2, \qquad u_{3,n} \longrightarrow u_3, \qquad z_n \longrightarrow z \quad (68)$$

uniformly on compact subsets of $\mathbb{R}$, thus pointwise on $\mathbb{R}$. Same arguments imply that $\{u_{2,n}'\}$, $\{u_{3,n}'\}$, and $\{z_n'\}$ are also uniformly convergent on compact subsets of $\mathbb{R}$ and pointwise convergent on $\mathbb{R}$. Consequently, we get

$$u_{2,n}' \longrightarrow u_2', \qquad u_{3,n}' \longrightarrow u_3', \qquad z_n' \longrightarrow z'. \quad (69)$$

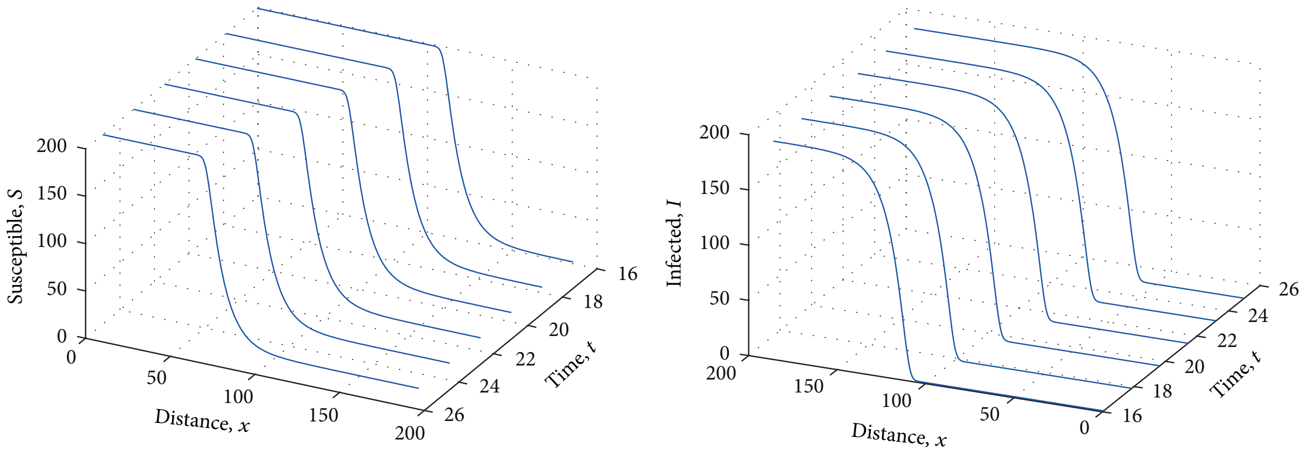

Figure 4: The wave profiles for S and I and their movements.

Since $(u_{2,n}, u_{3,n}, z_n)$ is the solution of (67), then $u(s) = (u_2(s), u_3(s), z(s))$ is the solution of (17) for $c = c^*$ and $u(s) \in \mathrm{cl}(\Pi_1)$, where $\mathrm{cl}(\Pi_1)$ is the closer of Π_1. Because system (67) is autonomous and $(u_{2,n}, u_{3,n}, z_n)$ satisfies boundary condition (62), we can assume that $u_{3,n}(0) = u^*/2$ for all n; thus, $u_3(0) > 0$. Then, similar to the proof of Lemma 13, we have that the solution $u(s)$ satisfies boundary condition (62). □

6. Simulations

In this section, we present some simulations to confirm the theoretical results. Set

$$f(I) = \frac{\beta_1 I}{K_h + I}, \qquad g(B) = \frac{\beta_2 B}{K_e + B}, \tag{70}$$

and assign numerical values to parameters as follows:

$$b = 0.01, \quad e = 1, \quad m = 0.5, \quad K_e = 6, \quad K_h = 2,$$
$$\beta_1 = 0.62, \quad \beta_2 = 0.001, \quad N = 200, \quad d = 2. \tag{71}$$

Obviously, such selection for $f(I)$ and $g(B)$ satisfies (A1) and (A2). Then, the traveling wave solution is described in Figure 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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