Research Article

On the Stochastic Restricted r-k Class Estimator and Stochastic Restricted r-d Class Estimator in Linear Regression Model

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The stochastic restricted r-k class estimator and stochastic restricted r-d class estimator are proposed for the vector of parameters in a multiple linear regression model with stochastic linear restrictions. The mean squared error matrix of the proposed estimators is derived and compared, and some properties of the proposed estimators are also discussed. Finally, a numerical example is given to show some of the theoretical results.

1. Introduction

The problem of multicollinearity or the ill-conditioned design matrix in linear regression model is very well known in statistics. In order to overcome this problem, different remedies have been introduced. One of the most important estimation methods is to consider biased estimators, such as the principal component regression (PCR) estimator [1], the ridge estimator (ORE) by Hoerl and Kennard [2], the r-k class estimator [3], the Liu estimator (LE) by Liu [4], the r-d class estimator [5], the r-k-d class estimator [6], and the principal component Liu-type estimator [7].

An alternative method to deal with multicollinearity problem is to consider parameter estimation with some restrictions on the unknown parameters, which may be exact or stochastic restrictions [8]. When stochastic additional restrictions on the parameter vector are supposed to hold, Durbin [9], Theil and Goldberger [10], and Theil [11] proposed the ordinary mixed estimator (OME). By grafting the ordinary regression ridge estimator and LE into the mixed estimation, Li and Yang [12] and Hubert and Wijekoon [13] introduced a stochastic restricted ridge estimator (SRRE) and stochastic restricted Liu estimator (SRLE), respectively, and Liu et al. [14] proposed the weighted mixed almost unbiased ridge estimator in linear regression model.

In this paper, in order to overcome multicollinearity, we introduce a stochastic restricted r-k class estimator and a stochastic restricted r-d class estimator for the vector of parameters in a linear regression model when additional stochastic linear restrictions are assumed to hold. Performance of the proposed estimators with respect to the mean squared error matrix (MSEM) criterion is discussed.

The rest of the paper is organized as follows. The model specifications and the new estimators are introduced in Section 2. Then, the superiority of the proposed estimators is discussed in Section 3 and a numerical example is given to illustrate the behavior of the estimators in Section 4. Finally, some conclusion remarks are given in Section 5.

2. Model Specifications and the Estimators

Consider the linear regression model

$$y = X\beta + \varepsilon, \tag{1}$$

where y is an $n \times 1$ vector of observation, X is an $n \times p$ known design matrix of rank p, β is a $p \times 1$ vector of unknown parameters, and ε is an $n \times 1$ vector of disturbances with expectation $E(\varepsilon) = 0$ and variance-covariance matrix $\mathrm{Cov}(\varepsilon) = \sigma^2 I_n$.

For the unrestricted model given by (1), the ORE proposed by Hoerl and Kennard [2] and the LE presented by Liu [4] are defined as follows:

$$\widehat{\beta}(k) = S_k^{-1} X' y = R_k \widehat{\beta}_{\text{OLS}},$$

$$\widehat{\beta}(d) = (S + I)^{-1} \left(X' y + d\widehat{\beta}_{\text{OLS}} \right) = F_d \widehat{\beta}_{\text{OLS}}, \quad (2)$$

where $k > 0$, $0 < d < 1$, $S = X'X$, $S_k = S + kI$, $R_k = S_k^{-1}S$, $F_d = (S + I)^{-1}(S + dI)$, and $\widehat{\beta}_{\text{OLS}} = S^{-1}X'y$ is the ordinary least squares (OLS) estimator of β.

Now let us consider the spectral decomposition of the matrix given as

$$X'X = \left(T_r, T_{p-r}\right) \begin{pmatrix} \Lambda_r & 0 \\ 0 & \Lambda_{p-r} \end{pmatrix} \begin{pmatrix} T_r' \\ T_{p-r}' \end{pmatrix}, \quad (3)$$

where Λ_r and Λ_{p-r} are diagonal matrices such that that the main diagonal elements of the $r \times r$ matrix Λ_r are the r largest eigenvalues of $X'X$, while Λ_{p-r} are the remaining $p-r$ eigenvalues. The $p \times p$ matrix $T = (T_r, T_{p-r})$ is orthogonal with $T_r = (t_1, t_2, \ldots, t_r)$ consisting of its first r columns and $T_{p-r} = (t_{r+1}, t_{r+2}, \ldots, t_p)$ consisting of the remaining $p - r$ columns of the matrix T. The PCR estimator for β can be written as

$$\widehat{\beta}_r = T_r \left(T_r' X' X T_r\right)^{-1} T_r' X' y. \quad (4)$$

The r-k class estimator proposed by Baye and Parker [3] and the r-d class estimator proposed by Kaçıranlar and Sakallıoğlu [5] are defined as

$$\widehat{\beta}_{rk}(r, k) = T_r \left(T_r' X' X T_r + kI_r\right)^{-1} T_r' X' y, \quad k > 0,$$

$$\widehat{\beta}_{rd}(r, d) = T_r \left(T_r' X' X T_r + I_r\right)^{-1} \quad (5)$$

$$\times \left(T_r' X' y + dT_r' \widehat{\beta}_r\right), \quad 0 < d < 1.$$

Followed by Xu and Yang [15], the r-k class estimator and the r-d class estimator could be rewritten as follows:

$$\widehat{\beta}_{rk}(r, k) = T_r \left(T_r' X' X T_r + kI_r\right)^{-1} T_r' X' y$$

$$= T_r T_r' \widehat{\beta}(k) = T_r T_r' R_k \widehat{\beta}_{\text{OLS}}, \quad (6)$$

$$\widehat{\beta}_{rd}(r, d) = T_r \left(T_r' X' X T_r + I_r\right)^{-1} \left(T_r' X' y + dT_r' \widehat{\beta}_r\right)$$

$$= T_r T_r' \widehat{\beta}(d) = T_r T_r' F_d \widehat{\beta}_{\text{OLS}}. \quad (7)$$

In addition to the model (1), let us be given some prior information about β in the form of a set of j independent stochastic linear restrictions as follows:

$$r = R\beta + e, \quad e \sim \left(0, \sigma^2 W\right), \quad (8)$$

where R is a $j \times p$ known matrix of rank j, e is a $j \times 1$ vector of disturbances with mean 0 and dispersion matrix $\sigma^2 W$, W is supposed to be known and positive definite, and the $j \times 1$ vector r can be interpreted as a random variable with

expectation $E(r) = R\beta$. Therefore the restriction (9) does not hold exactly but in the mean, and we suppose r to be known, that is, to be the realized value of the random vector, so that all the expectations are conditional on r [8]. In the following discussions, we do not mention this separately. Furthermore, it is also supposed that the random vector ε is stochastically independent of e.

For the restricted model specified by (1) and (8), the stochastic restricted ridge estimator (SRRE) proposed by Li and Yang [12] and the stochastic restricted Liu estimator (SRLE) proposed by Hubert and Wijekoon [13] are defined as

$$\widehat{\beta}_{\text{SRRE}}(k) = R_k \widehat{\beta}_{\text{OME}} = R_k \left(S + R' W^{-1} R\right)^{-1} \left(X' y + R' W^{-1} r\right), \quad (9)$$

$$\widehat{\beta}_{\text{SRLE}}(d) = F_d \widehat{\beta}_{\text{OME}} = F_d \left(S + R' W^{-1} R\right)^{-1} \left(X' y + R' W^{-1} r\right), \quad (10)$$

where $\widehat{\beta}_{\text{OME}} = (S + R' W^{-1} R)^{-1}(X' y + R' W^{-1} r)$ is the well-known ordinary mixed estimator (OME) of β.

We are now ready to propose a new stochastic restricted r-k class estimator which is defined by combing the OME and r-k class estimator and a new stochastic restricted r-d class estimator which is defined by combing the OME and r-d class estimator as follows:

$$\widehat{\beta}_{\text{SR}rk}(r, k) = T_r T_r' R_k \widehat{\beta}_{\text{OME}}$$

$$= T_r T_r' R_k \left(S + R' W^{-1} R\right)^{-1} \left(X' y + R' W^{-1} r\right),$$

$$\widehat{\beta}_{\text{SR}rd}(r, d) = T_r T_r' F_d \widehat{\beta}_{\text{OME}}$$

$$= T_r T_r' F_d \left(S + R' W^{-1} R\right)^{-1} \left(X' y + R' W^{-1} r\right). \quad (11)$$

From (11), we can see that

(1) when $R = 0$, we may conclude that $\widehat{\beta}_{\text{SR}rk}(r, k) = \widehat{\beta}_{rk}(r, k)$ and $\widehat{\beta}_{\text{SR}rd}(r, d) = \widehat{\beta}_{rd}(r, d)$;

(2) when $r = p$, we may conclude that $\widehat{\beta}_{\text{SR}rk}(r, k) = \widehat{\beta}_{\text{SRRE}}(k)$ and $\widehat{\beta}_{\text{SR}rd}(r, d) = \widehat{\beta}_{\text{SRLE}}(d)$.

At the end of this section, we will list some lemmas which are needed in the following proofs.

Lemma 1 (see [8]). *Assume that square matrices A and C are not singular and B and D are matrices with proper orders; then $(A + BCD)^{-1} = A^{-1} - A^{-1} B (C^{-1} + DA^{-1} B)^{-1} DA^{-1}$.*

Lemma 2 (see [16]). *Let A be a nonnegative definite matrix, namely, $A \geq 0$ and let α be some vector; then $A - \alpha\alpha' \geq 0$ if and only if $\alpha' A^+ \alpha \leq 1$, $\alpha \in \Re(A)$.*

Lemma 3 (see [17]). *Let $n \times n$ matrices be $A > 0$, $B \geq 0$, and $A - B \geq 0 \Leftrightarrow \lambda_1(BA^{-1}) \leq 1$, where $\lambda_1(BA^{-1})$ is the largest eigenvalue of BA^{-1}.*

In the next section, we will make comparison of the biased estimators.

3. Model Specifications and the Estimators

The mean squared error matrix (MSEM) of an estimator $\widehat{\beta}$ is defined as

$$M\left(\widehat{\beta}\right) = E\left(\widehat{\beta} - \beta\right)\left(\widehat{\beta} - \beta\right)' = \text{Cov}\left(\widehat{\beta}\right) + \text{Bias}\left(\widehat{\beta}\right)\text{Bias}\left(\widehat{\beta}\right)', \tag{12}$$

where $\text{Cov}(\widehat{\beta})$ is the dispersion matrix and $\text{Bias}(\widehat{\beta})$ is the bias vector. For the two given estimators $\widehat{\beta}_1$, $\widehat{\beta}_2$, the estimator $\widehat{\beta}_2$ is said to be superior to the estimator $\widehat{\beta}_1$ in the matrix MSE criterion if and only if

$$\Delta\left(\widehat{\beta}_1, \widehat{\beta}_2\right) = M\left(\widehat{\beta}_1\right) - M\left(\widehat{\beta}_2\right) \geq 0. \tag{13}$$

Note that the MSEM criterion is always superior to the scalar mean squared error criterion (MSE); we only consider the MSEM comparisons among the estimators.

3.1. MSEM Comparisons of the r-k Class Estimator and the Stochastic Restricted r-k (SRrk) Class Estimator.

In this subsection, we consider the MSEM comparison between the r-k class estimator and the stochastic restricted r-k (SRrk) class estimator.

Firstly, we can compute the bias vector and the variance of stochastic restricted r-k (SRrk) class estimator as follows:

$$\text{Bias}\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right) = E\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right) - \beta$$
$$= T_r T_r' R_k \beta - \beta = \left(T_r T_r' R_k - I\right)\beta, \tag{14}$$

$$\text{Cov}\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right) = \sigma^2 T_r T_r' R_k A R_k T_r T_r',$$

where $A = (S + R'W^{-1}R)^{-1}$.

Therefore, the MSEM of the stochastic restricted r-k class estimator is given by

$$\text{MSEM}\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right) = \sigma^2 T_r T_r' R_k A R_k T_r T_r'$$
$$+ \left(T_r T_r' R_k - I\right)\beta\beta'\left(T_r T_r' R_k - I\right)'. \tag{15}$$

From (6), we can compute the bias vector and the variance of r-k class estimator as follows:

$$\text{Bias}\left(\widehat{\beta}_{rk}\left(r,k\right)\right) = E\left(\widehat{\beta}_{rk}\left(r,k\right)\right) - \beta$$
$$= T_r T_r' R_k \beta - \beta = \left(T_r T_r' R_k - I\right)\beta, \tag{16}$$

$$\text{Cov}\left(\widehat{\beta}_{rk}\left(r,k\right)\right) = \sigma^2 T_r T_r' R_k S^{-1} R_k T_r T_r'.$$

Therefore, the MSEM of the r-k class estimator is given by

$$\text{MSEM}\left(\widehat{\beta}_{rk}\left(r,k\right)\right) = \sigma^2 T_r T_r' R_k S^{-1} R_k T_r T_r'$$
$$+ \left(T_r T_r' R_k - I\right)\beta\beta'\left(T_r T_r' R_k - I\right)'. \tag{17}$$

Now we consider the following difference of the MSEM:

$$\Delta_1 = \text{MSEM}\left(\widehat{\beta}_{rk}\left(r,k\right)\right) - \text{MSEM}\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right)$$

$$= \sigma^2 T_r T_r' R_k S^{-1} R_k T_r T_r'$$

$$+ \left(T_r T_r' R_k - I\right)\beta\beta'\left(T_r T_r' R_k - I\right)'$$

$$- \left\{\sigma^2 T_r T_r' R_k A R_k T_r T_r'\right.$$

$$\left. + \left(T_r T_r' R_k - I\right)\beta\beta'\left(T_r T_r' R_k - I\right)'\right\}$$

$$= \sigma^2 T_r T_r' R_k \left(S^{-1} - A\right) R_k T_r T_r'. \tag{18}$$

Theorem 4. *The stochastic restricted r-k class estimator $\widehat{\beta}_{\text{SR}rk}(r,k)$ always dominates the r-k class estimator $\widehat{\beta}_{rk}(r,k)$ in the MSEM criterion.*

Proof. By Lemma 1, we can obtain $A = (S + R'W^{-1}R)^{-1} = S^{-1} - S^{-1}R'(W + RS^{-1}R')^{-1}RS^{-1}$; then we obtain $S^{-1} - A = S^{-1}R'(W + RS^{-1}R')^{-1}RS^{-1} \geq 0$. Therefore, from (18), we may conclude that $\Delta_1 \geq 0$; that is to say, the stochastic restricted r-k class estimator $\widehat{\beta}_{\text{SR}rk}(r,k)$ always dominates the r-k class estimator $\widehat{\beta}_{rk}(r,k)$ in the MSEM criterion. $\square$

3.2. MSEM Comparisons of the Stochastic Restricted Ridge Estimator (SRRE) and the Stochastic Restricted r-k (SRrk) Class Estimator.

In this subsection, we consider the MSEM comparison between the stochastic restricted ridge estimator (SRRE) and the stochastic restricted r-k (SRrk) class estimator.

Firstly, from (9), we can compute the bias vector and the variance of stochastic restricted ridge estimator (SRRE) as follows:

$$\text{Bias}\left(\widehat{\beta}_{\text{SRRE}}\left(k\right)\right) = E\left(\widehat{\beta}_{\text{SRRE}}\left(k\right)\right) - \beta = \left(R_k - I\right)\beta,$$
$$\text{Cov}\left(\widehat{\beta}_{\text{SRRE}}\left(k\right)\right) = \sigma^2 R_k A R_k. \tag{19}$$

Then, we can obtain the MSEM of the SRRE as follows:

$$\text{MSEM}\left(\widehat{\beta}_{\text{SRRE}}\left(k\right)\right) = \sigma^2 R_k A R_k + \left(R_k - I\right)\beta\beta'\left(R_k - I\right)'. \tag{20}$$

Now let us consider the following difference:

$$\Delta_2 = \text{MSEM}\left(\widehat{\beta}_{\text{SRRE}}\left(k\right)\right) - \text{MSEM}\left(\widehat{\beta}_{\text{SR}rk}\left(r,k\right)\right)$$

$$= \sigma^2 R_k A R_k + \left(R_k - I\right)\beta\beta'\left(R_k - I\right)'$$

$$- \left\{\sigma^2 T_r T_r' R_k A R_k T_r T_r'\right.$$

$$\left. + \left(T_r T_r' R_k - I\right)\beta\beta'\left(T_r T_r' R_k - I\right)'\right\}$$

$$= \sigma^2 D + b_1 b_1' - b_2 b_2', \tag{21}$$

where $D = R_k A R_k - T_r T_r' R_k A R_k T_r T_r'$, $b_1 = (R_k - I)\beta$ and $b_2 = (T_r T_r' R_k - I)\beta$.

Table 1: Total Research and Development Expenditures as a percent of Gross National Product by Country: 1972–1986.

Year	Y	X_1	X_2	X_3	X_4
1972	2.3	1.9	2.2	1.9	3.7
1975	2.2	1.8	2.2	2.0	3.8
1979	2.2	1.8	2.4	2.1	3.6
1980	2.3	1.8	2.4	2.2	3.8
1981	2.4	2.0	2.5	2.3	3.8
1982	2.5	2.1	2.6	2.4	3.7
1983	2.6	2.1	2.6	2.6	3.8
1984	2.6	2.2	2.6	2.6	4.0
1985	2.7	2.3	2.8	2.8	3.7
1986	2.7	2.3	2.7	2.8	3.8

Theorem 5. *When $k > 0$ and $\lambda_1\{(T_r T_r' R_k A R_k T_r T_r')(R_k A R_k)^{-1}\} \leq 1$, then the stochastic restricted r-k class estimator $\widehat{\beta}_{SRrk}(r,k)$ is superior to the stochastic restricted ridge estimator $\widehat{\beta}_{SRRE}(k)$ in the MSEM criterion if and only if $b_2'(\sigma^2 D + b_1 b_1')^+ b_2 \leq 1$, $b_2 \in \Re\ (\sigma^2 D + b_1 b_1')$.*

Proof. For $k > 0$, it is easy to see that $R_k = S_k^{-1} S > 0$ and $A = (S + R' W^{-1} R)^{-1} > 0$. So we can conclude that $R_k A R_k > 0$.

On the other hand, $T_r T_r' R_k A R_k T_r T_r'$ can be rewritten as

$$T_r T_r' R_k A R_k T_r T_r' = T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T' R_k A R_k \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T'. \quad (22)$$

From (22), it is easy to see that $T_r T_r' R_k A R_k T_r T_r' \geq 0$. Thus, by Lemma 2, we can obtain that $D \geq 0$ when $\lambda_1\{(T_r T_r' R_k A R_k T_r T_r')(R_k A R_k)^{-1}\} \leq 1$. So from (21) and applying Lemma 3, we have Δ_2 if and only if $b_2'(\sigma^2 D + b_1 b_1')^+ b_2 \leq 1$, $b_2 \in \Re(\sigma^2 D + b_1 b_1')$. This theorem is proved. □

3.3. MSEM Comparisons of the r-d Class Estimator and the Stochastic Restricted r-d (SRrd) Class Estimator.

In this subsection, we consider the MSEM comparison between the r-d class estimator and the stochastic restricted r-d (SRrd) class estimator.

Firstly, we can compute the bias vector and the variance of stochastic restricted r-d (SRrd) class estimator as follows:

$$\text{Bias}\left(\widehat{\beta}_{SRrd}(r,d)\right) = E\left(\widehat{\beta}_{SRrd}(r,d)\right) - \beta$$
$$= T_r T_r' F_d \beta - \beta = \left(T_r T_r' F_d - I\right)\beta, \quad (23)$$
$$\text{Cov}\left(\widehat{\beta}_{SRrd}(r,d)\right) = \sigma^2 T_r T_r' F_d A F_d T_r T_r'.$$

Therefore, we can obtain the MSEM of the stochastic restricted r-d class estimator as follows:

$$\text{MSEM}\left(\widehat{\beta}_{SRrd}(r,d)\right) = \sigma^2 T_r T_r' F_d A F_d T_r T_r'$$
$$+ \left(T_r T_r' F_d - I\right)\beta\beta'\left(T_r T_r' F_d - I\right)'. \quad (24)$$

From (6), we can compute the bias vector and the variance of the r-d class estimator as follows:

$$\text{Bias}\left(\widehat{\beta}_{rd}(r,d)\right) = E\left(\widehat{\beta}_{rd}(r,d)\right) - \beta$$
$$= T_r T_r' F_d \beta - \beta = \left(T_r T_r' F_d - I\right)\beta, \quad (25)$$
$$\text{Cov}\left(\widehat{\beta}_{rd}(r,d)\right) = \sigma^2 T_r T_r' F_d S^{-1} F_d T_r T_r'.$$

Then, we can obtain the MSEM of the r-d class estimator as follows:

$$\text{MSEM}\left(\widehat{\beta}_{rd}(r,d)\right) = \sigma^2 T_r T_r' F_d S^{-1} F_d T_r T_r'$$
$$+ \left(T_r T_r' F_d - I\right)\beta\beta'\left(T_r T_r' F_d - I\right)'. \quad (26)$$

Now we consider the following difference:

$$\Delta_3 = \text{MSEM}\left(\widehat{\beta}_{rd}(r,d)\right) - \text{MSEM}\left(\widehat{\beta}_{SRrd}(r,d)\right)$$
$$= \sigma^2 T_r T_r' F_d S^{-1} F_d T_r T_r' + \left(T_r T_r' F_d - I\right)\beta\beta'\left(T_r T_r' F_d - I\right)'$$
$$- \left\{\sigma^2 T_r T_r' F_d A F_d T_r T_r'\right.$$
$$\left. + \left(T_r T_r' F_d - I\right)\beta\beta'\left(T_r T_r' F_d - I\right)'\right\}$$
$$= \sigma^2 T_r T_r' F_d\left(S^{-1} - A\right)F_d T_r T_r'. \quad (27)$$

Theorem 6. *The stochastic restricted r-d class estimator $\widehat{\beta}_{SRrd}(r,d)$ always dominates the r-d class estimator $\widehat{\beta}_{rd}(r,d)$ in the MSEM criterion.*

Proof. By Lemma 1, we can obtain $A = (S + R' W^{-1} R)^{-1} = S^{-1} - S^{-1} R'(W + R S^{-1} R')^{-1} R S^{-1}$; then we obtain $S^{-1} - A = S^{-1} R'(W + R S^{-1} R')^{-1} R S^{-1} \geq 0$. Therefore, from (27), we may conclude that $\Delta_3 \geq 0$; that is to say, the stochastic restricted r-d class estimator $\widehat{\beta}_{SRrd}(r,d)$ always dominates the r-d class estimator $\widehat{\beta}_{rd}(r,d)$ in the MSEM criterion. □

3.4. MSEM Comparisons of the Stochastic Restricted Liu Estimator (SRLE) and the Stochastic Restricted r-d (SRrd) Class Estimator.

In this subsection, we consider the MSEM comparison between the stochastic restricted Liu estimator (SRLE) and the stochastic restricted r-d (SRrd) class estimator.

Firstly, from (10), we can compute the bias vector and the variance of stochastic restricted Liu estimator (SRLE) as follows:

$$\text{Bias}\left(\widehat{\beta}_{SRLE}(d)\right) = E\left(\widehat{\beta}_{SRLE}(d)\right) - \beta = \left(F_d - I\right)\beta,$$
$$\text{Cov}\left(\widehat{\beta}_{SRLE}(d)\right) = \sigma^2 F_d A F_d. \quad (28)$$

Therefore, we can obtain the MSEM of the stochastic restricted Liu estimator (SRLE) as follows:

$$\text{MSEM}\left(\widehat{\beta}_{SRLE}(d)\right) = \sigma^2 F_d A F_d + \left(F_d - I\right)\beta\beta'\left(F_d - I\right). \quad (29)$$

TABLE 2: Estimated MSE values of rk, SRRE, and SRrk.

	$k = 0.000$	$k = 0.001$	$k = 0.003$	$k = 0.005$	$k = 0.007$	$k = 0.1$	$k = 0.2$
SRRE	0.0445	0.0423	0.0391	0.0372	0.0363	0.1214	0.1653
rk	0.0621	0.0607	0.0588	0.0587	0.0574	0.1335	0.1726
SRrk	0.0331	0.0330	0.0334	0.0344	0.0358	0.1308	0.1716

TABLE 3: Estimated MSE values of rd, SRLE, and SRrd.

	$d = 0.60$	$d = 0.70$	$d = 0.80$	$d = 0.85$	$d = 0.9$	$d = 0.95$	$d = 1.0$
SRLE	0.0540	0.0435	0.0384	0.0379	0.0388	0.0409	0.0445
rd	0.0732	0.0633	0.0582	0.0574	0.0577	0.0593	0.0621
SRrd	0.0622	0.0486	0.0392	0.0362	0.0340	0.0331	0.0331

Now we consider the following difference:

$$\Delta_4 = \text{MSEM}\left(\widehat{\beta}_{\text{SRLE}}(d)\right) - \text{MSEM}\left(\widehat{\beta}_{\text{SR}rd}(r,d)\right)$$

$$= \sigma^2 F_d A F_d + (F_d - I)\beta\beta'(F_d - I)$$

$$- \left\{\sigma^2 T_r T_r' F_d A F_d T_r T_r' \right. \tag{30}$$

$$\left. + \left(T_r T_r' F_d - I\right)\beta\beta'\left(T_r T_r' F_d - I\right)'\right\}$$

$$= \sigma^2 H + b_3 b_3' - b_4 b_4',$$

where $H = F_d A F_d - T_r T_r' F_d A F_d T_r T_r'$, $b_3 = (F_d - I)\beta$ and $b_4 = (T_r T_r' F_d - I)\beta$.

Theorem 7. When $0 < d < 1$ and $\lambda_1\{(T_r T_r' F_d A F_d T_r T_r')(F_d A F_d)^{-1}\} \leq 1$, then the stochastic restricted r-d class estimator $\widehat{\beta}_{\text{SR}rd}(r, d)$ is superior to the stochastic restricted Liu estimator $\widehat{\beta}_{\text{SRLE}}(d)$ in the MSEM criterion if and only if $b_4'(\sigma^2 H + b_3 b_3')^+ b_4 \leq 1$, $b_4 \in \Re(\sigma^2 H + b_3 b_3')$.

Proof. For $0 < d < 1$, it is easy to see that $F_d = (S + I)^{-1}(S + dI) > 0$ and $A = (S + R'W^{-1}R)^{-1} > 0$. So we can conclude that $F_d A F_d > 0$.

On the other hand, $T_r T_r' F_d A F_d T_r T_r'$ can be rewritten as

$$T_r T_r' F_d A F_d T_r T_r' = T\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T' F_d A F_d \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T'. \tag{31}$$

From (31), it is easy to see that $T_r T_r' F_d A F_d T_r T_r' \geq 0$. Thus, by Lemma 2, we can obtain that $H \geq 0$ when $\lambda_1\{(T_r T_r' F_d A F_d T_r T_r')(F_d A F_d)^{-1}\} \leq 1$. So from (30) and applying Lemma 3, we have Δ_4 if and only if $b_4'(\sigma^2 H + b_3 b_3')^+ b_4 \leq 1$, $b_4 \in \Re(\sigma^2 H + b_3 b_3')$. This theorem is proved. $\square$

Remark. For practical use, we may replace the unknown parameters in the theorems with their appropriate estimators.

4. Numerical Example

In order to illustrate our theoretical results, we now consider in this section the data set on Total National Research and

Development Expenditure as a Percent of Gross National Product originally due to Gruber [18] and later considered by Akdeniz and Erol [19]. In this paper, we use the same data, which is presented in Table 1.

Firstly, we obtain the ordinary least squares estimator of β:

$$\widehat{\beta}_{\text{OLS}} = S^{-1}X'y = (0.6455, 0.0896, 0.1436, 0.1526)' \tag{32}$$

with $\text{MSE}(\widehat{\beta}_{\text{OLS}}) = 0.0808$ and $\widehat{\sigma}^2_{\text{OLS}} = 0.0015$. Consider the following stochastic linear restrictions:

$$r = R\beta + e, \quad R = (1, -2, -2, -2), \quad e \sim \left(0, \widehat{\sigma}^2_{\text{OLS}}\right). \tag{33}$$

For the r-k class estimator (rk), the stochastic restricted ridge estimator (SRRE), and the stochastic restricted r-k class estimator (SRrk), their estimated mean squared error values are given in Table 2. For the r-d class estimator (rd), the stochastic restricted Liu estimator (SRLE) and the stochastic restricted r-d class estimator (SRrd), their estimated mean squared error values are given in Table 3. Their estimated mean squared error values are got by replacing in the corresponding theoretical MSE expressions all unknown model parameters by their OLS estimator.

From Table 2, we can see that the stochastic restricted r-k class estimator is always better than the r-k class estimator. As fact, the stochastic restricted r-k class estimator has more information about the unknown parameter, so this estimator is better than the r-k class estimator. When k is small, then the stochastic restricted r-k class estimator is superior over the stochastic restricted ridge estimator. However, when k becomes big, then the stochastic restricted ridge estimator is superior over the stochastic restricted r-k class estimator.

From Table 3, we can find that stochastic restricted r-d class estimator is always better than the r-d class estimator. When d is big, then the stochastic restricted r-d class estimator is superior over the stochastic restricted Liu estimator. However, when d becomes smaller, then the stochastic restricted Liu estimator is superior over the stochastic restricted r-d class estimator.

5. Conclusion

In this paper, the stochastic restricted r-k class estimator (SRrk) and stochastic restricted r-d class estimator (SRrd) are

proposed and some properties of the two estimators have also been discussed. In particular, we show that the proposed SRrk and SRrd are proved to have smaller mean squared error than the rk, SRRE, rd, and SRLE, respectively.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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