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Research Article

Complete Self-Shrinking Solutions for Lagrangian Mean Curvature Flow in Pseudo-Euclidean Space

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Let $f(x)$ be a smooth strictly convex solution of $\det(\partial^2 f/\partial x_i \partial x_j) = \exp\{(1/2)\sum_{i=1}^n x_i(\partial f/\partial x_i) - f\}$ defined on a domain $\Omega \subset \mathbb{R}^n$; then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinker of mean curvature flow in Pseudo-Euclidean space $\mathbb{R}^{2n}_n$ with the indefinite metric $\sum dx_i dy_i$. In this paper, we prove a Bernstein theorem for complete self-shrinkers. As a corollary, we obtain if the Lagrangian graph $M_{\nabla f}$ is complete in R^{2n}_n and passes through the origin then it is flat.

1. Introduction

Let M be an n-dimensional submanifold immersed into the Euclidean space $\mathbb{R}^{n+m}$. Mean curvature flow is a one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \to \mathbb{R}^{n+m}$ with corresponding images $M_t = X_t(M)$ such that

$$\frac{d}{dt}X(x,t) = H(x,t), \quad x \in M, \qquad X(x,0) = X(x) \tag{1}$$

is satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $X(x,t)$ in $\mathbb{R}^{n+m}$. Self-similar solutions to the mean curvature flow play an important role in understanding the behavior of the flow and the types of singularities. They satisfy a system of quasilinear elliptic PDE of the second order as follows:

$$H = -\frac{X^{\perp}}{2}, \tag{2}$$

where $(\cdots)^{\perp}$ stands for the orthogonal projection into the normal bundle NM.

Self-shrinkers in the ambient Euclidean space have been studied by many authors; for example, see [1–6] and so forth. For recent progress and related results, see the introduction in [7]. When the ambient space is a pseudo-Euclidean space, there are many classification works about self-shrinkers; for example, see [8–13] and so forth. But very little is known when self-shrinkers are complete not compact with respect to induced metric from pseudo-Euclidean space. In this paper, we will characterize self-shrinkers for Lagrangian mean curvature flow in the pseudo-Euclidean space from this aspect.

Let $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be null coordinates in $2n$-dimensional pseudo-Euclidean space $\mathbb{R}^{2n}_n$. Then, the indefinite metric (cf. [14]) is defined by $ds^2 = \sum_{i=1}^n dx_i dy_i$. Suppose $f(x)$ is a smooth strictly convex function defined on domain $\Omega \subset \mathbb{R}^n$. The graph $M_{\nabla f}$ of ∇f can be written as $(x_1, \ldots, x_n; \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. Then, the induced Riemannian metric on $M_{\nabla f}$ is given by

$$G = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j. \tag{3}$$

In particular, if function f satisfies

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \exp\left\{\frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} - f\right\}, \tag{4}$$

then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinking solution for mean curvature flow in $\mathbb{R}^{2n}_n$.

Huang and Wang [12] and Chau et al. [8] have used different methods to investigate the entire solutions to the above equation and showed that an entire smooth strictly convex solution to (4) in $\mathbb{R}^n$ is the quadratic polynomial under the decay condition on Hessian of f. Later Ding and Xin in [10] improve the previous ones in [8, 12] by removing the additional assumption and prove the following.

Theorem 1. *Any space-like entire graphic self-shrinking solution to Lagrangian mean curvature flow in R^{2n}_n with the indefinite metric $\sum_i dx_i dy_i$ is flat.*

These rigidity results assume that the self-shrinker graphs are entire. Namely, they are Euclidean complete. Here, we will characterize the rigidity of self-shrinker graphs from another completeness and pose the following problem.

If a graphic self-shrinker is complete with respect to induced metric from ambient space R^{2n}_n, then is it flat?

In this paper, we will use *affine technique* (see [15–18]) to prove the following Bernstein theorem. As a corollary, it gives a partial affirmative answer to the above problem.

Theorem 2. *Let $f(x)$ be a C^∞ strictly convex function defined on a convex domain $\Omega \subseteq \mathbb{R}^n$ satisfying the PDE (4). If there is a positive constant α depending only on n such that the hypersurface $M = \{(x, f(x))\}$ in $\mathbb{R}^{n+1}$ is complete with respect to the metric*

$$\widetilde{G} = \exp\left\{\alpha \sum x_i \frac{\partial f}{\partial x_i}\right\} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j, \tag{5}$$

then f is the quadratic polynomial.

Remark 3. If $f(x)$ is a strictly convex solution to (4), then the graph $\{(x, \nabla f/2n\alpha)\}$ is a minimal manifold in R^{2n}_n endowed with the conformal metric $ds^2 = \exp\{-\alpha x \cdot y\} dx \cdot dy$.

As a direct application of Theorem 2, we have the following.

Corollary 4. *Let f be a strictly convex C^∞-function defined on a convex domain $\Omega \subset \mathbb{R}^n$. If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}^{2n}_n$ is a complete space-like self-shrinker for mean curvature flow and the sum $\sum x_i(\partial f/\partial x_i)$ has a lower bound, then $M_{\nabla f}$ is flat.*

When the shrinker passes through the origin especially, we have the following corollary.

Corollary 5. *If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}^{2n}_n$ is a complete space-like self-shrinker for mean curvature flow and passes through the origin, then $M_{\nabla f}$ is flat.*

2. Preliminaries

Let $f(x_1, \ldots, x_n)$ be a strictly convex C^∞-function defined on a domain $\Omega \subset \mathbb{R}^n$. Consider the graph hypersurface

$$M := \{(x, f(x)) \mid x_{n+1} = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \Omega\}. \tag{6}$$

For M, we choose the canonical relative normalization $Y = (0, 0, \ldots, 1)$. Then, in terms of the language of the relative affine differential geometry, the *Calabi metric*

$$G = \sum f_{ij} dx_i dx_j \tag{7}$$

is the relative metric with respect to the normalization Y. For the position vector $y = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$, we have

$$y_{,ij} = \sum A^k_{ij} y_k + f_{ij} Y, \tag{8}$$

where "," denotes the covariant derivative with respect to the Calabi metric G. We recall some fundamental formulas for the graph M; for details, see [19]. The Levi-Civita connection with respect to the metric G has the *Christoffel symbols*

$$\Gamma^k_{ij} = \frac{1}{2} \sum f^{kl} f_{ijl}. \tag{9}$$

The *Fubini-Pick tensor* A_{ijk} satisfies

$$A_{ijk} = -\frac{1}{2} f_{ijk}. \tag{10}$$

Consequently, for the *relative Pick invariant*, we have

$$J = \frac{1}{4n(n-1)} \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}. \tag{11}$$

The *Gauss integrability conditions* and the *Codazzi equations* read

$$R_{ijkl} = \sum f^{mh} \left(A_{jkm} A_{hil} - A_{ikm} A_{hjl}\right), \tag{12}$$

$$A_{ijk,l} = A_{ijl,k}. \tag{13}$$

From (12), we get the Ricci tensor

$$R_{ik} = \sum f^{mh} f^{lj} \left(A_{iml} A_{hjk} - A_{imk} A_{hlj}\right). \tag{14}$$

Introduce the Legendre transformation of f

$$\xi_i = \frac{\partial f}{\partial x_i}, \quad i = 1, 2, \ldots, n,$$
$$u(\xi_1, \ldots, \xi_n) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} - f(x). \tag{15}$$

Define the functions

$$\rho := \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = \left[\det\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right)\right]^{1/(n+2)},$$
$$\Phi := \sum f^{ij} (\ln \rho)_i (\ln \rho)_j = \frac{\|\nabla \rho\|^2}{\rho^2}, \tag{16}$$

here and later the norm $\|\cdot\|$ is defined with respect to the Calabi metric. From the PDE (4), we obtain

$$\frac{\partial \ln \rho}{\partial x_i} = \frac{1}{2(n+2)} (f - u)_i = \frac{1}{2(n+2)} \{f_i - x_k f_{ki}\}. \tag{17}$$

That is,

$$x_i = \sum f^{ik} \left(f_k - 2(n+2)(\ln \rho)_k \right). \tag{18}$$

Using (17) and (18), we can get

$$\rho_{ij} = \frac{\rho_i \rho_j}{\rho} + f^{kl} f_{ijk} \rho_l - \frac{f^{kl} f_{ijk} f_l \rho}{2(n+2)}. \tag{19}$$

Put $\tau := (1/2) \sum f^{ij} (\rho_i / \rho) f_j$. From (19), we have

$$\Delta \rho = -\frac{n}{2} \frac{\|\nabla \rho\|^2}{\rho} + \tau \rho. \tag{20}$$

By (17), we get

$$4(n+2)^2 \Phi = \|\nabla u\|^2 + \|\nabla f\|^2 - 2(f+u), \tag{21}$$

and then

$$\|\nabla (f+u)\|^2 = 4(n+2)^2 \Phi + 4(f+u). \tag{22}$$

Using (17) yields

$$\Delta (f+u) = 2n + (n+2)^2 \Phi. \tag{23}$$

Define a conformal Riemannian metric $\widetilde{G} := \exp\{\alpha(f+u)\} G$, where α is a constant.

Conformal Ricci Curvature. Denote by $\widetilde{R}_{ij}$ the Ricci curvature with respect to the metric $\widetilde{G}$; then

$$\begin{aligned} \widetilde{R}_{ij} = R_{ij} &- \frac{(n-2)\alpha}{2} (f+u)_{,ij} + \frac{(n-2)\alpha^2}{4} (f+u)_{,i} (f+u)_{,j} \\ &- \frac{1}{2} \left(\alpha \Delta (f+u) + \frac{(n-2)\alpha^2}{2} \|\nabla (f+u)\|^2 \right) G_{ij}, \end{aligned} \tag{24}$$

where "," again denotes the covariant derivation with respect to the Calabi metric.

Using the above formulas, we can get the following crucial estimates.

Proposition 6. *Let $f(x_1, \ldots, x_n)$ be a C^∞ strictly convex function satisfying PDE (4). Then, the following estimate holds:*

$$\begin{aligned} \Delta \Phi \geq{} & A_1 \langle \nabla \Phi, \nabla \ln \rho \rangle + \frac{1}{4} \langle \nabla (f+u), \nabla \Phi \rangle - A_2 \frac{\|\nabla \Phi\|^2}{\Phi} \\ & + A_3 \Phi^2 + \Phi, \end{aligned} \tag{25}$$

where

$$\begin{aligned} A_1 &= \frac{6n^2 - n + 16}{(n-1)(3n+4)}, \qquad A_2 = \frac{3n^2 + 32n}{8(n-1)(3n+4)}, \\ A_3 &= \frac{64n^3 - 72n^2 - 46n - 72}{5n(n-1)(3n+4)}. \end{aligned} \tag{26}$$

Because its calculation is standard as in [16], we will give its proof in the appendix.

For affine hyperspheres, Calabi in [20] calculated the Laplacian of the Pick invariant J. Later, for a general convex function, Li and Xu proved the following lemma in [17].

Lemma 7. *The Laplacian of the relative Pick invariant J satisfies*

$$\begin{aligned} \Delta J \geq{} & \frac{n+2}{n(n-1)} \sum f^{il} f^{jm} f^{kn} A_{ijk} (\ln \rho)_{,lmn} \\ & + \frac{2}{n(n-1)} \|\nabla A\|^2 + 2J^2 - \frac{(n+2)^4}{4} \Phi^2, \end{aligned} \tag{27}$$

where "," denotes the covariant derivative with respect to the Calabi metric.

Using Lemma 7, we get the following corollary. For the proof, see the appendix.

Corollary 8. *Let $f(x_1, \ldots, x_n)$ be a C^∞ strictly convex function satisfying PDE (4); then*

$$\begin{aligned} \Delta J \geq{} & J^2 - 20(n+2)^8 \Phi^2 + \frac{1}{4} \langle \nabla J, \nabla (f+u) \rangle \\ & + J - \sqrt{n(n-1)} \|\nabla (f+u)\| J^{3/2}. \end{aligned} \tag{28}$$

3. Proof of Theorem 2

It is our aim to prove $\Phi \equiv 0$; thus, from definition of ρ,

$$\det(f_{ij}) = \text{const.} \tag{29}$$

everywhere on M. As in [8], by Euler homogeneous theorem, we get Theorem 2.

Denote by $s(p_0, p)$ the geodesic distance function from $p_0 \in M$ with respect to the metric $\widetilde{G}$. For any positive number a, let $B_a(p_0, \widetilde{G}) := \{p \in M \mid s(p_0, p) \leq a\}$. Denote

$$\begin{aligned} \mathscr{A} &:= \max_{B_a(p_0, \widetilde{G})} \left\{ \left(a^2 - s^2 \right)^2 \exp\{-\alpha(f+u)\} \Phi \right\}, \\ \mathscr{B} &:= \max_{B_a(p_0, \widetilde{G})} \left\{ \left(a^2 - s^2 \right)^2 \exp\{-\alpha(f+u)\} J \right\}. \end{aligned} \tag{30}$$

Lemma 9. *Let f be a strictly convex C^∞-function satisfying the PDE (4). Then, there exist positive constants α and C, depending only on n, such that*

$$\mathscr{A} \leq C \left(a^2 + a^3 \right). \tag{31}$$

Proof. Step 1. We will prove that there exists a constant C depending only on n such that

$$\mathscr{A} \leq C \left(\mathscr{B}^{1/2} a + a^2 + a^3 \right). \tag{32}$$

To this end, consider the function

$$F := \left(a^2 - s^2 \right)^2 \exp\{-\alpha(f+u)\} \Phi \tag{33}$$

defined on $B_a(p_0, \widetilde{G})$, where α is a positive constant to be determined later. Obviously, F attains its supremum at some interior point p^*. We may assume that s^2 is a C^2-function in a neighborhood of p^*. Choose an orthonormal frame field on M around p^* with respect to the Calabi metric G. Then, at p^*,

$$\frac{\Phi_{,i}}{\Phi} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2-s^2} = 0, \tag{34}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \alpha\Delta(f+u) - \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0, \tag{35}$$

where "," denotes the covariant derivative with respect to the Calabi metric G as before, and we used the fact $\|\nabla s\|_G^2 = \exp\{\alpha(f+u)\}$. Inserting Proposition 6 into (35), we get

$$\begin{aligned} &-(1+A_2)\frac{\sum(\Phi_{,i})^2}{\Phi^2} + A_3\Phi + \frac{1}{4}(f+u)_{,i}\frac{\Phi_{,i}}{\Phi} \\ &+ A_1\frac{\Phi_{,i}}{\Phi}\frac{\rho_{,i}}{\rho} + 1 - \alpha\left(2n + (n+2)^2\Phi\right) \\ &- \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0. \end{aligned} \tag{36}$$

Combining (34) with (36) and using the Schwarz inequality, we have

$$\begin{aligned} &\frac{1}{4}\sum(f+u)_{,i}\frac{\Phi_{,i}}{\Phi} \\ &\quad \ge \frac{1}{8}\alpha\sum\left[(f+u)_{,i}\right]^2 - \frac{2}{\alpha}\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}, \\ &A_1\sum\frac{\Phi_{,i}}{\Phi}\frac{\rho_{,i}}{\rho} \ge -\frac{A_3}{4}\Phi - \frac{A_1^2}{A_3}\frac{\Phi_{,i}^2}{\Phi^2}, \\ &\frac{\sum(\Phi_{,i})^2}{\Phi^2} \le 2\left(\alpha^2\sum\left[(f+u)_{,i}\right]^2 + 16\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}\right). \end{aligned} \tag{37}$$

Choose α small enough such that

$$2\left(1 + A_2 + \frac{A_1^2}{A_3}\right)\alpha \le \frac{1}{16}, \qquad \alpha(n+2)^2 \le \frac{A_3}{4}, \qquad 100n\alpha \le 1. \tag{38}$$

Then, by substituting the three estimates above, we get

$$\frac{A_3}{2}\Phi + \frac{1}{16}\alpha(f+u)_{,i}^2 - \exp\{\alpha(f+u)\}\frac{Ca^2}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0, \tag{39}$$

here and later C denotes positive constant depending only on n.

Denote $a^* = s(p_0, p^*)$. If $a^* = 0$, from (39), it is easy to complete the proof of the lemma. In the following, we assume that $a^* > 0$. Now, we calculate the term $4s\Delta s/(a^2-s^2)$. Firstly, we will give a lower bound of the Ricci curvature $\mathrm{Ric}(M, \widetilde{G})$. Assume that

$$\begin{aligned} \max_{B_{a^*}(p_0,\widetilde{G})}\{\exp\{-\alpha(f+u)\}\Phi\} &= \exp\{-\alpha(f+u)\}\Phi(\tilde{p}), \\ \max_{B_{a^*}(p_0,\widetilde{G})}\{\exp\{-\alpha(f+u)\}J\} &= \exp\{-\alpha(f+u)\}J(\tilde{q}). \end{aligned} \tag{40}$$

For any $p \in B_{a^*}(p_0, \widetilde{G})$, by a coordinate transformation, $f_{ij}(p) = \delta_{ij}$ and $R_{ij}(p) = 0$ hold for $i \ne j$. Then, at p,

$$\begin{aligned} R_{ii} &\ge \frac{1}{4}\left(\sum_m f_{mii}^2 + (n+2)\sum_m f_{mii}\frac{\partial}{\partial x_m}\ln\rho\right) \ge -\frac{(n+2)^2}{16}\Phi, \\ &\frac{(n-2)\alpha}{2}(f+u)_{,ii} \\ &\quad = \frac{(n-2)\alpha}{2}\left(2 - \frac{1}{2}f_{iik}(f+u)_k\right) \\ &\quad \le (n-2)\alpha + \frac{(n-2)\alpha^2}{4}\|\nabla(f+u)\|^2 + CJ. \end{aligned} \tag{41}$$

Then, using the Schwarz inequality and (22)–(24), we know that at the point p

$$\begin{aligned} &\mathrm{Ric}\left(M, \widetilde{G}\right) \\ &\quad \ge -\exp\{-\alpha(f+u)\} \\ &\qquad \times\{C\Phi + CJ + \alpha[3(n-2)\alpha(f+u) + 2(n-1)]\}\widetilde{G}. \end{aligned} \tag{42}$$

If $3(n-2)\alpha(f+u) + 2(n-1) \le 0$, then

$$-\exp\{-\alpha(f+u)\}\alpha[3(n-2)\alpha(f+u) + 2(n-1)] \ge 0. \tag{43}$$

Otherwise,

$$\exp\{-\alpha(f+u)\}\alpha[3(n-2)\alpha(f+u) + 2(n-1)] \le C. \tag{44}$$

Then, the Ricci curvature $\mathrm{Ric}(M, \widetilde{G})$ on $B_{a^*}(p_0, \widetilde{G})$ is bounded from below by

$$\begin{aligned} &\mathrm{Ric}\left(M, \widetilde{G}\right) \\ &\quad \ge -C\left(\frac{\Phi}{\exp\{\alpha(f+u)\}}(\tilde{p}) + \frac{J}{\exp\{\alpha(f+u)\}}(\tilde{q}) + 1\right)\widetilde{G}. \end{aligned} \tag{45}$$

By the Laplacian comparison theorem, we get

$$\begin{aligned}
&\frac{s\Delta s}{a^2-s^2}\\
&\quad= \exp\{\alpha(f+u)\}\frac{s\widetilde{\Delta}s}{a^2-s^2} - \frac{(n-2)\alpha}{2}\frac{s(f+u)_{,i}s_{,i}}{a^2-s^2}\\
&\quad\le C_3\frac{\exp\{\alpha(f+u)\}(p^*)}{a^2-s^2}\\
&\qquad\times\left(\sqrt{\exp\{-\alpha(f+u)\}\Phi(\tilde{p})}\right.\\
&\qquad\qquad\left.+\sqrt{\exp\{-\alpha(f+u)\}J(\tilde{q})}+1\right)s\\
&\qquad+\frac{\alpha}{16}(f+u)_{,i}^2 + C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},
\end{aligned}\tag{46}$$

where $\widetilde{\Delta}$ denotes the Laplacian with respect to the metric $\widetilde{G}$.

Substituting (46) into (39) yields

$$\begin{aligned}
&\exp\{-\alpha(f+u)\}\Phi\\
&\quad\le aC\left(\sqrt{\frac{\Phi}{\exp\{\alpha(f+u)\}}(\tilde{p})}\right.\\
&\qquad\left.+\sqrt{\frac{J}{\exp\{\alpha(f+u)\}}(\tilde{q})}+1\right)\times\left(a^2-s^2\right)^{-1}\\
&\qquad+\frac{Ca^2}{(a^2-s^2)^2}.
\end{aligned}\tag{47}$$

Note that

$$\begin{aligned}
\mathscr{A} &\ge \left[\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}\Phi\right](\tilde{p})\\
&\ge \left(a^2-s^2\right)^2(p^*)\exp\{-\alpha(f+u)\}(\tilde{p})\Phi(\tilde{p}),\\
\mathscr{B} &\ge \left[\left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}J\right](\tilde{q})\\
&\ge \left(a^2-s^2\right)^2(p^*)\exp\{-\alpha(f+u)\}(\tilde{q})J(\tilde{q}).
\end{aligned}\tag{48}$$

Multiplying by $(a^2-s^2)^2(p^*)$, at both sides of (47), yields

$$\mathscr{A} \le Ca\left(\mathscr{A}^{1/2}+\mathscr{B}^{1/2}\right)+C\left(a^2+a^3\right). \tag{49}$$

Using the Schwarz inequality, we complete Step 1.

Step 2. We will prove that there is a constant C depending only on n such that

$$\mathscr{B} \le C\left(\mathscr{A}+a^2+a^4\right). \tag{50}$$

Consider

$$H = \left(a^2-s^2\right)^2\exp\{-\alpha(f+u)\}J \tag{51}$$

defined on $B_a(p_0,\widetilde{G})$, where α is the constant in (38). Obviously, H attains its supremum at some interior point q^*. Choose an orthonormal frame field on M around q^* with respect to the Calabi metric G. Then, at q^*,

$$\frac{J_{,i}}{J} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2-s^2} = 0, \tag{52}$$

$$\begin{aligned}
&\frac{\Delta J}{J} - \frac{\sum(J_{,i})^2}{J^2} - \alpha\Delta(f+u) - \frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2}\\
&\quad-\frac{4s\Delta s}{a^2-s^2} \le 0,
\end{aligned}\tag{53}$$

where "," denotes the covariant derivative with respect to the Calabi metric G as before. Inserting Corollary 8 into (53), we get

$$\begin{aligned}
&J - 20(n+2)^8\frac{\Phi^2}{J} + \frac{1}{4}\sum\frac{J_{,i}}{J}(f+u)_{,i} + 1 - \frac{\sum(J_{,i})^2}{J^2}\\
&\quad-\alpha\left(2n+(n+2)^2\Phi\right) - \sqrt{n(n-1)}\left\|\nabla(f+u)\right\|J^{1/2}\\
&\quad-\frac{12a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - \frac{4s\Delta s}{a^2-s^2} \le 0.
\end{aligned}\tag{54}$$

Applying the Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{4}\sum\frac{J_{,i}}{J}(f+u)_{,i} \ge \frac{\alpha}{8}\sum\left[(f+u)_{,i}\right]^2 - C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},\\
&\sum\frac{(J_{,i})^2}{J^2} \le 2\alpha^2\sum\left[(f+u)_{,i}\right]^2 + \frac{32a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2},\\
&\sqrt{n(n-1)}\left\|\nabla(f+u)\right\|J^{1/2}\\
&\quad\le\frac{J}{4} + 4n(n-1)\left((n+2)^2\Phi+(f+u)\right).
\end{aligned}\tag{55}$$

Inserting these estimates into (54) yields

$$\begin{aligned}
&\frac{3}{4}J - 20(n+2)^8\frac{\Phi^2}{J} - C\Phi + \frac{\alpha}{16}\sum(f+u)_{,i}^2\\
&\quad-C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - C(f+u) - \frac{4s\Delta s}{a^2-s^2} \le 0,
\end{aligned}\tag{56}$$

here and later C denotes different positive constants depending only on n.

We discuss two subcases.

Case 1. If

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) \le \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*), \tag{57}$$

then $\mathscr{B} \le \mathscr{A}$. In this case, Step 2 is complete.

Case 2. Now, assume that

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) > \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*). \tag{58}$$

Then, $1 > (\Phi/J)(q^*)$. Thus,

$$\frac{3}{4}J - C\Phi + \frac{\alpha}{16}\sum\left[(f+u)_{,i}\right]^2 - C\frac{a^2\exp\{\alpha(f+u)\}}{(a^2-s^2)^2} - C(f+u) - \frac{4s\Delta s}{a^2-s^2} \le 0. \tag{59}$$

The rest of the estimate is almost the same as in Step 1. The only difference is to deal with the term $(f+u)$. If $(f+u)(q^*) \le 0$, then $-C(f+u)(q^*) \ge 0$. We can drop this term.

Otherwise, $\exp\{-\alpha(f+u)\}(f+u)$ has a uniform upper bound.

Using the same method as in Step 1, we can estimate the term $4s\Delta s/(a^2-s^2)$ and finally get

$$\mathscr{B} \le C\left(\mathscr{A} + a^2 + a^4\right). \tag{60}$$

Then, combining the conclusion of Step 1, we get

$$\mathscr{A} \le C\left(a^2 + a^3\right). \tag{61}$$

This completes the proof of Lemma 9. □

Proof of Theorem 2. For any point $q \in M$, choose sufficient large constant R_0 such that $q \in B_{R_0}(p_0, \widetilde{G})$. Then, for all $a \ge R_0$, $q \in B_a(p_0)$. Using Lemma 9, we know

$$\exp\{-\alpha(f+u)\}\Phi(q) \le \frac{C(n)\left(a^2+a^3\right)}{(a^2-s^2)^2}. \tag{62}$$

Now, let $a \to +\infty$, and we have

$$0 \le \exp\{-\alpha(f+u)\}\Phi(q) \le 0. \tag{63}$$

Consequently,

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(q) = \text{const}. \tag{64}$$

This completes the proof of Theorem 2. □

4. Appendix

Proof of Proposition 6. Let $p \in M$, and we choose a local orthonormal frame field of the metric G around p. Then,

$$\Phi = \frac{\sum(\rho_{,j})^2}{\rho^2}, \qquad \Phi_{,i} = 2\sum\frac{\rho_{,j}\rho_{,ji}}{\rho^2} - 2\rho_{,i}\frac{\sum(\rho_{,j})^2}{\rho^3},$$
$$\Delta\Phi = 2\frac{\sum(\rho_{,ji})^2}{\rho^2} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho^2} - 8\sum\frac{\rho_{,j}\rho_{,i}\rho_{,ji}}{\rho^3} + (n+6)\Phi^2 - 2\tau\Phi, \tag{65}$$

where we used (20). In the case $\Phi(p) = 0$, it is easy to get, at p,

$$\Delta\Phi \ge 2\frac{\sum(\rho_{,ij})^2}{\rho^2}. \tag{66}$$

Now, we assume that $\Phi(p) \ne 0$. Choose a local orthonormal frame field of the metric G around p such that $\rho_{,1}(p) = \|\nabla\rho\|(p) > 0$, $\rho_{,i}(p) = 0$, for all $i > 1$. Then,

$$\Delta\Phi = 2(1-\delta+\delta)\sum\frac{(\rho_{,ij})^2}{\rho^2} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho^2} - 8\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + (n+6)\Phi^2 - 2\tau\Phi, \tag{67}$$

where $1 > \delta > 0$ is a constant to be determined later. Applying (20), we obtain

$$2\frac{\sum(\rho_{,ij})^2}{\rho^2} \ge \frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} + \frac{2n}{n-1}\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + \frac{n^2}{2(n-1)}\Phi^2 - \frac{4}{n-1}\frac{\rho_{,11}}{\rho}\tau + \frac{2}{n-1}\tau^2 - \frac{2n}{n-1}\Phi\tau. \tag{68}$$

An application of the Ricci identity shows that

$$\frac{2}{\rho^2}\sum\rho_{,j}\rho_{,jii} = -2n\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + n\frac{(\rho_{,1})^4}{\rho^4} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}. \tag{69}$$

Substituting (68) and (69) into (67), we obtain

$$\begin{aligned}\Delta\Phi \ge{}& 2\delta\sum\frac{(\rho_{,ij})^2}{\rho^2} + \left(-2n - 8 + \frac{2n(1-\delta)}{n-1}\right)\times\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2}\\ &+ \left(\frac{n^2(1-\delta)}{2(n-1)} + 2(n+3)\right)\frac{(\rho_{,1})^4}{\rho^4} + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1} - \frac{4n-2-2n\delta}{n-1}\Phi\tau + (1-\delta)\\ &\times\left(\frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} - \frac{4}{n-1}\frac{\rho_{,11}}{\rho}\tau + \frac{2}{n-1}\tau^2\right).\end{aligned} \tag{70}$$

Note that

$$\frac{(\rho_{,11})^2}{\rho^2} = \frac{1}{4}\sum\frac{(\Phi_{,i})^2}{\Phi} - \frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} + 2\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} - \frac{(\rho_{,1})^4}{\rho^4}. \tag{71}$$

Then, (70) and (71) together give us

$$\Delta\Phi \geq 2\delta \sum \frac{(\rho_{,ij})^2}{\rho^2} + \frac{n(1-\delta)}{2(n-1)} \frac{\sum(\Phi_{,i})^2}{\Phi} + \left(\frac{6n(1-\delta)}{n-1} - 2(n+4)\right)\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + \left[\frac{(n^2-4n)(1-\delta)}{2(n-1)} + 2(n+3)\right]\frac{(\rho_{,1})^4}{\rho^4} + \frac{1-\delta}{n-1}\left(2\tau^2 - 4\frac{\rho_{,11}}{\rho}\tau\right) - \frac{4n-2-2n\delta}{n-1}\Phi\tau + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}. \quad (72)$$

Using the Schwarz inequality gives

$$2\frac{\rho_{,11}}{\rho}\tau \leq \frac{7}{3}\sum\frac{(\rho_{,ij})^2}{\rho^2} + \frac{3}{7}\tau^2. \quad (73)$$

Using

$$\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} = \frac{1}{2}\Phi_{,i}\frac{\rho_{,i}}{\rho} + \Phi^2, \quad (74)$$

and choosing $\delta = 7/(3n+4)$, we get

$$\Delta\Phi \geq \frac{n(1-\delta)}{2(n-1)}\frac{\sum(\Phi_{,i})^2}{\Phi} + \left(\frac{3n(1-\delta)}{n-1} - (n+4)\right) \times \sum\Phi_{,i}\frac{\rho_{,i}}{\rho} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + \left[\frac{(n^2+8n)(1-\delta)}{2(n-1)} - 2\right]\Phi^2 + \frac{8(1-\delta)}{7(n-1)}\tau^2 - \frac{4n-2-2n\delta}{n-1}\Phi\tau + 2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1}. \quad (75)$$

In the following, we will calculate the terms $R_{11}((\rho_{,1})^2/\rho^2)$ and $(\rho_{,1}/\rho^2)(\rho\tau)_{,1}$. Note that (17) is invariant under an affine transformation of coordinates that preserved the origin. So, we can choose the coordinates $x_1, x_2, \ldots, x_n$ such that $f_{ij}(p) = \delta_{ij}$ and $\partial\rho/\partial x_1 = \|\mathrm{grad}\rho\|(p) > 0$, $(\partial\rho/\partial x_i)(p) = 0$, for all $i > 1$. From (19), we easily obtain

$$\rho_{,ij} = \rho_{ij} + A_{ij1}\rho_{,1} = \frac{\rho_{,i}\rho_{,j}}{\rho} - A_{ij1}\rho_{,1} + \frac{A_{ijk}f_k\rho}{n+2}. \quad (76)$$

Thus, we get

$$\Phi_{,i} = \frac{2\rho_{,1}\rho_{,1i}}{\rho^2} - 2\frac{\rho_{,i}(\rho_{,1})^2}{\rho^3} = -2A_{i11}\frac{(\rho_{,1})^2}{\rho^2} + 2\frac{\rho_{,1}f_kA_{ki1}}{(n+2)\rho}, \quad (77)$$

$$\sum\Phi_{,i}\frac{\rho_{,i}}{\rho} = -2A_{111}\frac{(\rho_{,1})^3}{\rho^3} + 2\frac{f_kA_{k11}}{n+2}\frac{(\rho_{,1})^2}{\rho^2}. \quad (78)$$

By the same method, as deriving (69), we have

$$\sum(A_{ml1})^2 \geq \frac{n}{n-1}\sum(A_{i11})^2 - \frac{2}{n-1}A_{111}\sum A_{ii1} + \frac{1}{n-1}\left(\sum A_{ii1}\right)^2. \quad (79)$$

Note that $\sum A_{ii1} = ((n+2)/2)(\rho_1/\rho)$. Therefore, by (14), (77), (78), and (79), we obtain

$$2R_{11}\frac{(\rho_{,1})^2}{\rho^2} = 2\sum(A_{kj1})^2\frac{(\rho_{,1})^2}{\rho^2} - (n+2)A_{111}\frac{(\rho_{,1})^3}{\rho^3} \geq \frac{n}{2(n-1)}\frac{\sum(\Phi_{,i} - 2(\rho_{,1}f_kA_{ki1}/(n+2)\rho))^2}{\Phi} + \frac{(n+2)(n+1)}{2(n-1)}\sum\Phi_{,i}\frac{\rho_{,i}}{\rho} - \frac{n+1}{n-1}f_kA_{k11}\frac{(\rho_{,1})^2}{\rho^2} + \frac{(n+2)^2}{2(n-1)}\Phi^2. \quad (80)$$

On the other hand, we have

$$2\frac{\rho_{,1}}{\rho^2}(\rho\tau)_{,1} = 2\Phi\tau + \frac{1}{n+2}\sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho} + \Phi. \quad (81)$$

Then, inserting (80) and (81) into (75), we get

$$\Delta\Phi \geq \frac{2n-n\delta}{2(n-1)}\sum\frac{(\Phi_{,i})^2}{\Phi} - \frac{(n+2)(n-5)+6n\delta}{2(n-1)} \times \sum\Phi_{,i}\frac{\rho_{,i}}{\rho} + \Phi + \frac{2(n+2)^2 - (n^2+8n)\delta}{2(n-1)}\Phi^2 + \frac{8(1-\delta)}{7(n-1)}\tau^2 - \frac{2n(1-\delta)}{n-1}\Phi\tau + \frac{1}{n+2} \times \sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho} - \frac{n+1}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11} - \frac{2n}{(n-1)(n+2)}\frac{\sum\Phi_{,i}f_kA_{ki1}}{\sqrt{\Phi}} + \frac{2n}{(n-1)(n+2)^2} \times \sum(f_kA_{ki1})^2. \quad (82)$$

Using (77), we have

$$\frac{1}{n+2}\sum A_{1ik}f_kf_i\frac{\rho_{,1}}{\rho} - \frac{n+1}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11} = \frac{1}{2}f_i\Phi_{,i} - \frac{2}{n-1}\frac{(\rho_{,1})^2}{\rho^2}f_kA_{k11}. \quad (83)$$

One observes that the Schwarz inequality gives

$$\begin{aligned}
&\frac{2n}{(n-1)(n+2)}\frac{\sum \Phi_{,i} f_k A_{ki1}}{\sqrt{\Phi}} \\
&\quad \leq \frac{9n}{8(n-1)}\sum \frac{(\Phi_{,i})^2}{\Phi} + \frac{8n}{9(n-1)(n+2)^2} \\
&\quad\quad \times \sum (f_k A_{ki1})^2, \\
&\frac{2}{n-1}\frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11} \\
&\quad \leq \frac{9(n+2)^2}{10n(n-1)}\Phi^2 + \frac{10n}{9(n-1)(n+2)^2} \\
&\quad\quad \times \sum (f_k A_{ki1})^2, \\
&\quad\quad 2n\Phi\tau \leq \tau^2 + n^2\Phi^2.
\end{aligned} \tag{84}$$

Note that by (17) we have

$$\begin{aligned}
\frac{1}{4} f^{ij}\Phi_j f_i &= \frac{n+2}{2} f^{ij}\Phi_j (\ln\rho)_i + \frac{1}{4}\Phi_j x_j \\
&= \frac{n+2}{2} f^{ij}\Phi_j (\ln\rho)_i + \frac{1}{4} f^{ij}\frac{\partial \Phi}{\partial x_i}\frac{\partial u}{\partial x_j}.
\end{aligned} \tag{85}$$

Then, inserting these estimates into (82) yields Proposition 6. □

Proof of Corollary 8. Now, we will calculate the term $(\ln\rho)_{,ijk}$. In particular, if f satisfies PDE (4), choose the coordinate $(x_1, x_2, \ldots, x_n)$ such that $f_{ij}(p) = \delta_{ij}$; then we have

$$\begin{aligned}
(\ln\rho)_{,ijk} &= \frac{1}{n+2}\left(A_{ijk} + A_{ijk,p} f_{,p}\right) - (\ln\rho)_{,l} A_{ijk,l} \\
&\quad + A_{ijl}A_{klp}\left(3(\ln\rho)_{,p} - \frac{2}{n+2} f_{,p}\right).
\end{aligned} \tag{86}$$

Using (17), we have

$$3(\ln\rho)_{,p} - \frac{2}{n+2} f_{,p} = -\frac{1}{n+2}(f+u)_{,p} + (\ln\rho)_{,p}. \tag{87}$$

By the Young inequality and the Schwarz inequality, we have

$$\begin{aligned}
&\frac{n+2}{n(n-1)}\sum A_{ijk}A_{ijl}A_{klh}(\ln\rho)_{,h} \\
&\quad \leq \frac{1}{2}J^2 + 16n^2(n-1)^2(n+2)^4\Phi^2, \\
&\frac{1}{n(n-1)}\sum A_{ijk}A_{ijl,k} f_l \\
&\quad = \frac{1}{2}\sum J_{,l} f_{,l} = \frac{1}{4}\langle \nabla J, \nabla(f+u)\rangle + \frac{n+2}{2}\langle \nabla J, \nabla \ln\rho\rangle, \\
&\frac{n+2}{n(n-1)}\sum A_{ijk}A_{ijl,k}(\ln\rho)_{,l} \\
&\quad = \frac{n+2}{2}\sum J_{,i}(\ln\rho)_{,i} \\
&\quad \leq \frac{1}{n(n-1)}\sum \left(A_{ijk,l}\right)^2 + \frac{(n+2)^2}{4}J\Phi \\
&\quad \leq \frac{1}{n(n-1)}\sum \left(A_{ijk,l}\right)^2 + \frac{1}{4}J^2 + \frac{(n+2)^4}{16}\Phi^2, \\
&\frac{1}{n(n-1)}\sum A_{ijk}A_{jil}A_{klp}(f+u)_{,p} \\
&\quad \leq \sqrt{n(n-1)}\,\|\nabla(f+u)\|\,J^{3/2}.
\end{aligned} \tag{88}$$

Thus, by inserting (88) into Lemma 7, we obtain Corollary 8. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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