Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 156353, 10 pages
http://dx.doi.org/10.1155/2014/156353

Research Article

Composition Operators in Hyperbolic Bloch-Type and $F(p,q,s)$ Spaces

Marko Kotilainen[1] and Fernando Pérez-González[2]

[1] *Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland*
[2] *Departamento de Análisis Matemático, Universidad de La Laguna, La Laguna, 38271 Tenerife, Spain*

Correspondence should be addressed to Fernando Pérez-González; fpergon@ull.es

Received 14 October 2013; Accepted 11 March 2014; Published 10 April 2014

Academic Editor: Marek Wisla

Composition operators C_φ from Bloch-type $\mathcal{B}_\alpha$ spaces to $F(p,q,s)$ classes, from $F(p,q,s)$ to $\mathcal{B}_\alpha$, and from $F(p_1,q_1,0)$ to $F(p_2,q_2,s_2)$ are considered. The criteria for these operators to be bounded or compact are given. Our study also includes the corresponding hyperbolic spaces.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$, and let $B(\mathbb{D})$ be the subset of $\mathcal{H}(\mathbb{D})$ consisting of those h for which $h(\mathbb{D}) \subset \mathbb{D}$. Every $\varphi \in B(\mathbb{D})$ induces the composition operator C_φ acting on $\mathcal{H}(\mathbb{D})$, defined by $C_\varphi(f) := f \circ \varphi$. By Littlewood's subordination principle any such composition operator maps every Hardy and Bergman space into itself. For the theory of composition operators in analytic function spaces see [1, 2]. Clearly every composition operator C_φ maps also $B(\mathbb{D})$ into itself. Hyperbolic classes are subsets of $B(\mathbb{D})$ and are defined by using the hyperbolic derivative $h^*(z) := |h'(z)|/(1-|h(z)|^2)$ of $h \in B(\mathbb{D})$. The hyperbolic derivative of the composition $h \circ \varphi$ satisfies the equality $(h\circ\varphi)^*(z) = h^*(\varphi(z))|\varphi'(z)|$ which can be understood as a kind of chain rule.

For $0 < \alpha < \infty$, the α-Bloch space $\mathcal{B}_\alpha$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_\alpha} := \sup_{z\in\mathbb{D}} \left|f'(z)\right| \left(1-|z|^2\right)^\alpha < \infty. \tag{1}$$

The closure of polynomials in $\mathcal{B}_\alpha$ is the little space α-Bloch $\mathcal{B}_{\alpha,0}$ which consists of those $f \in \mathcal{H}(\mathbb{D})$ for which $|f'(z)|(1-|z|^2)^\alpha \to 0$, as $|z| \to 1^-$. The spaces $\mathcal{B}_1$ and $\mathcal{B}_{1,0}$ are the classical Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$, respectively. For the theory of Bloch spaces, see the classical reference [3] and also [4, 5].

The hyperbolic α-Bloch classes $\mathcal{B}^*_\alpha$ and $\mathcal{B}^*_{\alpha,0}$ are the sets of those $h \in B(\mathbb{D})$ for which

$$\|h\|_{\mathcal{B}^*_\alpha} := \sup_{z\in\mathbb{D}} h^*(z)\left(1-|z|^2\right)^\alpha < \infty \tag{2}$$

and $\lim_{|z|\to 1} h^*(z)(1-|z|^2)^\alpha = 0$, respectively. In the special case $\alpha = 1$ it is simply denoted $\mathcal{B}^*_1 := \mathcal{B}^*$ and $\mathcal{B}^*_{1,0} := \mathcal{B}^*_0$. Clearly $\mathcal{B}^*_\alpha$ and $\mathcal{B}^*_{\alpha,0}$ are not linear spaces since the sum of two functions in $B(\mathbb{D})$ does not necessarily belong to $B(\mathbb{D})$. Moreover, the Schwarz-Pick lemma implies $\mathcal{B}^*_\alpha = B(\mathbb{D})$ for all $\alpha \geq 1$, and therefore the hyperbolic α-Bloch classes are only considered when $0 < \alpha \leq 1$.

Let Green's function of $\mathbb{D}$ be defined as $g(z,a) := -\log|\varphi_a(z)|$, where $\varphi_a(z) := (a-z)/(1-\overline{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the points zero and $a \in \mathbb{D}$. For any $a \in \mathbb{D}$, the automorphism φ_a is its own inverse and satisfies the fundamental equalities

$$\left|\varphi_a'(z)\right|\left(1-|z|^2\right) = \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\overline{a}z|^2} = 1-\left|\varphi_a(z)\right|^2 \tag{3}$$

which can be verified by straightforward calculations.

For $0 < p < \infty$, $-2 < q < \infty$, and $0 \le s < \infty$, the spaces $F(p,q,s)$ and $F_0(p,q,s)$ consist of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{F(p,q,s)} := \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \right)^{1/p} < \infty, \tag{4}$$

and

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) = 0, \quad 0 < s < \infty, \tag{5}$$

respectively. Here $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbb{D}$. The family $F(p,q,s)$ of function spaces, introduced by Zhao in [6], is known as the general family of function spaces. For $p \ge 1$, the space $F(p,q,s)$ is a Banach space with respect to the norm $\|f\|_{F(p,q,s)} + |f(0)|$, and so is $F_0(p,q,s)$ as a closed subspace of $F(p,q,s)$; see [6, Section 2]. When $0 < p < 1$, the space $F(p,q,s)$ is a complete metric space with the (invariant) metric defined by $d(f,g) = \|f-g\|^p_{F(p,q,s)} + |f(0)-g(0)|^p$. The metric is also p-homogeneous; that is, $d(\lambda f, 0) = |\lambda|^p d(f,0)$ for $\lambda \in \mathbb{C}$, and therefore the space $F(p,q,s)$ is a quasi-Banach space for $0 < p < 1$. If $q+s \le -1$, then the space $F(p,q,s)$ reduces to the space of constant functions by [6, Proposition 2.12]. Therefore from now on it is always assumed that the parameters p, q, and s of the spaces $F(p,q,s)$ or $F_0(p,q,s)$ in question satisfy $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, and $q + s > -1$ without mentioning it every time. Many classical function spaces can be found among the family $F(p,q,s)$ by choosing the parameters appropriately. In order to connect the results of the present paper to the ones in the existing literature some information on this matter is gathered in Table 1. For example, A^p stands for the classical Bergman space and $\mathscr{D}_q$ denotes the weighted Dirichlet space. The interested reader is invited to see [6, 7] for more information and the definitions of the spaces.

The class $F^*(p,q,s)$ is defined as the set of those $h \in B(\mathbb{D})$ for which

$$\|h\|_{F^*(p,q,s)} := \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(h^*(z)\right)^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \right)^{1/p} < \infty. \tag{6}$$

Similarly, $h \in F_0^*(p,q,s)$, if (5) with $|f'|$ replaced by h^*, is satisfied. It is sometimes convenient to set $q = \alpha p - 2$, where $\alpha > 0$. By Schwarz-Pick lemma $F_0^*(p,\alpha p-2,s) = B(\mathbb{D})$, if $\alpha > 1$, and hence the classes $F^*(p,\alpha p-2,s)$ and $F_0^*(p,\alpha p-2,s)$ are considered only when $\alpha \le 1$.

A composition operator $C_\varphi : \mathscr{B}^*_\alpha \to F^*(p,q,s)$ is said to be bounded if there exists a positive constant C such that

$$\left\|C_\varphi(h)\right\|_{F^*(p,q,s)} \le C\|h\|_{\mathscr{B}^*_\alpha} \tag{7}$$

Table 1: General function family $F(p,q,s)$ and the classical function spaces.

p	q	s	$F(p,q,s)$	$F_0(p,q,s)$
$(0,\infty)$	$(-2,\infty)$	$(1,\infty)$	$\mathscr{B}^{(q+2)/p}$	$\mathscr{B}_0^{(q+2)/p}$
2	0	$(0,\infty)$	Q_s	$Q_{s,0}$
2	0	1	BMOA	VMOA
2	1	0	H^2	H^2
$[1,\infty)$	p	0	A^p	A^p
2	$(-1,\infty)$	0	$\mathscr{D}_q$	$\mathscr{D}_q$
$(1,\infty)$	$p-2$	0	B_p	B_p

for all $h \in \mathscr{B}^*_\alpha$. Further, $C_\varphi : \mathscr{B}^*_{\alpha,0} \to F^*(p,q,s)$ is said to be bounded if (7) is satisfied for all $h \in \mathscr{B}^*_{\alpha,0}$. On the other hand, $C_\varphi : \mathscr{B}^*_\alpha \to F_0^*(p,q,s)$ is said to be bounded if (7) is satisfied for all $h \in \mathscr{B}^*_\alpha$ and $C_\varphi(\mathscr{B}^*_\alpha) \subset F_0^*(p,q,s)$. Hereafter a bounded operator C_φ mapping from one hyperbolic class into another is understood in an analogous manner.

The purpose of this paper is, on one hand, to complete in part certain results in the existing literature and, on the other hand, to continue the line of research of [8] on composition operators in hyperbolic function classes. The spaces/classes of interest in this work are α-Bloch spaces, Dirichlet-type spaces, and $F(p,q,s)$-spaces as well as their hyperbolic counterparts. The boundedness of the composition operator is discussed in several different cases, using the standard tools such as the change of variable formula by Stanton and different kind of characterizations of Carleson measures.

The remainder of this paper is organized as follows. In Section 2, the main results are presented together with necessary definitions. In Section 3 some auxiliary results on hyperbolic classes are given and the necessary background material involving Carleson measures and Nevanlinna counting function is introduced. Sections 4–11 contain the proofs of the main results in chronological order.

2. Main Results

Bounded and compact composition operators mapping from $\mathscr{B}_\alpha$ or $\mathscr{B}_{\alpha,0}$ into $F(p,q,s)$ or $F_0(p,q,s)$ have been studied in many particular cases in [9, 10]; see also [11–15], the most general result being found in [16]. The first result in the present paper extends in part these results on bounded composition operators to the corresponding hyperbolic classes under certain conditions on the parameters.

Theorem 1. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, and $\varphi \in B(\mathbb{D})$. Then the following statements are equivalent:*

(1) $C_\varphi : \mathscr{B}_\alpha \to F(p,q,s)$ *is bounded;*

(2) $C_\varphi : \mathscr{B}_{\alpha,0} \to F(p,q,s)$ *is bounded;*

(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (|\varphi'(z)|^p/(1-|\varphi(z)|^2)^{p\alpha})(1-|z|^2)^q g^s(z,a)\,dA(z) < \infty$.

Moreover, if $0 < \alpha \le 1$, *then (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ *is bounded;*

(5) $C_\varphi : \mathcal{B}^*_{\alpha,0} \to F^*(p,q,s)$ *is bounded.*

Theorem 2. *Let* $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_\alpha \to F_0(p,q,s)$ *is bounded;*

(2) $C_\varphi : \mathcal{B}_\alpha \to F_0(p,q,s)$ *is compact;*

(3) $\lim_{|a|\to 1} \int_{\mathbb{D}} (|\varphi'(z)|^p/(1-|\varphi(z)|^2)^{p\alpha})(1-|z|^2)^q g^s(z,a)\, dA(z) = 0$.

In case $0 < \alpha \le 1$ *the conditions (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_\alpha \to F^*_0(p,q,s)$ *is bounded.*

Theorem 3. *Let* $0 < \alpha < \infty$, $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_{\alpha,0} \to F_0(p,q,s)$ *is bounded;*

(2) $\varphi \in F_0(p,q,s)$ *and*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \left(1-|z|^2\right)^q g^s(z,a)\, dA(z) < \infty. \tag{8}$$

Moreover, if $0 < \alpha \le 1$, *then (1) and (2) are equivalent to*

(3) $C_\varphi : \mathcal{B}^*_{\alpha,0} \to F^*_0(p,q,s)$ *is bounded.*

It is now proceeded to study the case when the target space is $\mathcal{B}_\beta$ or $\mathcal{B}_{\beta,0}$. The following result should be compared with [8, Theorem 1.3] and [10, Theorem 2.2.1(iii)].

Theorem 4. *Let* $0 < \alpha, \beta < \infty$, $0 < p < \infty$, $0 < s \le 1$, *and* $\varphi \in B(\mathbb{D})$. *Then, the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ *is bounded;*

(2) $C_\varphi : F(p,\alpha p-2,s) \to \mathcal{B}_\beta$ *is bounded;*

(3) $\sup_{z\in\mathbb{D}} (|\varphi'(z)|/(1-|\varphi(z)|^2)^\alpha)(1-|z|^2)^\beta < \infty$.

Moreover, if $0 < \alpha, \beta \le 1$, *then (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_\alpha \to \mathcal{B}^*_\beta$ *is bounded;*

(5) $C_\varphi : F^*(p,\alpha p-2,s) \to \mathcal{B}^*_\beta$ *is bounded.*

Since the test functions used in the proof of Theorem 4 are included in the little versions of the domain spaces, the next corollary follows.

Corollary 5. *Let* $0 < \alpha, \beta < \infty$, $0 < p < \infty$, $0 < s \le 1$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_{\alpha,0} \to \mathcal{B}_\beta$ *is bounded;*

(2) $C_\varphi : F_0(p,\alpha p-2,s) \to \mathcal{B}_\beta$ *is bounded;*

(3) $\sup_{z\in\mathbb{D}} (|\varphi'(z)|/(1-|\varphi(z)|^2)^\alpha)(1-|z|^2)^\beta < \infty$.

Moreover, if $0 < \alpha, \beta \le 1$, *then (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_{\alpha,0} \to \mathcal{B}^*_\beta$ *is bounded;*

(5) $C_\varphi : F^*_0(p,\alpha p-2,s) \to \mathcal{B}^*_\beta$ *is bounded.*

Theorem 6. *Let* $0 < \alpha, \beta < \infty$, $0 < s \le 1$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_\alpha \to \mathcal{B}_{\beta,0}$ *is bounded;*

(2) $C_\varphi : F(p,\alpha p-2,s) \to \mathcal{B}_{\beta,0}$ *is bounded;*

(3) $\lim_{|z|\to 1} (|\varphi'(z)|/(1-|\varphi(z)|^2)^\alpha)(1-|z|^2)^\beta = 0$.

Moreover, if $0 < \alpha, \beta \le 1$, *then (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_\alpha \to \mathcal{B}^*_{\beta,0}$ *is bounded;*

(5) $C_\varphi : F^*(p,\alpha p-2,s) \to \mathcal{B}^*_{\beta,0}$ *is bounded.*

Theorem 7. *Let* $0 < \alpha, \beta < \infty$, $0 < s \le 1$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_{\alpha,0} \to \mathcal{B}_{\beta,0}$ *is bounded;*

(2) $C_\varphi : F_0(p,\alpha p-2,s) \to \mathcal{B}_{\beta,0}$ *is bounded;*

(3) $\varphi \in \mathcal{B}_{\beta,0}$ *and*

$$\sup_{z\in\mathbb{D}} \frac{\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^\alpha} \left(1-|z|^2\right)^\beta < \infty. \tag{9}$$

Moreover, if $0 < \alpha, \beta \le 1$, *then (1)–(3) are equivalent to*

(4) $C_\varphi : \mathcal{B}^*_{\alpha,0} \to \mathcal{B}^*_{\beta,0}$ *is bounded;*

(5) $C_\varphi : F^*_0(p,\alpha p-2,s) \to \mathcal{B}^*_{\beta,0}$ *is bounded.*

If the domain and the target class both are some $F^*(p,q,s)$ classes with $0 < s \le 1$, then the situation seems to be more complicated. The following result characterizes bounded composition operators mapping from $F^*(p_1,q_1,0)$ into $F^*(p_2,q_2,s_2)$ when $p_2 \ge p_1$. Note that in the hyperbolic case the condition $q_1 + 2 \le p_1$ is indeed needed in the proof while the condition $q_2 + s_2 + 1 \le p_2$ only guarantees that the target class is not the whole class $B(\mathbb{D}) = \mathcal{B}^*$.

Theorem 8. *Let* $0 < p_1 \le p_2 < \infty$, $-2 < q_1, q_2 < \infty$, $0 < s_2 \le 1$, *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : F(p_1,q_1,0) \to F(p_2,q_2,s_2)$ *is bounded;*

(2) $\sup_{a,b\in\mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(\varphi(z))|^{(p_2/p_1)(q_1+2)} |\varphi'(z)|^{p_2} (1-|z|^2)^{q_2} g^{s_2}(z,b)\, dA(z) < \infty$.

Moreover, if $q_1 + 2 \le p_1$ *and* $q_2 + s_2 + 1 \le p_2$, *then (1) and (2) are equivalent to*

(3) $C_\varphi : F^*(p_1,q_1,0) \to F^*(p_2,q_2,s_2)$ *is bounded.*

Remark 9. The condition (1) in Theorem 8 is a special case of (2) in Theorem 4. If $s_2 > 1$, then $F(p_2,\alpha p_2-2,s_2) = \mathcal{B}_\alpha$ by Lemma 12, and Theorem 8 implies that $C_\varphi : F(p,q,0) \to \mathcal{B}_\alpha$ is bounded if and only if

$$\sup_{a,b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi'_a(\varphi(z))\right|^{q+2} \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^{\alpha p-2} \times g^s(z,b)\, dA(z) < \infty, \quad 1 < s < \infty. \tag{10}$$

However, a straightforward calculation shows that (10) is satisfied if and only if

$$\sup_{z\in\mathbb{D}} \frac{\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(q+2)/p}}\left(1-|z|^2\right)^{\alpha} < \infty. \tag{11}$$

Thus, by taking $p = 2$ and $q = 0$, we see that [10, Theorem 2.2.1(iii)] remains valid also when the domain space is the classical Dirichlet space $\mathcal{D} = F(2, 0, 0)$.

Theorem 10. *Let $p_2 \geq p_1$ and $\varphi \in B(\mathbb{D})$. Then the following conditions are equivalent:*

(1) *$C_\varphi : F(p_1, q_1, 0) \to F_0(p_2, q_2, s_2)$ is bounded (compact);*

(2) *$C_\varphi : F(p_1, q_1, 0) \to F(p_2, q_2, s_2)$ is bounded (compact) and $\varphi \in F_0(p_2, q_2, s_2)$.*

Function $\varphi : \mathbb{D} \to \mathbb{D}$ belongs to $F_0(p_2, q_2, s_2)$ if and only if

$$\lim_{|b|\to 1}\sup_{|a|\leq r}\int_{\mathbb{D}} \left|\varphi_a'(\varphi(z))\right|^{\gamma}\left|\varphi'(z)\right|^{p_2} \times\left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(z) = 0 \tag{12}$$

for every $0 \leq r < 1$ and $\gamma > 0$. In particular, this holds for $\gamma = p_2(q_1 + 2)/p_1$. In this sense, Theorem 10 is related to [9, Theorem 5.2].

Theorem 11. *Let $p_2 \geq p_1$, $q_1 + 2 \leq p_1$, $q_2 + s_2 + 1 \leq p_2$, and $\varphi \in B(\mathbb{D})$. Then the following conditions are equivalent:*

(1) *$C_\varphi : F^*(p_1, q_1, 0) \to F_0^*(p_2, q_2, s_2)$ is bounded;*

(2) *$C_\varphi : F^*(p_1, q_1, 0) \to F^*(p_2, q_2, s_2)$ is bounded and $\varphi \in F_0(p_2, q_2, s_2)$.*

For suitable choice of parameters the nonhyperbolic cases of Theorems 8–11 reduce to [9, Theorem 5.2]. The reasoning there, however, is slightly different.

3. Auxiliary Results and Background Material

Some basic properties of the hyperbolic classes $F^*(p, p\alpha-2, s)$ are gathered in the following lemma.

Lemma 12. *Let $h \in B(\mathbb{D})$ and $0 \leq s < \infty$. Then the following assertions hold:*

(1) *$\|h\|_{\mathcal{B}_\alpha^*} \leq C\|h\|_{F^*(p,\alpha p-2,s)}$, where C is a positive constant independent of h;*

(2) *$\|h\|_{\mathcal{B}_\alpha^*} \simeq \|h\|_{F^*(p,\alpha p-2,s)}$ if $1 < s < \infty$;*

(3) *$\|h\|_{F^*(p,\alpha p-2,s)} \simeq (\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(h^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|)^s dA(z))^{1/p}$.*

Lemma 12 can be proved in a similar manner as the corresponding results for the spaces $F(p, p\alpha-2, s)$; see [6, 17] for details. Note that (1) implies the inclusion $F^*(p, \alpha p - 2, s) \subset \mathcal{B}_\alpha^*$, $0 \leq s < \infty$.

A positive Borel measure μ on $\mathbb{D}$ is said to be a *bounded s-Carleson measure*, if

$$\sup_I \frac{\mu(S(I))}{|I|^s} < \infty, \quad 0 < s < \infty, \tag{13}$$

where $|I|$ denotes the arc length of a subarc I of $\mathbb{T}$;

$$S(I) = \left\{z \in \mathbb{D} : \frac{z}{|z|} \in I, 1-|I| \leq |z|\right\} \tag{14}$$

is the Carleson box based on I, and the supremum is taken over all subarcs I of $\mathbb{T}$ such that $|I| \leq 1$. Moreover, if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0, \quad 0 < s < \infty, \tag{15}$$

then μ is said to be a *compact s-Carleson measure*. If $s = 1$, then a bounded (resp., compact) 1-Carleson measure is just a standard bounded (resp., compact) Carleson measure.

For $a \in \mathbb{D}$ and $0 < r < 1$, let the pseudohyperbolic disc be defined by $D(a, r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$. The pseudohyperbolic disc $D(a, r)$ is an Euclidean disc centered at $(1-r^2)a/(1-|a|^2r^2)$ with radius $(1-|a|^2)r/(1-|a|^2r^2)$; see [18, page 3].

In the following lemma we have gathered some well-known and useful characterizations of bounded s-Carleson measures. For the proof, see [19, Theorem 13], [20, Lemma 2.1], [21, pp. 89–90], and [22, Proposition 2.1].

Lemma A. *Let μ be a positive Borel measure on $\mathbb{D}$, $1 < s < \infty$, $0 < r < 1$ and $0 < \tau < \infty$. Then the following statements are equivalent:*

(1) *$K_1 := \sup_I(\mu(S(I))/|I|^s) < \infty$;*

(2) *$K_2 := \sup_{z\in\mathbb{D}}(\mu(D(z,r))/(1-|z|^2)^s) < \infty$;*

(3) *$K_3 := \sup_{z\in\mathbb{D}}\int((1-|z|^2)^\tau/|1-\overline{z}w|^{1+\tau})^s\, d\mu(w) < \infty$.*

Moreover, the expressions K_1, K_2, and K_3 are comparable.

Another auxiliary result needed is Luecking's [21] characterization of Carleson measures in terms of functions in the weighted Bergman spaces.

Theorem B. *Let μ be a positive measure on $\mathbb{D}$, and let $0 < p \leq q < \infty$. Then μ is a bounded $(q/p)(2+\alpha)$-Carleson measure if and only if there is a positive constant C, depending only on p and q, such that*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C\|f\|_{A_\alpha^p}^q \tag{16}$$

for all analytic functions f in $\mathbb{D}$, in particular for all $f \in A_\alpha^p$, the standard weighted Bergman space. Moreover, if μ is a bounded $(q/p)(2+\alpha)$-Carleson measure, then $C = C_1C_2$, where $C_1 > 0$ depends only on p, q and α, and

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^{(q/p)(2+\alpha)}}. \tag{17}$$

The following change of variables formula by Stanton, [23, 24], was apparently first used by Shapiro [25] in the study of composition operators. It also plays a key role in some of our proofs.

Lemma C. *Let g and u be positive measurable functions on $\mathbb{D}$, and let $\varphi \in B(\mathbb{D})$. Then*

$$\int_{\mathbb{D}} (g \circ \varphi)(z) \left|\varphi'(z)\right|^2 u(z)\, dA(z) = \int_{\mathbb{D}} g(w)\, U(\varphi, w)\, dA(w), \tag{18}$$

where

$$U(\varphi, w) = \sum_{z \in \varphi^{-1}\{w\}} u(z), \quad w \in \mathbb{D} \setminus \{\varphi(0)\}. \tag{19}$$

If $u(z) = (-\log|z|)^s$, then $U(\varphi, w)$ is the generalized Nevanlinna counting function

$$N_{\varphi,s}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|}\right)^s. \tag{20}$$

For the study of compactness we need the following well-known result; see [1, Proposition 3.11] for a similar result. The following can be deduced by a result of Tjani; see [26].

Lemma D. *Let $\varphi \in B(\mathbb{D})$. Then $C_\varphi : F(p_1, q_1, s_1) \to F(p_2, q_2, s_2)$ is compact if and only if for any bounded sequence $\{f_n\}$ in $F(p_1, q_1, s_1)$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\|f_n \circ \varphi\|_{F(p_2,q_2,s_2)} \to 0$ as $n \to \infty$.*

4. Proof of Theorem 1

It will be shown first that the conditions (3), (4), and (5) are equivalent by proving the implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3). Since (4) clearly implies (5), (see Lemma 12), it suffices to prove the other two implications.

4.1. Proof of (3) $\Rightarrow$ (4). Let $0 < \alpha \le 1$ and $h \in \mathcal{B}^*_\alpha$; that is, $\|h\|_{\mathcal{B}^*_\alpha} = \sup_{z\in\mathbb{D}} h^*(z)(1-|z|^2)^\alpha < \infty$. Then

$$\begin{aligned}\|h \circ \varphi\|^p_{F^*(p,q,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left((h\circ\varphi)^*(z)\right)^p \left(1-|z|^2\right)^q g^s(z,a)\, dA(z) \\ &\le \|h\|^p_{\mathcal{B}^*_\alpha} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \times \left(1-|z|^2\right) g^s(z,a)\, dA(z),\end{aligned} \tag{21}$$

and therefore $C_\varphi : \mathcal{B}^*_\alpha \to F^*(p,q,s)$ is bounded if (3) is satisfied.

4.2. Proof of (5) $\Rightarrow$ (3). Let first $\alpha = 1$. Suppose that $C_\varphi : \mathcal{B}^*_0 \to F^*(p,q,s)$ is bounded, and define $h_b(z) := bz$ for $b \in \mathbb{D}$. Then $h^*_b(z) = |b|(1-|bz|^2)^{-1}$ and therefore $h_b \in \mathcal{B}^*_0$ for all $b \in \mathbb{D}$. Since $C_\varphi : \mathcal{B}^*_0 \to F^*(p,q,s)$ is bounded, there is a positive constant C such that

$$\begin{aligned}\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|b|^p \left|\varphi'(z)\right|^p}{\left(1-\left|b\varphi(z)\right|^2\right)^p} \left(1-|z|^2\right)^q g^s(z,a)\, dA(z) \\ \le C\|h_b\|^p_{\mathcal{B}^*} \le C|b|^p.\end{aligned} \tag{22}$$

Taking limit as $|b| \to 1^-$, Fatou's lemma yields (3) with $\alpha = 1$.

If $0 < \alpha < 1$, by [10, Theorem 2.1.1], there are functions f_1 and f_2 in $\mathcal{B}_\alpha$, such that $|f_1'(z)| + |f_2'(z)| \simeq (1-|z|)^{-\alpha}$. Since

$$\begin{aligned}\left|f_i(z)\right| &\le \int_0^{|z|} \left|f_i'(w)\right| |dw| \\ &\le \|f_i\|_{\mathcal{B}^\alpha} \int_0^1 (1-r)^{-\alpha}\, dr = \frac{\|f_i\|_{\mathcal{B}^\alpha}}{1-\alpha}, \quad i = 1, 2.\end{aligned} \tag{23}$$

Hence the functions f_i are bounded, and $h_i(z) = f_i(z)/2\|f_i\|_{H^\infty}$, $i = 1, 2$, satisfy $\|h_i\|_{H^\infty} \le 1/2$, $i = 1, 2$. Therefore $h^*_i(z) \simeq |h_i'(z)|$ in $\mathbb{D}$ for $i = 1, 2$ and also $h^*_1(z) + h^*_2(z) \simeq (1-|z|)^{-\alpha}$. Applying the assumption that $C_\varphi : \mathcal{B}^*_0 \to F^*(p,q,s)$ is bounded for the functions h_1 and h_2, and using the asymptotic inequalities

$$\begin{aligned}\left((h_1\circ\varphi)^*(z)\right)^p + \left((h_2\circ\varphi)^*(z)\right)^p &\simeq \left|\varphi'(z)\right|^p \left(h^*_1(\varphi(z)) + h^*_2(\varphi(z))\right)^p \\ &\simeq \left|\varphi'(z)\right|^p \left(1-\left|\varphi(z)\right|^2\right)^{-\alpha p},\end{aligned} \tag{24}$$

the condition (3) with $0 < \alpha < 1$ follows.

4.3. The Rest of the Assertions. It was shown in [16, Theorems 1.1 and 1.4] that (1), (2) and (3) are equivalent for $p \ge 2$. In fact, He and Jiang required this restriction just to see that (2) implies (3). Now, we prove that such implication holds for *any* $p > 0$. Suppose that $C_{\varphi p} : \mathcal{B}_{\alpha,0} \to F(p,q,s)$ is bounded. Let $\{\beta_n\}$ be a sequence in $\mathbb{D}$ such that $|b_n| \to 1$, and consider the functions

$$f_n(z) = \frac{1}{|b_n|} \sum_{k=0}^{\infty} 2^{k(\alpha-1)} (b_n z)^{2^k}, \quad n = 1, 2, \ldots. \tag{25}$$

By [27, Theorem 1], each $f_n \in \mathcal{B}_{\alpha,0}$ and there is a constant $K > 0$ such that $\sup_n \|f_n\|_{\mathcal{B}_\alpha} \le K$. Since C_φ is bounded,

$$\begin{aligned}K^p \left\|C_\varphi\right\|^p &\ge \left\|C_\varphi\right\|^p \|f_n\|^p_{B_\alpha} \\ &\ge \int_{\mathbb{D}} \left|f_n'(\varphi(z))\right|^p \left|\varphi'(z)\right|^p \times \left(1-|z|^2\right)^q g^s(z,a)\, dA(z),\end{aligned} \tag{26}$$

for any $a \in \mathbb{D}$. Let now $f_{n,\theta}(z) := f_n(e^{i\theta}z)$. Then

$$K^p\left\|C_\varphi\right\|^p \geq \int_{\mathbb{D}} \left|f_n'\left(e^{i\theta}\varphi(z)\right)\right|^p \left|\varphi'(z)\right|^p \times \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \tag{27}$$

for any $a \in \mathbb{D}$ and $n = 1, 2, \ldots$. An integration with respect to θ using Fubini theorem and a Zygmund's result on gap series (see Theorem 8.20 on page 215 of Volume I of [28]) we get

$$\begin{aligned} K^p\left\|C_\varphi\right\|^p &\geq \frac{1}{2\pi}\int_{\mathbb{D}}\int_0^{2\pi} \left|f_n'\left(e^{i\theta}\varphi(z)\right)\right|^p \left|\varphi'(z)\right|^p \times \left(1-|z|^2\right)^q g^s(z,a)\,d\theta\,dA(z) \\ &= \frac{1}{2\pi}\int_{\mathbb{D}}\int_0^{2\pi} \left|\sum_{k=1}^{\infty} 2^{\alpha k}\left(b_n\varphi(z)\right)^{2k-1} e^{i(2^k-1)\theta}\right|^p \times \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\,d\theta\,dA(z) \\ &\asymp \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} 2^{2\alpha k}\left|d_n\varphi(z)\right|^{2(2^k-1)}\right)^{p/2} \times \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z). \end{aligned} \tag{28}$$

Since $\sum_{k=1}^{\infty} 2^{2\alpha k} r^{2^{k+1}} > C(\alpha)/(1-r^2)^{2\alpha}$ for any $r \in (0,1)$ we have

$$\int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|b_n\varphi(z)\right|^2\right)^{\alpha p}} \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \leq C, \tag{29}$$

C being a constant independent of neither a nor n. An application of Fatou's lemma in the above inequality yields (3).

5. Proof of Theorem 2

The equivalence of (1), (2), and (3) is proved in [16]. Hence it remains to show that these together are equivalent to (4).

5.1. The Necessity of (4). Let $0 < \alpha \leq 1$. It follows from (1) that $C_\varphi : \mathcal{B}_\alpha \to F(p,q,s)$ is bounded and hence $C_\varphi : \mathcal{B}_\alpha^* \to F^*(p,q,s)$ is bounded by Theorem 1. Furthermore, (3) implies that

$$\begin{aligned} &\lim_{|a|\to 1}\int_{\mathbb{D}} \left(\left(f\circ\varphi\right)^*(z)\right)^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \\ &\leq \|f\|_{\mathcal{B}_\alpha^*}^p \lim_{|a|\to 1}\int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \times \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) = 0 \end{aligned} \tag{30}$$

for every $f \in \mathcal{B}_\alpha^*$; that is, $C_\varphi(\mathcal{B}_\alpha^*) \subset F_0^*(p,q,s)$.

5.2. The Sufficiency of (4). Let $C_\varphi : \mathcal{B}_\alpha^* \to F_0^*(p,q,s)$ be bounded. Choosing h_1 and h_2 as in the proof of Theorem 1 one obtains

$$\begin{aligned} &\lim_{|a|\to 1}\int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \left(1-|z|^2\right) g^s(z,a)\,dA(z) \\ &\asymp \lim_{|a|\to 1}\int_{\mathbb{D}} \sum_{j=1}^{2}\left(\left(h_j\circ\varphi\right)^*(z)\right)^p \times \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) = 0. \end{aligned} \tag{31}$$

Thus (3) holds, and the proof is complete.

6. Proof of Theorem 3

Since (1) and (2) are equivalent by [16], it remains to prove the necessity and the sufficiency of (3).

6.1. The Necessity of (3). By (1) the operator $C_\varphi : \mathcal{B}_{\alpha,0} \to F(p,q,s)$ is bounded, which implies by Theorem 1 that $C_\varphi : \mathcal{B}_{\alpha,0}^* \to F^*(p,q,s)$ is bounded and

$$M := \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \times \left(1-|z|^2\right) g^s(z,a)\,dA(z) < \infty. \tag{32}$$

For every $f \in \mathcal{B}_{\alpha,0}^*$ and $\varepsilon > 0$ there exists $0 < r < 1$ such that

$$\frac{\left|f'(z)\right|}{1-\left|f(z)\right|^2}\left(1-|z|^2\right)^\alpha < \varepsilon \tag{33}$$

when $|z| > r$. For this fixed r,

$$\begin{aligned} &\int_{\mathbb{D}} \left(\left(f\circ\varphi\right)^*(z)\right)^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \\ &= \left(\int_{\{|\varphi(z)|>r\}} + \int_{\{|\varphi(z)|\leq r\}}\right)\left(\left(f\circ\varphi\right)^*(z)\right)^p \times \left(1-|z|^2\right)^q g^s(z,a)\,dA(z) \\ &\leq \varepsilon^p \int_{\{|\varphi(z)|>r\}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \left(1-|z|^2\right) g^s(z,a)\,dA(z) \\ &\quad + \sup_{|w|\leq r}\frac{\left|f'(w)\right|^p}{\left(1-\left|f(w)\right|^2\right)^p} \times \int_{\{|\varphi(z)|\leq r\}} \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\,dA(z), \end{aligned}$$

$$\le \varepsilon^p M + \sup_{|w|\le r} \frac{\left|f'(w)\right|^p}{\left(1-\left|f(w)\right|^2\right)^p} \times \int_{\mathbb{D}} \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\, dA(z). \tag{34}$$

By (2), the right hand side tends to zero as $|a| \to 1-$. Hence $C_\varphi(f) \in F_0^*(p,q,s)$ for every $f \in \mathcal{B}_{\alpha,0}^*$.

6.2. The Sufficiency of (3). It is enough to show that (3) implies (2). The condition (3) implies that $C_\varphi : \mathcal{B}_{\alpha,0}^* \to F^*(p,q,s)$ is bounded, whence by Theorem 1

$$\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>r\}} \frac{\left|\varphi'(z)\right|^p}{\left(1-\left|\varphi(z)\right|^2\right)^{p\alpha}} \times \left(1-|z|^2\right) g^s(z,a)\, dA(z) < \infty. \tag{35}$$

Furthermore, since the function $h(z) = z/2$ belongs to $\mathcal{B}_{\alpha,0}^*$, one obtains

$$\lim_{|a|\to 1-} \int_{\mathbb{D}} \left|\varphi'(z)\right|^p \left(1-|z|^2\right)^q g^s(z,a)\, dA(z) \le 2^p \lim_{|a|\to 1-} \int_{\mathbb{D}} \left((h\circ\varphi)^*(z)\right)^p \times \left(1-|z|^2\right)^q g^s(z,a)\, dA(z) = 0 \tag{36}$$

and (2) is satisfied.

7. Proof of Theorem 4

It will be shown that the conditions (3), (4), and (5) are equivalent by proving the implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3). Since (4) clearly implies (5) by Lemma 12, it suffices to prove the other two implications. The equivalence of (1), (2), and (3) is proved for example, in [29]; see Corollaries 2.10 and 2.12.

7.1. Proof of (3) $\Rightarrow$ (4). If $h \in \mathcal{B}_\alpha^*$, then

$$(h\circ\varphi)^*(z)\left(1-|z|^2\right)^\beta \le \|h\|_{\mathcal{B}_\alpha^*} \frac{\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^\alpha}\left(1-|z|^2\right)^\beta, \tag{37}$$

and it follows that (3) implies (4).

7.2. Proof of (5) $\Rightarrow$ (3). Suppose that $C_\varphi : F^*(p,\alpha p-2,s) \to \mathcal{B}_\beta^*$ is bounded, and define $\phi_{\gamma,a}(z) := 1-(1-\overline{a}z)^\gamma$, where $0<\gamma<1$ and $a \in \mathbb{D}$. By the assumption there exists a positive constant C_1 such that

$$\frac{\gamma|a|}{2}\frac{\left|\varphi'(z)\right|}{\left|1-\overline{a}\varphi(z)\right|}\left(1-|z|^2\right)^\beta \le \left(\phi_{\gamma,a\circ\varphi}\right)^*(z)\left(1-|z|^2\right)^\beta \le \varepsilon\left\|\phi_{\gamma,a}\right\|_{F^*(p,\alpha p-2,s)}^p \tag{38}$$

for all $a \in \mathbb{D}$. By Lemma 12,

$$\left\|\phi_{\gamma,a}\right\|_{F^*(p,\alpha p-2,s)}^p \simeq \gamma^p|\overline{a}|^p \sup_{b\in\mathbb{D}}\left(1-|b|^2\right)^s \times \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{\alpha p-2+s}}{|1-\overline{a}z|^p\left|1-\overline{b}z\right|^{2s}}\, dA(z), \tag{39}$$

and an application of [30, Lemma 2.5] shows that there exists a positive constant C_2 such that $\|\phi_{\gamma,a}\|_{F^*(p,\alpha p-2,s)} \le C_2\gamma|a|(1-|a|^2)^{\alpha-1}$ for all $a \in \mathbb{D}$ and $0<\gamma<1$ if $s \notin \{\alpha p, p(1-\alpha)\}$. But since the right hand side of (39) is a decreasing function of $s>0$, it follows that $\|\phi_{\gamma,a}\|_{F^*(p,\alpha p-2,s)} \le C_2\gamma|a|(1-|a|^2)^{\alpha-1}$ for all $0<s\le 1$, $a \in \mathbb{D}$ and $0<\gamma<1$. This together with (38) yields

$$\frac{\left|\varphi'(z)\right|}{\left|1-\overline{a}\varphi(z)\right|}\left(1-|z|^2\right)^\beta \le 2C_1C_2\left(1-|a|^2\right)^{\alpha-1} \tag{40}$$

for all $a \in \mathbb{D}$, and the condition (3) follows by choosing $a = \varphi(z)$.

It is worth noticing that the implication (5) $\Rightarrow$ (3) in the case $0<\alpha<1$ can also be proved by using the functions $\phi_a(z) := 2^{\alpha-2}(1-\overline{a}z)^{1-\alpha}$ for which $|\phi_a(z)| < 1/2$, $z \in \mathbb{D}$, and $\|\phi_a\|_{F^*(p,\alpha p-2,s)} \simeq \|\phi_a\|_{F(p,\alpha p-2,s)} \le C$, where C is a positive constant, for all $a \in \mathbb{D}$.

8. Proof of Theorem 6

Since (1) clearly implies (2) and (2) implies (3) by Theorem 4 and the fact that identity function belongs to $F(p,\alpha p-2,s)$ with the present parameters, the first thing to show is that (3) implies (1). By Theorem 4 it remains to prove the inclusion of $f\circ\varphi$ to the space $\mathcal{B}_{\beta,0}$ with every $f \in \mathcal{B}_\alpha$. But this follows from the (3) and the inequality

$$\left|f'(\varphi(z))\right|\left|\varphi'(z)\right|\left(1-|z|^2\right)^\beta \le \|f\|_{\mathcal{B}_\alpha}\frac{\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^\alpha}\left(1-|z|^2\right)^\beta. \tag{41}$$

In a similar manner (3) implies (4) and (4) clearly implies (5). It remains to show that (5) implies (3). Let $\phi_{\gamma,a}(z) := 1-(1-\overline{a}z)^\gamma$, where $0<\gamma<1$ and $a \in \mathbb{D}$. Given $\varepsilon>0$,

$$\frac{\left|\varphi'(z)\right|}{\left|1-\overline{a}\varphi(z)\right|}\left(1-|z|^2\right)^\beta \le 2\varepsilon C_2\left(1-|a|^2\right)^{\alpha-1} \tag{42}$$

for all $a \in \mathbb{D}$, C_2 being a positive constant independent of a and $|z|$ close enough to the unit circle. The condition (3) follows by choosing $a = \varphi(z)$.

9. Proof of Theorem 8

It will be shown that (1) and (2) are equivalent, and (2) and (3) are equivalent.

9.1. Proof of (2) $\Rightarrow$ (1). By Lemma C,

$$I_b(f) := \int_{\mathbb{D}} \left|(f\circ\varphi)'(z)\right|^{p_2}\left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(z) = \int_{\mathbb{D}} \left|f'(w)\right|^{p_2} d\mu_b(w), \tag{43}$$

where $d\mu_b(w) = \sum_{z\in\varphi^{-1}\{w\}} |\varphi'(z)|^{p_2-2}(1-|z|^2)^{q_2} g^{s_2}(z,b)\,dA(w)$. By Theorem B and Lemmas A and C,

$$\begin{aligned} I_b(f) &\le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{p_2(q_1+2)/p_1} d\mu_b(w)\, \|f'\|^{p_2}_{A^{p_1}_{q_1}} \\ &\le \sup_{a,b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(\varphi(z))\right|^{p_2(q_1+2)/p_1} \left|\varphi'(z)\right|^{p_2} \times \left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(w)\, \|f\|^{p_2}_{F(p_1,q_1,0)}, \end{aligned} \tag{44}$$

and it follows that $C_\varphi : F(p_1,q_1,0) \to F(p_2,q_2,s_2)$ is bounded if (2) is satisfied.

9.2. Proof of (1) $\Rightarrow$ (2). This implication can be proved by using the test functions f_a such that $f_a'(z) = (\varphi_a'(z))^{(2+q_1)/p_1}$. These functions satisfy the inequality $\|f_a\|_{F(p_1,q_1,0)} \le C < \infty$ for all $a \in \mathbb{D}$ by Forelli-Rudin estimates [5, Lemma 4.2.2].

9.3. Proof of (2) $\Rightarrow$ (3). This implication can be proved in a similar manner as the implication (2) $\Rightarrow$ (1) since the assertion in Theorem B holds also when $|f|^p$ is replaced with a subharmonic function $(h^*)^p$, $h \in B(\mathbb{D})$, and $\|f\|_{A^p_\alpha}$ is replaced by $\|h\|_{F^*(p,\alpha,0)}$. The proof of this fact will be presented next since some of the details will be needed later on. Since $(f^*)^{p_1}$ is subharmonic, there exists a positive constant C, depending only on p_1 and p_2, such that

$$\left(f^*(w)\right)^{p_2} \le \frac{C}{\left(1-|w|^2\right)^{2p_2/p_1}} \left(\int_{D(w,1/2)} \left(f^*(z)\right)^{p_1} dA(z)\right)^{p_2/p_1}. \tag{45}$$

This together with Lemma C implies

$$\begin{aligned} &\int_{\mathbb{D}} \left(\left(f\circ\varphi\right)^*(z)\right)^{p_2}\left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(z) \\ &= \int_{\mathbb{D}} \left(f^*(w)\right)^{p_2} d\mu_b(w) \le C \int_{\mathbb{D}} \left(1-|w|^2\right)^{-2p_2/p_1} \\ &\quad \times \left(\int_{D(w,1/2)} \left(f^*(z)\right)^{p_1} dA(z)\right)^{p_2/p_1} d\mu_b(w). \end{aligned} \tag{46}$$

Then the symmetry $\chi_{D(z,r)}(w) = \chi_{D(w,r)}(z)$ of the characteristic functions of pseudohyperbolic discs and the Minkowski inequality yield

$$\begin{aligned} &\int_{\mathbb{D}} \left(\left(f\circ\varphi\right)^*(z)\right)^{p_2}\left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(z) \\ &\le C\left(\int_{\mathbb{D}} \left(f^*(z)\right)^{p_1} \times \left(\int_{D(w,1/2)} \frac{d\mu_b(w)}{\left(1-|w|^2\right)^{2p_2/p_1}}\right)^{p_1/p_2} \times dA(z)\right)^{p_2/p_1}. \end{aligned} \tag{47}$$

The desired inequality

$$\begin{aligned} &\int_{\mathbb{D}} \left(\left(f\circ\varphi\right)^*(z)\right)^{p_2}\left(1-|z|^2\right)^{q_2} g^{s_2}(z,b)\,dA(z) \\ &\le C'\left(\int_{\mathbb{D}} \left(f^*(z)\right)^{p_1}\left(1-|z|^2\right)^{q_1} \times \left(\int_{D(w,1/2)} \left|\varphi_z'(w)\right|^{(p_2/p_1)(q_1+2)} d\mu_b(w)\right)^{p_1/p_2} \times dA(z)\right)^{p_2/p_1} \\ &\le C' \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{(p_2/p_1)(q_1+2)} d\mu_b(w) \times \left(\int_{\mathbb{D}} \left(f^*(z)\right)^{p_1}\left(1-|z|^2\right)^{q_1} dA(z)\right)^{p_2/p_1} \end{aligned} \tag{48}$$

follows by the fact that $1-|w|^2 \simeq 1-|z|^2 \simeq |1-\overline{z}w|$ for $w \in D(z,1/2)$. Thus (2) implies (3) by Lemma C.

9.4. Proof of (3) $\Rightarrow$ (2). To prove this implication, define

$$h_a(z) := \frac{(4+2q_1-p_1)\,a}{6p_1} \int_0^z \left(\varphi_a'(w)\right)^{(2+q_1)/p_1} dw. \tag{49}$$

Then $\|h_a\|_{H^\infty} \le 1/2$ for all $a \in \mathbb{D}$, and therefore $h_a^*(z) \simeq |h_a'(z)|$ in $\mathbb{D}$. The implication (3) $\Rightarrow$ (2) now follows by using the functions h_a in a similar manner as the functions f_a in the proof of (1) $\Rightarrow$ (2).

10. Proof of Theorem 10

10.1. Proof of (1) $\Rightarrow$ (2). Suppose first that $C_\varphi : F(p_1,q_1,0) \to F_0(p_2,q_2,s_2)$ is bounded; that is, $C_\varphi : F(p_1,q_1,0) \to F(p_2,q_2,s_2)$ is bounded and $C_\varphi(F(p_1,q_1,0)) \subset F_0(p_2,q_2,s_2)$.

Then, by using the function $f(z) = z$, $f \in F(p_1, q_1, 0)$ the inclusion $C_\varphi(F(p_1, q_1, 0)) \subset F_0(p_2, q_2, s_2)$ implies that $\varphi \in F_0(p_2, q_2, s_2)$.

10.2. Proof of (2) $\Longrightarrow$ (1). Suppose now that $C_\varphi : F(p_1, q_1, 0) \to F(p_2, q_2, s_2)$ is bounded, $\varphi \in F_0(p_2, q_2, s_2)$, and $f \in F(p_1, q_1, 0)$. To show that $f \circ \varphi \in F_0(p_2, q_2, s_2)$, we argue as in the proof of [9, Theorem 5.2]. Let $\varepsilon > 0$. Since polynomials are dense in $F(p, q, 0)$, there exists a polynomial P such that

$$\|f - P\|_{F(p_1,q_1,0)} < \left(\frac{\varepsilon}{2^{p_2+1}}\right)^{1/p_2} \frac{1}{\|C_\varphi\|} \tag{50}$$

and due to the boundedness

$$\|C_\varphi(f - P)\|_{F(p_2,q_2,s_2)} < \left(\frac{\varepsilon}{2^{p_2+1}}\right)^{1/p_2}. \tag{51}$$

It is possible to find $\delta \in (0, 1)$ such that

$$\int_{\mathbb{D}} |\varphi'(z)|^{p_2} \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) < \frac{\varepsilon}{2^{p_2+1}\|P'\|_\infty^{p_2}}, \tag{52}$$

as $|b| > \delta$. The claim follows from (51) and (52), since for $|b| > \delta$

$$\begin{aligned}
&\int_{\mathbb{D}} \left|\left(C_\varphi(f)\right)'(z)\right|^{p_2} |\varphi'(z)|^{p_2} \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) \\
&\le \sup_{b\in\mathbb{D}} 2^{p_2} \int_{\mathbb{D}} \left|\left(C_\varphi(f) - C_\varphi(P)\right)'(z)\right|^{p_2} \\
&\qquad \times \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) \\
&\quad + 2^{p_2} \int_{\mathbb{D}} \left|\left(C_\varphi(P)\right)'(z)\right|^{p_2} \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) \\
&\quad < \frac{\varepsilon}{2} + 2^{p_2}\|P'\|_\infty^{p_2} \int_{\mathbb{D}} |\varphi'(z)|^{p_2} \left(1 - |z|^2\right)^{q_2} \\
&\qquad \times g^{s_2}(z, b)\, dA(z) < \varepsilon.
\end{aligned} \tag{53}$$

11. Proof of Theorem 11

11.1. Proof of (1) $\Longrightarrow$ (2). Assume $C_\varphi : F^*(p_1, q_1, 0) \to F_0^*(p_2, q_2, s_2)$ is bounded. Then C_φ from $F^*(p_1, q_1, 0)$ to $F^*(p_2, q_2, s_2)$ is bounded. Since $f(z) = z/2$ belongs to $F^*(p_1, q_1, 0)$, it follows that

$$\begin{aligned}
&\lim_{|b|\to 1} \int_{\mathbb{D}} |\varphi'(z)|^{p_2} \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) \\
&\le 2^{p_2} \lim_{|b|\to 1} \int_{\mathbb{D}} \left((f \circ \varphi)^*(z)\right)^{p_2} \\
&\qquad \times \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) = 0.
\end{aligned} \tag{54}$$

11.2. Proof of (2) $\Longrightarrow$ (1). Assume that $C_\varphi : F^*(p_1, q_1, 0) \to F^*(p_2, q_2, s_2)$ is bounded and $\varphi \in F_0(p_2, q_2, s_2)$. By Theorem 8 there exists $M > 0$ such that

$$\sup_{a,b\in\mathbb{D}} \int_{\mathbb{D}} \left(\varphi_a'(w)\right)^{(p_2/p_1)(q_1+2)} d\mu_b(w) \le M. \tag{55}$$

It follows by (48) that for any $r \in (0, 1)$

$$\begin{aligned}
&\int_{\mathbb{D}} \left((f \circ \varphi)^*(z)\right)^{p_2} \left(1 - |z|^2\right)^{q_2} g^{s_2}(z, b)\, dA(z) \\
&\le C\Bigg(\int_{\mathbb{D}} (f^*(z))^{p_1} \left(1 - |z|^2\right)^{q_1} \\
&\qquad \times \left(\int_{D(w,1/2)} (\varphi_z'(w))^{(p_2/p_1)(q_1+2)} d\mu_b(w) \right)^{p_1/p_2} \\
&\qquad \times dA(z) \Bigg)^{p_2/p_1} \\
&\le C\Bigg(\int_{\Delta(0,r)} (f^*(z))^{p_1} \left(1 - |z|^2\right)^{q_1} \\
&\qquad \times \left(\int_{D(w,1/2)} \left(\varphi_z'(w)\right)^{(p_2/p_1)(q_1+2)} d\mu_b(w) \right)^{p_1/p_2} dA(z) \\
&\qquad + M^{p_1/p_2} \int_{\mathbb{D}\setminus\Delta(0,r)} (f^*(z))^{p_1} \left(1 - |z|^2\right)^{q_1} dA(z) \Bigg)^{p_2/p_1} \\
&=: C\left(A(r, b) + M^{p_1/p_2} B(\delta)\right)^{p_2/p_1}.
\end{aligned} \tag{56}$$

For given $\varepsilon > 0$, use the Lebesgue Dominated Convergence Theorem to fix r such that

$$B(\delta) < \frac{1}{2M^{p_1/p_2}} \left(\frac{\varepsilon}{C}\right)^{p_1/p_2}. \tag{57}$$

On the other hand, with fixed r, there exists δ such that

$$\begin{aligned}
A(\delta, b) &= \int_{\Delta(0,r)} (f^*(z))^{p_1} \left(1 - |z|^2\right)^{q_1} \\
&\qquad \times \left(\int_{D(w,1/2)} \left(\varphi_z'(w)\right)^{(p_2/p_1)(q_1+2)} d\mu_b(w) \right)^{p_1/p_2} dA(z) \\
&\le (1 - r)^{-q_1-2} \int_{\Delta(0,r)} (f^*(z))^{p_1} \\
&\qquad \times \left(1 - |z|^2\right)^{q_1} dA(z) \left(\int_{D(w,1/2)} d\mu_b(w) \right)^{p_1/p_2} \\
&\le (1 - r)^{-q_1-2} \|f\|_{F^*(p_1,q_1,0)}^{p_1} \\
&\qquad \times \left(\int_{\mathbb{D}} |\varphi'(z)|^{p_2} \left(1 - |z|^2\right)^{q_1} g^{s_2}(z, b) dA(z) \right)^{p_1/p_2} \\
&< \frac{1}{2} \left(\frac{\varepsilon}{C}\right)^{p_1/p_2},
\end{aligned} \tag{58}$$

when $|b| > \delta$. Thus $f \circ \varphi$ belongs to $F_0^*(p_2, q_2, s_2)$ by (56), (57), and (58).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank Jouni Rättyä for many valuable comments and advice. This research was supported in part by the Grant of the MEC-Spain MTM2011-26358, the Finnish Cultural Foundation, Academy of Finland Project no. 268009, and Faculty of Science and Forestry of University of Eastern Finland Project no. 930349.

References

[1] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[2] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics, Springer, New York, NY, USA, 1993.

[3] J. M. Anderson, J. Clunie, and Ch. Pommerenke, "On Bloch functions and normal functions," *Journal für die Reine und Angewandte Mathematik*, vol. 270, pp. 12–37, 1974.

[4] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, Germany, 1992.

[5] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, NY, USA, 1990.

[6] R. Zhao, "On a general family of function spaces," *Annales Academiæ Scientiarium Fennicæ. Mathematica*, no. 105, pp. 1–56, 1996.

[7] J. Rättyä, "On some complex function spaces and classes," *Annales Academiæ Scientiarium Fennicæ. Mathematica*, no. 124, pp. 1–73, 2001.

[8] X. Li, F. Pérez-González, and J. Rättyä, "Composition operators in hyperbolic Q-classes," *Annales Academiæ Scientiarum Fennicæ. Mathematica*, vol. 31, no. 2, pp. 391–404, 2006.

[9] W. Smith and R. Zhao, "Composition operators mapping into the Q_p spaces," *Analysis. International Mathematical Journal of Analysis and its Applications*, vol. 17, no. 2-3, pp. 239–263, 1997.

[10] J. Xiao, *Holomorphic Q Classes*, vol. 1767 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2001.

[11] W. Yang, "Composition operators from $F(p, q, s)$ spaces to the nth weighted-type spaces on the unit disc," *Applied Mathematics and Computation*, vol. 218, no. 4, pp. 1443–1448, 2011.

[12] X. Zhu, "Composition operators from Bloch type spaces to $F(p, q, s)$ spaces," *Univerzitet u Nišu. Prirodno-Matematički Fakultet. Filomat*, vol. 21, no. 2, pp. 11–20, 2007.

[13] S. Ye, "Weighted composition operators from $F(p, q, s)$ into logarithmic Bloch space," *Journal of the Korean Mathematical Society*, vol. 45, no. 4, pp. 977–991, 2008.

[14] F. Colonna and S. Li, "Weighted composition operators from the minimal Möbius invariant space into the Bloch space," *Mediterranean Journal of Mathematics*, vol. 10, no. 1, pp. 395–409, 2013.

[15] J. Liu, Z. Lou, and A. K. Sharma, "Weighted differentiation composition operators to Bloch-type spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 151929, 9 pages, 2013.

[16] Y. He and L. Jiang, "Composition operators from B^α to $F(p, q, s)$," *Acta Mathematica Scientia B*, vol. 23, no. 2, pp. 252–260, 2003.

[17] R. Zhao, "On α-Bloch functions and VMOA," *Acta Mathematica Scientia B*, vol. 16, no. 3, pp. 349–360, 1996.

[18] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, NY, USA, 1981.

[19] J. Arazy, S. D. Fisher, and J. Peetre, "Möbius invariant function spaces," *Journal für die Reine und Angewandte Mathematik*, vol. 363, pp. 110–145, 1985.

[20] R. Aulaskari, D. A. Stegenga, and J. Xiao, "Some subclasses of *BMOA* and their characterization in terms of Carleson measures," *The Rocky Mountain Journal of Mathematics*, vol. 26, no. 2, pp. 485–506, 1996.

[21] D. H. Luecking, "Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives," *American Journal of Mathematics*, vol. 107, no. 1, pp. 85–111, 1985.

[22] F. Pérez-González and J. Rättyä, "Forelli-Rudin estimates, Carleson measures and $F(p, q, s)$-functions," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 2, pp. 394–414, 2006.

[23] M. Essén, D. F. Shea, and C. S. Stanton, "A value-distribution criterion for the class L log L, and some related questions," *Annales de l'Institute Fourier*, vol. 35, no. 4, pp. 127–150, 1985.

[24] C. S. Stanton, "Counting functions and majorization for Jensen measures," *Pacific Journal of Mathematics*, vol. 125, no. 2, pp. 459–468, 1986.

[25] J. H. Shapiro, "The essential norm of a composition operator," *Annals of Mathematics*, vol. 125, no. 2, pp. 375–404, 1987.

[26] M. Tjani, "Compact composition operators on Besov spaces," *Transactions of the American Mathematical Society*, vol. 355, no. 11, pp. 4683–4698, 2003.

[27] S. Yamashita, "Gap series and α-Bloch functions," *Yokohama Mathematical Journal*, vol. 28, no. 1-2, pp. 31–36, 1980.

[28] A. Zygmund, *Trigonometric Series*, Cambridge University Press, London, UK, 1959.

[29] M. Kotilainen, "On composition operators in Q_K type spaces," *Journal of Function Spaces and Applications*, vol. 5, no. 2, pp. 103–122, 2007.

[30] J. M. Ortega and J. Fàbrega, "Pointwise multipliers and corona type decomposition in BMOA," *Annales de l'Institute Fourier*, vol. 46, no. 1, pp. 111–137, 1996.