

Research Article

Exponential Stability of Stochastic Differential Equation with Mixed Delay

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Received 24 November 2013; Revised 21 January 2014; Accepted 21 January 2014; Published 4 March 2014

Academic Editor: Oluwole Daniel Makinde

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This paper focuses on a class of stochastic differential equations with mixed delay based on Lyapunov stability theory, Itô formula, stochastic analysis, and inequality technique. A sufficient condition for existence and uniqueness of the adapted solution to such systems is established by employing fixed point theorem. Some sufficient conditions of exponential stability and corollaries for such systems are obtained by using Lyapunov function. By utilizing Doob's martingale inequality and Borel-Cantelli lemma, it is shown that the exponentially stable in the mean square of such systems implies the almost surely exponentially stable. In particular, our theoretical results show that if stochastic differential equation is exponentially stable and the time delay is sufficiently small, then the corresponding stochastic differential equation with mixed delay will remain exponentially stable. Moreover, time delay upper limit is solved by using our theoretical results when the system is exponentially stable, and they are more easily verified and applied in practice.

1. Introduction

The nondeterministic (i.e., stochastic) phenomena are frequently encountered in many practical systems. These systems should be described by stochastic differential equations (SDEs for short) instead of ordinary ones. On the other hand, time delays are included in many practical systems, such as networks control systems, traffic systems, production process control systems, and population and economic dynamic systems, that is, the current and future states of the systems dependent on their departed states. In current years, the study of analysis and synthesis of stochastic time delay systems, which are described by stochastic delayed differential equations (SDDEs for short), is a popular topic in the field of control theory [1–7]. Because the existence of time delay is often the reason of instability and deteriorates the control performance, the studies on time delay systems stability and control have important theoretical and practical values.

A real dynamic system is influenced by both stochastic disturbances and time delays, so when we consider the behavior of a dynamic system, we use the stochastic delayed

differential equation and the stochastic functional differential equation (SFDE for short) as modeling tools to investigate stability of stochastic dynamic systems with discrete delays or distributed delays. So far, these topics have received a lot of attention and there are so many references about them. For example, Cong [1] and Li et al. [2] obtained exponential stability conditions of linear stochastic neutral delay systems. Mao [8], Mao and Shah [9], Zhu and Hu [10], Zhu and Hu [11], S. Xie and L. Xie [12], and Zhu et al. [13] established some stability criteria of the stochastic system with discrete delays. An improved delay-dependent stability criterion is derived for stochastic delay systems by a strict LMI in [14]. Hu and Wu [15], Wu et al. [16], Yin et al. [17], and Zhou et al. [18] established some stability criteria of the stochastic system with distributed delays. However, discrete delays and distributed delays always coexist in real dynamic systems; thus, it is reasonable to consider them together and it leads us to investigate stochastic differential equations with mixed delays (SMDDEs for short).

Although stochastic differential systems with mixed delays received increasing attention recently, there is a little previous literature, as systematic research on such system

has not been developed yet. For example, Zhu and Song [19] obtained some exponential stability results for a class of impulsive nonlinear stochastic differential equations with mixed delays by Razumikhin technique, but these sufficient conditions only ensure the exponential stability of the trivial solution in the mean square and did not give a bound for the time delay δ . Deng et al. [20] and L. Xu and D. Xu [21] focused on the corresponding study of exponential stability of neural network model. Thus, this paper aims to fill the gap in a sense. In this paper, we investigate not only the exponential stability in the mean square but also the almost surely exponential stability for a class of SMDDEs based on Lyapunov stability theory, Itô formula, stochastic analysis, inequality technique, and so on. We first consider the existence and uniqueness of the adapted solution by employing fixed point theorem. Next, some sufficient conditions of exponential stability and corollaries for stochastic differential systems with mixed delays are obtained by using Lyapunov function. By utilizing Doob's martingale inequality and Borel-Cantelli lemma, it is shown that the exponentially stable in the mean square of SMDDE implies the almost surely exponentially stable. The obtained results generalize and improve some recent results (for instance, [19–21]). In particular, our theoretical results show that if SDE is exponentially stable and the time delay is sufficiently small, then the corresponding SMDDE will remain exponentially stable. Moreover, the time delay upper limit is solved by using our theoretical results when the system is exponentially stable, and they are more easily verified and applied in practice. It should be mentioned that the approach provided here is different from those used in [19–21]. Finally, we present a simple example to illustrate the effectiveness of our stable results.

The rest of this paper is organized as follows. In Section 2, we give the preliminary results about SMDDEs. Main results and proofs for SMDDEs are provided in Section 3. Section 4 presents a simple example to illustrate our stable results. Section 5 lists some concluding remarks.

2. Preliminaries

Throughout this paper and unless specified, we let $W(t) = (W_1(t), \dots, W_m(t))^T$ be an m -dimensional Brownian motion defined on a complete probability space (Ω, \mathcal{F}, P) with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ (i.e., $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$) and augmented by all the P -null sets in \mathcal{F}). Denote by $|\cdot|$ the Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T . If A is a matrix, denote by $\|A\|$ the operator norm of A ; that is, $\|A\| = \sup\{|Ax| : |x| = 1\}$. $\xi(\cdot) \in C[-\delta, 0]$ is the initial path of x , where $\delta > 0$ is a given finite time delay and $C[-\delta, 0]$ is the set of continuous functions from $[-\delta, 0]$ into R^n . Moreover, denote by $L^p_{\mathcal{F}_0}(-\delta, 0; R^n)$ the family of R^n -valued adapted stochastic processes $\xi(s)$, $-\delta \leq s \leq 0$ such that $\xi(s)$ is \mathcal{F}_0 -measurable and $E \sup_{-\delta \leq t \leq 0} |\xi(t)|^p < +\infty$ ($p > 1$).

We also use the notation $L^2_{\mathcal{F}}(s, r; R^n) = \{\phi(t) : \{\phi(t), s \leq t \leq r\}$ which is R^n -valued adapted stochastic processes s.t. $\int_s^r E|\phi(t)|^2 dt < \infty\}$.

We consider the following stochastic differential equations with mixed delays:

$$\begin{aligned} dx(t) &= b(t, x(t), y(t), z(t)) dt \\ &+ \sigma(t, x(t), y(t), z(t)) dW(t), \quad t \geq 0, \quad (1) \\ x(t) &= \xi(t), \quad t \in [-\delta, 0], \end{aligned}$$

where $b \in C(R^+ \times R^n \times R^n \times R^n, R^n)$ and $\sigma \in C(R^+ \times R^n \times R^n \times R^n, R^{n \times m})$ represent the nonlinear uncertainties and

$$y(t) = \int_{-\delta}^0 e^{\mu s} x(t+s) ds, \quad z(t) = x(t-\delta) \quad (2)$$

represent given functional of the path segment $x_t := \{x(t+s); s \in [-\delta, 0]\}$ of x ; $\mu \in R^+$ is the given averaging parameter. Furthermore, we always assume that $b(t, 0, 0, 0) = \sigma(t, 0, 0, 0) \equiv 0$ for the stability purpose of this paper.

For simplicity, in what follows, we write $\psi = \psi(t)$ sometimes, where $\psi = x, y, z, \xi$.

To develop our theories and results, we need to introduce the following concepts and important inequalities. For stochastic system, exponential stability in mean square and almost surely exponential stability are generally used [13].

Definition 1. The trivial solution of (1) is said to be p th moment exponentially stable, if there exists a positive constant ε such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln (E|x(t; \xi)|^p) \leq -\varepsilon \quad (3)$$

for any $\xi \in L^p_{\mathcal{F}_0}(-\delta, 0; R^n)$.

Especially, $p = 2$, and it is called mean square exponentially stable.

Definition 2. The trivial solution of (1) is said to be almost surely exponentially stable. If there exists a positive constant η such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |x(t; \xi)| \leq -\eta \quad \text{a.s.} \quad (4)$$

for any $\xi \in L^p_{\mathcal{F}_0}(-\delta, 0; R^n)$.

Lemma 3 (see [22]). *For any real matrices $\xi_1, \xi_2 \in R^n$ and a constant $\theta > 0$, the following matrix inequality holds:*

$$2\xi_1^T \xi_2 \leq \theta \xi_1^T \xi_1 + \frac{1}{\theta} \xi_2^T \xi_2. \quad (5)$$

Lemma 4 (Cauchy-Schwarz inequality). *Let f and g be real functions which are continuous on the closed interval $[a, b]$. Then,*

$$\left(\int_a^b f(t)g(t)dt \right)^2 \leq \int_a^b f(t)^2 dt \cdot \int_a^b g(t)^2 dt. \quad (6)$$

3. Main Results

3.1. Existence and Uniqueness Result of the Solution for SMD-DEs. We make the following assumptions for the coefficients of (1).

(H3.1) Let $T \in (0, \infty)$ be fixed time duration, $\forall x, y, z, x', y', z' \in R^n, t \in [0, T]$, and there exists a constant $C > 0$ such that

$$\begin{aligned} &|b(t, x, y, z) - b(t, x', y', z')| \\ &+ |\sigma(t, x, y, z) - \sigma(t, x', y', z')| \\ &\leq C(|x - x'| + |y - y'| + |z - z'|) \end{aligned} \quad (7)$$

(H3.2) $\sup_{0 \leq t \leq T} (|b(t, 0, 0, 0) + \sigma(t, 0, 0, 0)|) < +\infty$.

Theorem 5. Let (H3.1) and (H3.2) hold. Then for any $\xi(t) \in L^2_{\mathcal{F}_0}(-\delta, 0; R^n)$, (1) has a unique t -continuous adapted solution, denoted by $x(t; \xi)$ and $E \sup_{0 \leq t \leq T} |x(t)|^2 < +\infty$. So (1) has a trivial solution $x(t; 0) = 0$.

Inspired by the literature [23], we present the proof of the Theorem 5 as follows.

Proof. Let us define a norm in Banach space $L^2_{\mathcal{F}}(-\delta, T; R^n)$ as follows:

$$|\chi(\cdot)|_{\theta} = \left(E \left[\int_{-\delta}^T e^{-\theta s} |\chi(s)|^2 ds \right] \right)^{1/2}, \quad \theta > 0. \quad (8)$$

Clearly it is equivalent to the original norm of $L^2_{\mathcal{F}}(-\delta, T; R^n)$. We consider

$$\begin{aligned} x(t) &= \xi(0) + \int_0^t b(s, \chi, y_{\chi}, z_{\chi}) ds \\ &+ \int_0^t \sigma(s, \chi, y_{\chi}, z_{\chi}) dW(s), \quad t \geq 0, \\ x(t) &= \xi(t), \quad t \in [-\delta, 0], \end{aligned} \quad (9)$$

where $y_{\chi} = \int_{-\delta}^0 e^{\mu s} \chi(t+s) ds, z_{\chi} = \chi(t-\delta)$. Define a mapping $T : L^2_{\mathcal{F}}(-\delta, T; R^n) \rightarrow L^2_{\mathcal{F}}(-\delta, T; R^n)$ such that $T(\chi(\cdot)) = x(\cdot)$. We desire to prove that T is a contraction mapping under the norm $|\chi(\cdot)|_{\theta}$. For arbitrary $\chi(\cdot), \chi'(\cdot) \in L^2_{\mathcal{F}}(-\delta, T; R^n)$, set $T(\chi(\cdot)) = x(\cdot), T(\chi'(\cdot)) = x'(\cdot)$ and $\tilde{\chi}(\cdot) = \chi(\cdot) - \chi'(\cdot), \tilde{x}(\cdot) = x(\cdot) - x'(\cdot)$. Then, $\tilde{x}(\cdot)$ satisfies

$$\begin{aligned} \tilde{x}(t) &= \int_0^t [b(s, \chi, y_{\chi}, z_{\chi}) - b(s, \chi', y'_{\chi}, z'_{\chi})] ds \\ &+ \int_0^t [\sigma(s, \chi, y_{\chi}, z_{\chi}) - \sigma(s, \chi', y'_{\chi}, z'_{\chi})] dW(s), \\ &t \geq 0, \\ \tilde{x}(t) &= 0, \quad t \in [-\delta, 0]. \end{aligned} \quad (10)$$

Applying Itô's formula to $e^{-\theta t} |\tilde{x}(t)|^2$, we have

$$\begin{aligned} &d(e^{-\theta t} |\tilde{x}(t)|^2) \\ &= -\theta e^{-\theta t} |\tilde{x}(t)|^2 dt + e^{-\theta t} d|\tilde{x}(t)|^2 \\ &= -\theta e^{-\theta t} |\tilde{x}(t)|^2 dt + e^{-\theta t} \\ &\quad \times \left\{ 2\tilde{x}(t) \right. \\ &\quad \times [(b(s, \chi, y_{\chi}, z_{\chi}) - b(s, \chi', y'_{\chi}, z'_{\chi})) dt \\ &\quad + (\sigma(s, \chi, y_{\chi}, z_{\chi}) - \sigma(s, \chi', y'_{\chi}, z'_{\chi})) dW(t)] \\ &\quad \left. + |\sigma(s, \chi, y_{\chi}, z_{\chi}) - \sigma(s, \chi', y'_{\chi}, z'_{\chi})|^2 dt \right\}. \end{aligned} \quad (11)$$

Integrating from 0 to T and taking the expectation in the above, we get

$$\begin{aligned} &\theta E \int_0^T e^{-\theta t} |\tilde{x}(t)|^2 dt \\ &= 2E \int_0^T e^{-\theta t} \tilde{x}(t) [b(t, \chi, y_{\chi}, z_{\chi}) - b(t, \chi', y'_{\chi}, z'_{\chi})] dt \\ &+ E \int_0^T e^{-\theta t} |\sigma(s, \chi, y_{\chi}, z_{\chi}) - \sigma(s, \chi', y'_{\chi}, z'_{\chi})|^2 dt. \end{aligned} \quad (12)$$

Lemma 3 yields

$$\begin{aligned} &\theta E \int_0^T e^{-\theta t} |\tilde{x}(t)|^2 dt \\ &\leq E \int_0^T e^{-\theta t} (|\tilde{x}(t)|^2 + |b(t, \chi, y_{\chi}, z_{\chi}) - b(t, \chi', y'_{\chi}, z'_{\chi})|^2) dt \\ &+ E \int_0^T e^{-\theta t} |\sigma(s, \chi, y_{\chi}, z_{\chi}) - \sigma(s, \chi', y'_{\chi}, z'_{\chi})|^2 dt. \end{aligned} \quad (13)$$

Then by (H3.1), we obtain

$$\begin{aligned} &\theta E \int_0^T e^{-\theta t} |\tilde{x}(t)|^2 dt \\ &\leq E \int_0^T e^{-\theta t} (|\tilde{x}(t)|^2 + (C(|\tilde{\chi}(t)| + |\tilde{y}_{\chi}(t)| + |\tilde{z}_{\chi}(t)|))^2) dt \\ &+ E \int_0^T e^{-\theta t} (C(|\tilde{\chi}(t)| + |\tilde{y}_{\chi}(t)| + |\tilde{z}_{\chi}(t)|))^2 dt. \end{aligned} \quad (14)$$

Thus,

$$\begin{aligned} &(\theta - 1) E \int_0^T e^{-\theta t} |\tilde{x}(t)|^2 dt \\ &\leq 6C^2 \cdot E \left[\int_0^T e^{-\theta t} |\tilde{\chi}(t)|^2 dt + \int_0^T e^{-\theta t} |\tilde{y}_{\chi}(t)|^2 dt \right. \\ &\quad \left. + \int_0^T e^{-\theta t} |\tilde{z}_{\chi}(t)|^2 dt \right], \end{aligned} \quad (15)$$

where

$$\begin{aligned} & \int_0^T e^{-\theta t} |\widehat{z}_\chi(t)|^2 dt \\ &= \int_{-\delta}^{T-\delta} e^{-\theta(r+\delta)} |\widehat{\chi}(r)|^2 dr \quad (r = t - \delta) \\ &= e^{-\theta\delta} \int_{-\delta}^{T-\delta} e^{-\theta r} |\widehat{\chi}(r)|^2 dr \\ &\leq \int_{-\delta}^{T-\delta} e^{-\theta t} |\widehat{\chi}(t)|^2 dt \leq \int_{-\delta}^T e^{-\theta t} |\widehat{\chi}(t)|^2 dt. \end{aligned} \tag{16}$$

Lemma 4 yields

$$\begin{aligned} |y(t)|^2 &= \left| \int_{-\delta}^0 e^{\mu s} x(t+s) ds \right|^2 \\ &\leq \int_{-\delta}^0 e^{2\mu s} ds \cdot \int_{-\delta}^0 |x(t+s)|^2 ds \\ &= \frac{(1 - e^{-2\mu\delta})}{2\mu} \cdot \int_{t-\delta}^t |x(r)|^2 dr \quad (r = t + s) \\ &\leq \frac{1}{2\mu} \cdot \int_{t-\delta}^t |x(r)|^2 dr. \end{aligned} \tag{17}$$

It then follows from (17) that

$$\begin{aligned} & \int_0^T e^{-\theta t} |\widehat{y}_\chi(t)|^2 dt \\ &\leq \frac{1}{2\mu} \cdot \int_0^T e^{-\theta t} \left(\int_{t-\delta}^t |\widehat{\chi}(r)|^2 dr \right) dt \\ &= \frac{1}{2\mu} \cdot \int_{-\delta}^T |\widehat{\chi}(r)|^2 \left(\int_{0 \vee r}^{(r+\delta) \wedge T} e^{-\theta t} dt \right) dr \\ &\leq \frac{1}{2\mu} \cdot \int_{-\delta}^T |\widehat{\chi}(r)|^2 e^{-\theta r} ((r + \delta) \wedge T - 0 \vee r) dr \\ &\leq \frac{\delta}{2\mu} \cdot \int_{-\delta}^T e^{-\theta r} |\widehat{\chi}(r)|^2 dr. \end{aligned} \tag{18}$$

Then,

$$\begin{aligned} & (\theta - 1) E \int_0^T e^{-\theta t} |\widehat{x}(t)|^2 dt \\ &\leq 6C^2 \cdot E \left[\int_{-\delta}^T e^{-\theta t} |\widehat{\chi}(t)|^2 dt + \frac{\delta}{2\mu} \cdot \int_{-\delta}^T e^{-\theta r} |\widehat{\chi}(r)|^2 dr \right. \\ &\quad \left. + \int_{-\delta}^T e^{-\theta t} |\widehat{\chi}(t)|^2 dt \right] \\ &\leq 3C^2 \left(4 + \frac{\delta}{\mu} \right) \cdot E \int_{-\delta}^T e^{-\theta t} |\widehat{\chi}(t)|^2 dt. \end{aligned} \tag{19}$$

Let $\theta = 6C^2(4 + (\delta/\mu)) + 1$; then the above yields

$$E \int_{-\delta}^T e^{-\theta t} |\widehat{x}(t)|^2 dt \leq \frac{1}{2} E \int_{-\delta}^T e^{-\theta t} |\widehat{\chi}(t)|^2 dt. \tag{20}$$

That is,

$$|\widehat{x}(\cdot)|_\theta = \frac{1}{\sqrt{2}} |\widehat{\chi}(\cdot)|_\theta. \tag{21}$$

This implies that T is a strict contraction mapping. Then it follows from the fixed point theorem that (1) has a unique solution in $L^2_{\mathcal{F}}(-\delta, T; R^n)$. Since b and σ satisfy (H3.1) and (H3.2), we can easily derive that $E \sup_{0 \leq t \leq T} |x(t)|^2 < +\infty$ and $x(t; \xi)$ is continuous with respect to $t \in [0, T]$. Furthermore, by $b(t, 0, 0, 0) = \sigma(t, 0, 0, 0) \equiv 0$, (1) has a trivial solution $x(t; 0) = 0$. \square

For simplicity, in what follows we write $x(t; \xi) = x(t)$.

3.2. *Exponential Stability for SMDDEs.* We make the following assumptions for the coefficients of (1).

(H3.3) There exist nonnegative constants α, β , for any $t \geq 0$ such that

$$|b(t, x, y, z)| \leq \alpha (|x| + |y| + |z|) \tag{22}$$

and for any $(t, x, y, z) \in R^+ \times R^n \times R^n \times R^n$ such that

$$tr \left[\sigma^T(t, x, y, z) \sigma(t, x, y, z) \right] \leq \beta (|x|^2 + |y|^2 + |z|^2). \tag{23}$$

(H3.4) By (H3.1), one has $|b(t, x, y, z) - b(t, x, x, x)| \leq C(|x - y| + |x - z|)$.

In the study of mean square exponential stability, it is often to use a quadratic function as the Lyapunov function; that is, $V(t, x) = x^T G x$, where G is a symmetric positive definite $n \times n$ matrix.

Theorem 6. *Let (H3.3) and (H3.4) hold; then the trivial solution of (1) is exponentially stable in the mean square. Assume that there exist symmetric positive definite $n \times n$ matrices G and a constant $\lambda > 0$ such that*

$$2x^T G b(t, x, x, x) \leq -\lambda |x|^2, \tag{24}$$

$$\forall (t, x, x, x) \in R^+ \times R^n \times R^n \times R^n,$$

$$\begin{aligned} \lambda > \|G\| \beta \left(2 + \frac{\delta}{\mu} \right) \\ + 4C \|G\| \sqrt{1 + \frac{\delta}{\mu} + (6\delta\alpha^2 + 2\beta) \left(2 + \frac{\delta}{\mu} \right)} \delta. \end{aligned} \tag{25}$$

In order to prove Theorem 6, we need three lemmas, proofs of which are left in appendix.

Lemma 7. Fix the initial data $\xi(t)$ arbitrarily. Then,

$$\int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau)|^2 d\tau ds \leq \delta e^{\varepsilon \delta} \int_0^t e^{\varepsilon \tau} \cdot E|x(\tau)|^2 d\tau, \quad (26)$$

$$\int_0^t e^{\varepsilon s} \cdot E|x(s-\delta)|^2 ds \leq c_{11} e^{\varepsilon \delta} + e^{\varepsilon \delta} \cdot \int_0^{t-\delta} e^{\varepsilon s} \cdot E|x(s)|^2 ds \quad (27)$$

for any $t \geq \delta$, where c_{11} is a constant larger than $\int_{-\delta}^0 E|\xi(s)|^2 ds$.

Lemma 8. Fix the initial data $\xi(t)$ arbitrarily. Then,

$$\begin{aligned} & \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|y(\tau)|^2 d\tau ds \\ & \leq \frac{\delta^2}{\mu} c_{12} e^{2\varepsilon \delta} + \frac{\delta^2}{\mu} e^{2\varepsilon \delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds, \end{aligned} \quad (28)$$

$$\int_0^t e^{\varepsilon s} \cdot E|y(s)|^2 ds \leq \frac{\delta}{\mu} c_{12} e^{\varepsilon \delta} + \frac{\delta}{\mu} e^{\varepsilon \delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \quad (29)$$

for any $t \geq \delta$.

Lemma 9. Let (H3.3) and (H3.4) hold. Then,

$$\int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau-\delta)|^2 d\tau ds \quad (30)$$

$$\leq \delta c_1 e^{2\varepsilon \delta} + \delta e^{2\varepsilon \delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds,$$

$$\begin{aligned} & \int_0^t e^{\varepsilon s} \cdot E|x(s) - x(s-\delta)|^2 ds \\ & \leq c_2 + (6\delta\alpha^2 + 2\beta) \cdot \delta e^{\varepsilon \delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \end{aligned} \quad (31)$$

$$+ (6\delta\alpha^2 + 2\beta)$$

$$\cdot \left(\frac{\delta}{\mu} + 1 \right) \left[c_1 \delta e^{2\varepsilon \delta} + \delta e^{2\varepsilon \delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \right]$$

for any $t \geq \delta$, where c_2 is a constant larger than $\int_0^{\delta} e^{\varepsilon s} E|x(s) - x(s-\delta)|^2 ds$.

Based on the above Lemmas 7–9, we now carry out a proof for Theorem 6.

Proof of Theorem 6. Fix the initial data $\xi(t)$ arbitrarily. Applying Itô's formula to $x^T Gx$, we have

$$\begin{aligned} d(x^T Gx) &= dx^T \cdot Gx + x^T G \cdot dx + dx^T \cdot G \cdot dx \\ &= 2x^T Gb(t, x, y, z) dt + 2x^T G\sigma(t, x, y, z) dW(t) \\ &\quad + \text{tr} [\sigma^T(t, x, y, z) G\sigma(t, x, y, z)] dt. \end{aligned} \quad (32)$$

Combining Lemmas 3 and (24) as well as (H3.4), we can estimate the first item of (32) as follows:

$$\begin{aligned} & 2x^T Gb(t, x, y, z) \\ &= 2x^T Gb(t, x, x, x) + 2x^T G[b(t, x, y, z) - b(t, x, x, x)] \\ &\leq -\lambda|x|^2 + \theta|x|^2 + \left(\frac{\|G\|^2}{\theta} \right) \cdot |b(t, x, y, z) - b(t, x, x, x)|^2 \\ &\leq -\lambda|x|^2 + \theta|x|^2 + \left(\frac{\|G\|^2}{\theta} \right) \cdot 2C^2 (|x-y|^2 + |x-z|^2) \\ &\leq -\lambda|x|^2 + \theta|x|^2 \\ &\quad + \frac{4C^2}{\theta} \|G\|^2 (|x|^2 + |y|^2) + \frac{2C^2}{\theta} \|G\|^2 |x-z|^2, \end{aligned} \quad (33)$$

where $\theta = 2C\|G\|\sqrt{1 + (\delta/\mu) + (6\delta\alpha^2 + 2\beta)(2 + (\delta/2\mu))\delta}$.

By (24), the last item of (32) yields

$$\text{tr} [\sigma^T(t, x, y, z) G\sigma(t, x, y, z)] \leq \|G\| \beta (|x|^2 + |y|^2 + |z|^2). \quad (34)$$

Substituting the above two into (32), we get

$$\begin{aligned} d[x^T Gx] &\leq \left[-\lambda|x|^2 + \theta|x|^2 + \frac{4C^2}{\theta} \|G\|^2 (|x|^2 + |y|^2) \right. \\ &\quad \left. + \frac{2C^2}{\theta} \|G\|^2 |x-z|^2 \right] dt \\ &\quad + \|G\| \beta (|x|^2 + |y|^2 + |z|^2) dt \\ &\quad + 2x^T G\sigma(t, x, y, z) dW(t) \\ &\leq \left(-\lambda + \theta + \|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) |x|^2 dt \\ &\quad + \left(\|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) |y|^2 dt + \|G\| \beta |z|^2 dt \\ &\quad + \frac{2C^2}{\theta} \|G\|^2 |x-z|^2 dt \\ &\quad + 2x^T G\sigma(t, x, y, z) dW(t). \end{aligned} \quad (35)$$

For small enough $\varepsilon > 0$, we derive

$$\begin{aligned} & \|G\| (\beta + \varepsilon) + \theta + \frac{4C^2}{\theta} \|G\|^2 + \|G\| \beta \left(1 + \frac{\delta}{\mu} \right) e^{\varepsilon \delta} \\ & \quad + \frac{1}{\theta} (2C\|G\|)^2 \end{aligned}$$

$$\begin{aligned}
& \cdot \left[\left(6\delta\alpha^2 + 2\beta + \frac{1}{\mu} \right) \delta e^{\varepsilon\delta} \right. \\
& \quad \left. + (6\delta\alpha^2 + 2\beta) \cdot \left(1 + \frac{\delta}{\mu} \right) \delta e^{2\varepsilon\tau} \right] \\
& \geq \|G\| (\beta + \varepsilon) \\
& \quad + \theta + \frac{4C^2}{\theta} \|G\|^2 + \|G\| \beta \left(1 + \frac{\delta}{\mu} \right) + \frac{(2C \|G\|)^2}{\theta} \\
& \cdot \left[\left(6\delta\alpha^2 + 2\beta + \frac{1}{\mu} \right) \delta + (6\delta\alpha^2 + 2\beta) \right. \\
& \quad \left. \cdot \left(1 + \frac{\delta}{\mu} \right) \delta \right] \\
& > \|G\| \beta \left(2 + \frac{\delta}{\mu} \right) + \theta + \frac{(2C \|G\|)^2}{\theta} \\
& \cdot \left[1 + \frac{\delta}{\mu} + (6\delta\alpha^2 + 2\beta) \left(2 + \frac{\delta}{\mu} \right) \delta \right]. \tag{36}
\end{aligned}$$

If (25) holds, then we can choose $\varepsilon > 0$ small enough such that

$$\begin{aligned}
\lambda = & \|G\| (\beta + \varepsilon) + \theta + \frac{4C^2}{\theta} \|G\|^2 + \|G\| \beta \left(1 + \frac{\delta}{\mu} \right) e^{\varepsilon\delta} \\
& + \frac{(2C \|G\|)^2}{\theta} \left[\left(6\delta\alpha^2 + 2\beta + \frac{1}{\mu} \right) \delta e^{\varepsilon\delta} + (6\delta\alpha^2 + 2\beta) \right. \\
& \quad \left. \cdot \left(1 + \frac{\delta}{\mu} \right) \delta e^{2\varepsilon\delta} \right]. \tag{37}
\end{aligned}$$

Applying Itô's formula to $e^{\varepsilon t} x^T Gx$, we derive

$$\begin{aligned}
e^{\varepsilon t} x^T Gx = & \xi^T(0) G\xi(0) + \varepsilon \int_0^t e^{\varepsilon s} x^T Gx ds + \int_0^t e^{\varepsilon s} d[x^T Gx], \\
& \text{for any } t \geq 0. \tag{38}
\end{aligned}$$

Substituting (35) into the above yields

$$\begin{aligned}
e^{\varepsilon t} x^T Gx \leq & \xi^T(0) G\xi(0) + \varepsilon \int_0^t e^{\varepsilon s} x^T Gx ds \\
& + \int_0^t e^{\varepsilon s} \cdot 2x^T G\sigma(s, x, y, z) dW(s) \\
& + \int_0^t e^{\varepsilon s} \left[\left(-\lambda + \theta + \|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) |x|^2 \right. \\
& \quad + \left(\|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) |y|^2 \\
& \quad \left. + \|G\| \beta |z|^2 + \frac{2C^2}{\theta} \|G\|^2 |x - z|^2 \right] ds. \tag{39}
\end{aligned}$$

Taking the expectation in the above, we have

$$\begin{aligned}
& E(e^{\varepsilon t} x^T Gx) \\
& \leq E(\xi^T(0) G\xi(0)) \\
& \quad - \left[\lambda - \theta - \|G\| (\beta + \varepsilon) - \frac{4C^2}{\theta} \|G\|^2 \right] \int_0^t e^{\varepsilon s} \cdot E|x|^2 ds \\
& \quad + \left(\|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) \int_0^t e^{\varepsilon s} \cdot E|y|^2 ds \\
& \quad + \|G\| \beta \int_0^t e^{\varepsilon s} \cdot E|z|^2 ds \\
& \quad + \frac{2C^2}{\theta} \|G\|^2 \int_0^t e^{\varepsilon s} \cdot E|x - z|^2 ds. \tag{40}
\end{aligned}$$

Now we apply Lemmas 7–9 to the last three terms on the right-hand side of (40) to get an estimate of $E(e^{\varepsilon t} x^T Gx)$ as follows:

$$\begin{aligned}
& E(e^{\varepsilon t} x^T Gx) \\
& \leq E(\xi^T(0) G\xi(0)) - \left[\lambda - \theta - \|G\| (\beta + \varepsilon) - \frac{4C^2}{\theta} \|G\|^2 \right] \\
& \quad \times \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \\
& \quad + \left(\|G\| \beta + \frac{4C^2}{\theta} \|G\|^2 \right) \frac{\delta}{\mu} \left(c_1 e^{\varepsilon\delta} + e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \right) \\
& \quad + \|G\| \beta \left(c_1 e^{\varepsilon\delta} + e^{\varepsilon\delta} \int_0^{t-\delta} e^{\varepsilon s} \cdot E|x(s)|^2 ds \right) \\
& \quad + \frac{2C^2}{\theta} \|G\|^2 \left\{ c_2 + (6\delta\alpha^2 + 2\beta) \cdot \delta e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \right. \\
& \quad \quad \left. + (6\delta\alpha^2 + 2\beta) \cdot \left(\frac{\delta}{\mu} + 1 \right) \right. \\
& \quad \quad \left. \times \left[c_1 \delta e^{2\varepsilon\delta} + \delta e^{2\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \right] \right\} \\
& \leq E(\xi^T(0) G\xi(0)) - \left[\lambda - \theta - \|G\| (\beta + \varepsilon) - \frac{4C^2}{\theta} \|G\|^2 \right] \\
& \quad \times \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \\
& \quad + \|G\| \beta \left(1 + \frac{\delta}{\mu} \right) \left(c_1 e^{\varepsilon\delta} + e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \right) \\
& \quad + \frac{(2C \|G\|)^2}{\theta} \left\{ c_2 + \frac{\delta}{\mu} c_1 e^{\varepsilon\delta} + \left(6\delta\alpha^2 + 2\beta + \frac{1}{\mu} \right) \right. \\
& \quad \quad \left. \times \delta e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds + (6\delta\alpha^2 + 2\beta) \right\}
\end{aligned}$$

$$\begin{aligned}
 & \cdot \left(\frac{\delta}{\mu} + 1 \right) \\
 & \times \left(c_1 \delta e^{2\epsilon\delta} + \delta e^{2\epsilon\delta} \int_0^t e^{\epsilon s} E|x(s)|^2 ds \right) \Big\} \\
 = & c_4 \tag{41}
 \end{aligned}$$

for $t \geq \tau$, where

$$\begin{aligned}
 c_4 = & E \left(\xi^T(0) G \xi(0) \right) + \|G\| \beta \left(\frac{\delta}{\mu} + 1 \right) c_1 e^{\epsilon\delta} + \frac{(2C \|G\|)^2}{\theta} \\
 & \cdot \left[c_2 + \frac{\delta}{\mu} c_1 e^{\epsilon\delta} + (6\delta\alpha^2 + 2\beta) \left(\frac{\delta}{\mu} + 1 \right) c_1 \delta e^{2\epsilon\delta} \right]. \tag{42}
 \end{aligned}$$

Since G is positive definite,

$$x^T G x \geq \lambda_{\min}(G) |x|^2, \tag{43}$$

where $\lambda_{\min}(G) > 0$ is the smallest eigenvalue of G .

Then,

$$E \left(e^{\epsilon t} x^T(t) G x(t) \right) \geq E \left(e^{\epsilon t} \lambda_{\min}(G) |x(t)|^2 \right). \tag{44}$$

It then follows from (41) that

$$E|x(t)|^2 \leq \left[\frac{c_4}{\lambda_{\min}(G)} \right] \cdot e^{-\epsilon t} \quad \text{for any } t \geq \tau. \tag{45}$$

Hence,

$$\begin{aligned}
 \frac{1}{t} \ln \left(E|x(t)|^2 \right) & \leq \frac{1}{t} \ln \left(\left[\frac{c_4}{\lambda_{\min}(G)} \right] \cdot e^{-\epsilon t} \right) \\
 & = -\epsilon + \frac{1}{t} \ln \left[\frac{c_4}{\lambda_{\min}(G)} \right]. \tag{46}
 \end{aligned}$$

This easily yields

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \left(E|x(t)|^2 \right) \leq -\epsilon. \tag{47}$$

Then (1) is exponentially stable in the mean square. □

Theorem 10. Let $\epsilon > 0$, under the same assumption as Theorem 6, if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \left(E|x(t)|^2 \right) \leq -\epsilon. \tag{48}$$

Then,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln (|x(t)|) \leq -\frac{\epsilon}{2} \quad \text{a.s.} \tag{49}$$

Proof. let $\epsilon > 0$, under the same assumption as Theorem 6. It follows from (48) that

$$\frac{1}{t} \ln \left(E|x(t)|^2 \right) \leq -\epsilon + \frac{1}{t} \cdot \ln M \tag{50}$$

for all $t \geq \delta$, $M > 0$. Then

$$E|x(t)|^2 \leq M e^{-\epsilon t}. \tag{51}$$

For $t \in [k\delta, (k+1)\delta]$, $k = 2, 3, \dots$, we have

$$E \left(\sup_{k\delta \leq t \leq (k+1)\delta} E|x(t)|^2 \right) \leq M e^{-\epsilon k\delta}. \tag{52}$$

Let $\epsilon_0 \in (0, \epsilon)$ be arbitrary. By Doob's martingale inequality, it follows from (52) that

$$P \left(\omega : \sup_{k\delta \leq t \leq (k+1)\delta} |x(t)| > e^{-(\epsilon-\epsilon_0)k\delta/2} \right) \leq c_4 e^{-\epsilon_0 k\delta}. \tag{53}$$

Thus, it follows from the Borel-Cantelli lemma that, for almost all $\omega \in \Omega$, there exists $k_0(\omega)$, and $k \geq k_0(\omega)$,

$$P \left(\omega : \sup_{k\delta \leq t \leq (k+1)\delta} |x(t)| \leq e^{-(\epsilon-\epsilon_0)k\delta/2} \right) = 1. \tag{54}$$

So, for $k\delta \leq t \leq (k+1)\delta$, $k \geq k_0(\omega)$,

$$\ln |x(t)| \leq -\frac{(\epsilon - \epsilon_0) k\delta}{2} \quad \text{a.s.} \tag{55}$$

This easily yields

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |x(t)| \leq -\frac{\epsilon - \epsilon_0}{2} \quad \text{a.s.} \tag{56}$$

Since ϵ_0 is arbitrary, we must have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |x(t)| \leq -\frac{\epsilon}{2} \quad \text{a.s.} \tag{57} \quad \square$$

Remark 11. The exponentially stable in the mean square of SMDDE (1) implies the almost surely exponentially stable. In general, Theorem 10 is still true for p th moment exponential stability.

Let us single out two important special cases.

Case 1. If $\sigma(t, x, y, z) \equiv 0$, then (1) reduces to nonlinear deterministic differential equation with mixed delay

$$\begin{aligned}
 dx(t) & = b(t, x, y, z) dt, \quad t \geq 0, \\
 x(t) & = \xi(t), \quad t \in [-\delta, 0]. \tag{58}
 \end{aligned}$$

Applying Theorem 6 to (58), we obtain the following useful result.

Corollary 12. Let (H3.4) and condition (22) hold, then the trivial solution of (58) is exponentially stable in the mean square. Assume that there exists a symmetric positive definite $n \times n$ matrices G and a constant $\lambda > 0$ such that

$$2x^T G b(t, x, x, x) \leq -\lambda |x|^2, \tag{59}$$

$$\forall (t, x, x, x) \in R^+ \times R^n \times R^n \times R^n,$$

$$\lambda > 4C \|G\| \sqrt{1 + \frac{\delta}{\mu} + 6\delta^2\alpha^2 \left(2 + \frac{\delta}{\mu} \right)}. \tag{60}$$

Remark 13. The bound for the time delay δ when SMDDE (1) is exponentially stable follows from (25), and the bound for the corresponding deterministic case follows from (60).

Case 2. If the time delay $\delta = 0$, then (1) reduces to nonlinear SDE

$$dx(t) = b(t, x(t), 0, x(t)) dt + \sigma(t, x(t), 0, x(t)) dW(t), \quad t \geq 0. \tag{61}$$

Corollary 14. *If SDE (61) is exponentially stable and the time delay δ is sufficiently small, then the corresponding SMDDE (1) will remain exponentially stable.*

Proof. SMDDE (1) can be rewritten as

$$\begin{aligned} dx(t) = & b(t, x(t), 0, x(t)) dt + \sigma(t, x(t), 0, x(t)) dW(t) \\ & - [b(t, x(t), 0, x(t)) \\ & - b(t, x(t), y(t), z(t))] dt \\ & - [\sigma(t, x(t), 0, x(t)) \\ & - \sigma(t, x(t), y(t), z(t))] dW(t). \end{aligned} \tag{62}$$

It is clear that if the time delay δ is sufficiently small, then SMDDE (1) is regarded as the perturbed system of the corresponding SDE (61) (without delay). On the other hand, the perturbation term

$$\begin{aligned} & [b(t, x(t), 0, x(t)) - b(t, x(t), y(t), z(t))] dt \\ & + [\sigma(t, x(t), 0, x(t)) - \sigma(t, x(t), y(t), z(t))] dW(t) \end{aligned} \tag{63}$$

could be so small that the perturbed equation (1) would behave in a similar way as (61) asymptotically. Applying Theorem 5 and Remark 12 in [15], we derive SMDDE (1) which will remain exponentially stable. \square

4. Example

Let us now present a simple example to illustrate our results which can help us find the time delay upper limit.

Example 1. Let us now consider a two-dimensional SMDDE

$$\begin{aligned} dx(t) = & b(t, x(t), y(t), z(t)) dt \\ & + \sigma(t, x(t), y(t), z(t)) dW(t), \end{aligned} \tag{64}$$

where

$$b(t, x(t), y(t), z(t)) := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -0.5x_1 - 0.5y_1 - 0.5z_1 + 0.5x_2 \sin x_1 x_2 \\ -0.7x_2 - 0.3y_2 - 0.5z_2 - 0.5x_1 \sin x_1 x_2 \end{pmatrix} \tag{65}$$

for $x = (x_1, x_2)^T, y = (y_1, y_2)^T, z = (z_1, z_2)^T$ in R^2 , and $\sigma(t, x(t), y(t), z(t)): R^+ \times R^2 \times R^2 \times R^2 \rightarrow R^{2 \times 2}$ represent the nonlinear uncertainties. $W(t)$ is a given two-dimensional Brownian motion,

$$y(t) = \int_{-\delta}^0 e^{\mu s} x(t+s) ds, \quad z(t) = x(t-\delta) \tag{66}$$

represent given functionals of the path segment $x_t := \{x(t+s); s \in [-\delta, 0]\}$ of $x, \mu \in R^+$ is the given averaging parameter, and $\delta > 0$ is a given finite time delay.

For convenience, let us choose G to be the second-order identity matrix, and $\mu = 1$. We assume that, for any $(t, x, y, z) \in R^+ \times R^2 \times R^2 \times R^2$,

$$tr[\sigma^T(t, x, y)\sigma(t, x, y)] \leq 0.1(|x|^2 + |y|^2 + |z|^2). \tag{67}$$

Via a simple calculation, it is easy to estimate that

$$\begin{aligned} 2x^T b(t, x, x, x) & \leq -3|x|^2, \\ |b(t, x, y, z) - b(t, x, x, x)| & \leq 0.5(|x - y| + |x - z|), \tag{68} \\ |b(t, x, y, z)| & \leq 1.72(|x| + |y| + |z|); \end{aligned}$$

that is, $\|G\| = 1, \lambda = 3, C = 0.5, \alpha = 1.72, \beta = 0.1$, and $\mu = 1$. By plugging these into (25), it is easy to find $\delta < 0.1396$; that is, if $\delta < 0.1396$, then (64) is exponentially stable in the mean square and is also almost surely exponentially stable.

If $\sigma(t, x(t), y(t), z(t)) \equiv 0$, by plugging $\|G\| = 1, \lambda = 3, C = 0.5, \alpha = 1.72$, and $\mu = 1$ into (60), then, we would conclude that (64) is exponentially stable provided $\delta < 0.1677$.

Remark 15. Figures 1 and 2 give the simulation results of Example 1 when $\sigma \neq 0, \delta < 0.1396$ (Figure 1) and $\sigma \equiv 0, \delta < 0.1677$ (Figure 2). The parameter values used in the calculations are $\|G\| = 1, \lambda = 3, C = 0.5, \alpha = 1.72, \mu = 1$, and $\delta = 0.1$.

5. Concluding Remarks

In this paper, we investigate not only the exponential stability in the mean square but also the almost surely exponential stability for a class of SMDDEs based on Lyapunov stability theory, Itô formula, stochastic analysis, inequality technique, and so on. We first consider the existence and uniqueness of the adapted solution by employing fixed point theorem. Next, some sufficient conditions of exponential stability and corollaries for stochastic differential systems with mixed delays are obtained by using Lyapunov function. Theorem 10 shows that the exponentially stable in the mean square of SMDDE implies the almost surely exponentially stable. The obtained results generalize and improve some recent results (for instance, [19–21]). In particular, our theoretical results show that if SDE is exponentially stable and the time delay is sufficiently small, then the corresponding SMDDE will remain exponentially stable. Moreover, the time delay upper

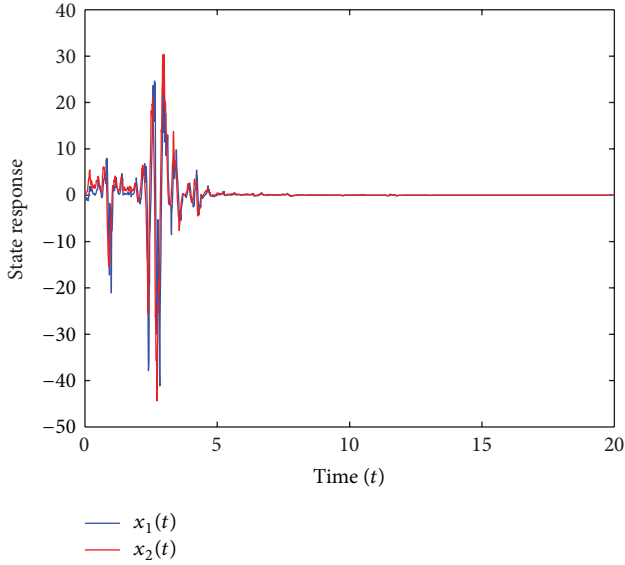


FIGURE 1: The simulation results of Example 1 when $\sigma \neq 0$, $\delta < 0.1396$.

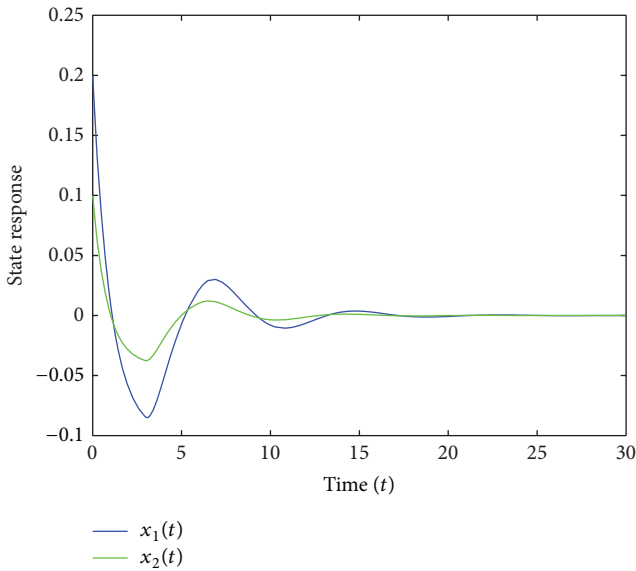


FIGURE 2: The simulation results of Example 1 when $\sigma \equiv 0$, $\delta < 0.1677$.

limit is solved by using our theoretical results when the system is exponentially stable, and they are more easily verified and applied in practice. It should be mentioned that the approach provided here is different from those used in [19–21]. Finally, we present a simple example to illustrate the effectiveness of our stable results. Another challenging problem is to study a class of stochastic differential equations with mixed variable delays or a class of stochastic control systems with correlated state and observation noises (for instance, [24]). We hope to study these problems in forthcoming papers.

Appendix

We now present proofs of Lemmas 7–9.

Proof of Lemma 7. For any $t \geq \delta$, we easily get

$$\begin{aligned}
 & \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau)|^2 d\tau ds \\
 &= \int_0^t E|x(\tau)|^2 \left(\int_{\tau \vee \delta}^{(\tau+\delta) \wedge t} e^{\varepsilon s} ds \right) d\tau \\
 &\leq \int_0^t E|x(\tau)|^2 \cdot e^{\varepsilon[(\tau+\delta) \wedge t]} [((\tau+\delta) \wedge t) - (\tau \vee \delta)] d\tau \\
 &\leq \delta e^{\varepsilon \delta} \int_0^t e^{\varepsilon \tau} \cdot E|x(\tau)|^2 d\tau, \\
 & \int_0^t e^{\varepsilon s} \cdot E|x(s-\delta)|^2 ds \\
 &= \int_0^{\delta} e^{\varepsilon s} \cdot E|x(s-\delta)|^2 ds + \int_{\delta}^t e^{\varepsilon s} \cdot E|x(s-\delta)|^2 ds \\
 &\leq e^{\varepsilon \delta} \int_0^{\delta} E|x(s-\delta)|^2 ds + e^{\varepsilon \delta} \int_{\delta}^t e^{\varepsilon(s-\delta)} \cdot E|x(s-\delta)|^2 ds \\
 &e^{\varepsilon \delta} \int_{-\delta}^0 E|\xi(\tau)|^2 d\tau + e^{\varepsilon \delta} \int_0^{t-\delta} e^{\varepsilon \tau} \cdot E|x(\tau)|^2 d\tau \quad (\tau = s - \delta) \\
 &\leq c_{11} e^{\varepsilon \delta} + e^{\varepsilon \delta} \int_0^{t-\delta} e^{\varepsilon \tau} \cdot E|x(\tau)|^2 d\tau
 \end{aligned} \tag{A.1}$$

for any $t \geq \delta$, where $c_{11} \geq \int_{-\delta}^0 E|\xi(\tau)|^2 d\tau$. □

Proof of Lemma 8. Similar to (26), for any $t \geq \delta$, we have

$$\int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|y(\tau)|^2 d\tau ds \leq \delta e^{\varepsilon \delta} \int_0^t e^{\varepsilon \tau} \cdot E|y(\tau)|^2 d\tau. \tag{A.2}$$

For $t \geq \delta$, it then follows from (17) that

$$\begin{aligned}
 & \int_0^t e^{\varepsilon s} \cdot E|y(s)|^2 ds \\
 &\leq \frac{1}{2\mu} \int_0^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau)|^2 d\tau ds \\
 &= \frac{1}{2\mu} \left[\int_0^{\delta} e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau)|^2 d\tau ds \right. \\
 &\quad \left. + \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau)|^2 d\tau ds \right].
 \end{aligned} \tag{A.3}$$

Similar to (26) and substituting (26) into the above, we obtain (29); that is

$$\begin{aligned}
 & \int_0^t e^{\varepsilon s} \cdot E|y(s)|^2 ds \\
 &\leq \frac{\delta}{2\mu} e^{\varepsilon \delta} \int_{-\delta}^{\delta} e^{\varepsilon s} \cdot E|x(s)|^2 ds
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\delta}{2\mu} e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \\
 & \leq \frac{\delta}{2\mu} c_{12} e^{\varepsilon\delta} + \frac{\delta}{2\mu} e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds \\
 & \leq \frac{\delta}{\mu} c_{12} e^{\varepsilon\delta} + \frac{\delta}{\mu} e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds,
 \end{aligned}
 \tag{A.4}$$

$$\begin{aligned}
 & \leq 2\delta \cdot E \int_{s-\delta}^s 3\alpha^2 (|x|^2 + |y|^2 + |z|^2) dt \\
 & + 2\beta \cdot E \int_{s-\delta}^s (|x|^2 + |y|^2 + |z|^2) dt \\
 & = (6\delta\alpha^2 + 2\beta) \int_{s-\delta}^s (E|x|^2 + E|y|^2 + E|z|^2) dt.
 \end{aligned}
 \tag{A.9}$$

where $c_{12} \geq \int_{-\delta}^{\delta} e^{\varepsilon s} \cdot E|x(s)|^2 ds$.

Substituting (29) into (A.2) yields

$$\begin{aligned}
 & \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|y(\tau)|^2 d\tau ds \\
 & \leq \frac{\delta^2}{\mu} c_{12} e^{2\varepsilon\delta} + \frac{\delta^2}{\mu} e^{2\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds.
 \end{aligned}
 \tag{A.5}$$

The relation (28) in Lemma 8 is then proved. □

Proof of Lemma 9. Similar to (26), for any $t \geq \delta$, we have

$$\int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau - \delta)|^2 d\tau ds \leq \delta e^{\varepsilon\delta} \int_0^t e^{\varepsilon\tau} \cdot E|x(\tau - \delta)|^2 d\tau.
 \tag{A.6}$$

Substituting (27) into the above inequality yields

$$\begin{aligned}
 & \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x(\tau - \delta)|^2 d\tau ds \\
 & \leq \delta e^{\varepsilon\delta} \left(c_{11} e^{\varepsilon\delta} + e^{\varepsilon\delta} \int_0^{t-\delta} e^{\varepsilon s} \cdot E|x(s)|^2 ds \right) \\
 & < \delta c_{11} e^{2\varepsilon\delta} + \delta e^{2\varepsilon\delta} \int_0^t e^{\varepsilon s} \cdot E|x(s)|^2 ds.
 \end{aligned}
 \tag{A.7}$$

The relation (30) in Lemma 9 is then proved.

On the other hand, for $s \geq \delta$, we have

$$\begin{aligned}
 & x(s) - x(s - \delta) \\
 & = \int_{s-\delta}^s b(t, x, y, z) dt + \int_{s-\delta}^s \sigma(t, x, y, z) dW(t).
 \end{aligned}
 \tag{A.8}$$

By (22) and (23), it follows from Lemmas 3-4 that

$$\begin{aligned}
 & E|x - z|^2 \\
 & \leq 2E \left| \int_{s-\delta}^s b(t, x, y, z) dt \right|^2 + 2E \left| \int_{s-\delta}^s \sigma(t, x, y, z) dW(t) \right|^2 \\
 & \leq 2\delta \cdot E \int_{s-\delta}^s |b(t, x, y, z)|^2 dt \\
 & + 2E \int_{s-\delta}^s \text{tr} [\sigma^T(t, x, y, z) \sigma(t, x, y, z)] dt
 \end{aligned}$$

Similar to (27), for $t \geq \delta$, we have

$$\begin{aligned}
 & \int_0^t e^{\varepsilon s} \cdot E|x - z|^2 ds \\
 & = \int_0^{\delta} e^{\varepsilon s} \cdot E|x(s) - x(s - \delta)|^2 ds \\
 & + \int_{\delta}^t e^{\varepsilon s} \cdot E|x(s) - x(s - \delta)|^2 ds \\
 & \leq c_2 + (6\delta\alpha^2 + 2\beta) \\
 & \cdot \left(\int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|x|^2 dt ds \right. \\
 & \left. + \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|y|^2 dt ds + \int_{\delta}^t e^{\varepsilon s} \int_{s-\delta}^s E|z|^2 dt ds \right),
 \end{aligned}
 \tag{A.10}$$

where $c_2 \geq \int_0^{\delta} e^{\varepsilon s} \cdot E|x(s) - x(s - \delta)|^2 ds$.

Substituting (26), (28), and (30) into (A.10), for $t \geq \delta$, we get

$$\begin{aligned}
 & \int_0^t e^{\varepsilon s} \cdot E|x(s) - x(s - \delta)|^2 ds \\
 & \leq c_2 + (6\delta\alpha^2 + 2\beta) \cdot \delta e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \\
 & + (6\delta\alpha^2 + 2\beta) \cdot \left(\frac{\delta}{\mu} c_{12} + c_{11} \right) \\
 & \times \left[\delta e^{2\varepsilon\delta} + \delta e^{2\varepsilon\delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \right] \\
 & \leq c_2 + (6\delta\alpha^2 + 2\beta) \cdot \delta e^{\varepsilon\delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \\
 & + (6\delta\alpha^2 + 2\beta) \cdot \left(\frac{\delta}{\mu} + 1 \right) \\
 & \times \left[c_1 \delta e^{2\varepsilon\delta} + \delta e^{2\varepsilon\delta} \int_0^t e^{\varepsilon s} E|x(s)|^2 ds \right],
 \end{aligned}
 \tag{A.11}$$

where $c_1 = \max\{c_{11}, c_{12}\}$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities (JBK130401).

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